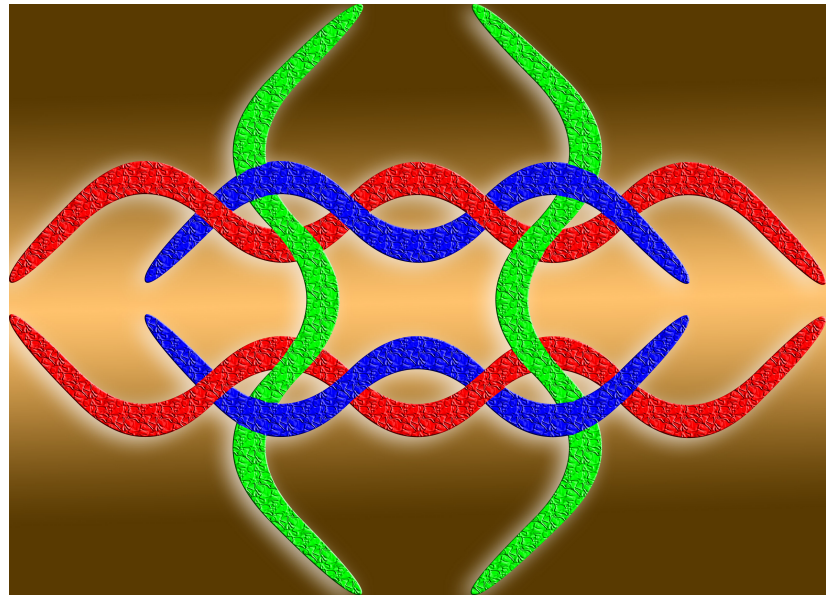

From Modal Logic to (Co)Algebraic Reasoning

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(actual version: http://fldit-www.cs.tu-dortmund.de/∼peter/CTL.pdf)

Contents

Kripke models and modal formulas

A **Kripke model** $K = (Q, Lab, At, trans, transL, value, valueL)$ consists of

- a set Q of **states**,
- a set Lab of **labels** (actions, input, etc.),
- a set At of **atoms**,
- **transition relations** $trans : Q \to \mathcal{P}(Q)$ and $transL : Q \times Lab \to \mathcal{P}(Q)$,
- **value relations** $value : At \to \mathcal{P}(Q)$ and $valueL : At \times Lab \to \mathcal{P}(Q)$.

State and path formulas

are generated by the following CFG rules: Let $S = \{sf, pf\}$ and V be an S-sorted set of variables.

$$sf \;\to\; at \mid val(at, lab) \qquad \text{for all } at \in At \text{ and } lab \in Lab \tag{1}$$

$$sf \;\to\; true \mid false \mid \neg sf \mid sf \vee sf \mid sf \wedge sf \mid sf \Rightarrow sf \tag{2}$$

$$sf \rightarrow EX\ sf \mid AX\ sf \mid \langle lab \rangle sf \mid [lab]sf \qquad \text{for all } lab \in L \qquad (4)$$

$$sf \rightarrow x \mid \mu x.sf \mid \nu x.sf \qquad \text{for all } x \in V \qquad (5)$$

$$sf \rightarrow EF\ sf \mid AF\ sf \mid EG\ sf \mid AG\ sf$$

$$sf \rightarrow sf\ EU\ sf \mid sf\ AU\ sf \mid sf\ EW\ sf \mid sf\ AW\ sf$$

$$sf \rightarrow sf\ ER\ sf \mid sf\ AR\ sf$$

$$pf \rightarrow at \mid at(lab) \qquad \text{for all } at \in At \text{ and } lab \in L \qquad (6)$$

$$pf \rightarrow true \mid false \mid \neg pf \mid pf \vee pf \mid pf \wedge pf \mid pf \Rightarrow pf \qquad (7)$$

$$pf \rightarrow next\ pf \mid \langle lab \rangle pf \mid [lab]pf \qquad \text{for all } lab \in L \qquad (8)$$

$$pf \rightarrow x \mid \mu x.pf \mid \nu x.pf \qquad \text{for all } x \in V \qquad (9)$$

$$pf \rightarrow F\ pf \mid G\ pf \mid pf\ U\ pf \mid pf\ W\ pf \mid pf\ R\ pf$$

$$pf \rightarrow pf \rightsquigarrow pf$$

Assumptions

Both the local and global semantics of modal formulas defined below require that formulas resp. transition relations meet the following constraints:

- For every free variable x of a subformula ψ of a formula φ, φ encloses a smallest abstraction $\mu x.\vartheta$ or $\nu x.\vartheta$ (see rules (5) and (9)) that encloses ψ.
  $\Longrightarrow$ *Abstractions can be evaluated hierarchically.*

- Every free occurrences of a variable x in $\mu x.\varphi$ resp. $\nu x.\varphi$ has **positive polarity**, i.e., the number of negation symbols of $\mu x.\varphi$ resp. $\nu x.\varphi$ above the occurrence is even.
  $\Longrightarrow$ *All logical operators are monotone.*

- $\rightarrow$ is **image finite**, i.e., for all $q \in Q$ and $lab \in Lab$, $trans(q)$ and $transL(lab)(q)$ are finite.
  $\Longrightarrow$ *All logical operators are ω-bicontinuous.*
  $\Longrightarrow$ *Fixpoints can be computed incrementally as Kleene closures of the accompanying step functions* (see below).

Local semantics

The set of all paths of K

$$
\begin{aligned}
path(K) =_{def} \{p \in Q^{\mathbb{N}} \mid \quad \forall\, i \in \mathbb{N} : \quad & p_{i+1} \in trans(p_i) \\
\vee \quad & \exists\, lab \in Lab : p_{i+1} \in transL(p_i)(lab) \\
\vee \quad & (trans(p_i) = \emptyset \wedge \\
& \forall\, lab \in Lab : transL(p_i)(lab) = \emptyset \wedge \\
& \forall\, k > i : p_i = p_k)\}
\end{aligned}
$$

The set of all paths of K that start with $q \in Q$

$$path(K, q) =_{def} \{p \in path(K) \mid p_0 = q\}$$

Let $q \in Q$, $at \in At$, $lab \in Lab$, $b : V \to \mathcal{P}(Q)$ and $c : V \to \mathcal{P}(path(K))$.

Validity of state formulas

$$
\begin{array}{lll}
q \models at & \Longleftrightarrow_{def} & q \in value(at) \\
q \models at(lab) & \Longleftrightarrow_{def} & q \in valueL(at, lab) \\
q \models true & & \text{for all } q \in Q \cup path(K) \\
q \models false & & \text{for no } q \in Q \cup path(K) \\
q \models \neg\varphi & \Longleftrightarrow_{def} & q \not\models \varphi
\end{array}
$$

$$
\begin{array}{lll}
q \models \varphi \vee \psi & \Longleftrightarrow_{def} & q \models \varphi \text{ or } q \models \psi \\
q \models \varphi \wedge \psi & \Longleftrightarrow_{def} & q \models \varphi \text{ and } q \models \psi \\
q \models \varphi \Rightarrow \psi & \Longleftrightarrow_{def} & q \models \neg\varphi \vee \psi \\
q \models x & \Longleftrightarrow_{def} & q \in b(x) \\
q \models \mu x.\varphi(x) & \Longleftrightarrow_{def} & q \in \text{least solution of } x = \varphi(x) \\
q \models \nu x.\varphi(x) & \Longleftrightarrow_{def} & q \in \text{greatest solution of } x = \varphi(x) \\
q \models EX\varphi & \Longleftrightarrow_{def} & \exists\, q' \in trans(q) : q' \models \varphi \\
q \models AX\varphi & \Longleftrightarrow_{def} & \forall\, q' \in trans(q) : q' \models \varphi \\
q \models \langle lab \rangle\varphi & \Longleftrightarrow_{def} & \exists\, q' \in transL(q)(lab) : q' \models \varphi \\
q \models [lab]\varphi & \Longleftrightarrow_{def} & \forall\, q' \in transL(q)(lab) : q' \models \varphi \\
q \models EF\varphi & \Longleftrightarrow_{def} & \exists\, p \in path(K, q)\ \exists\, i \in \mathbb{N} : p_i \models \varphi \\
q \models AF\varphi & \Longleftrightarrow_{def} & \forall\, p \in path(K, q)\ \exists\, i \in \mathbb{N} : p_i \models \varphi \\
q \models EG\varphi & \Longleftrightarrow_{def} & \exists\, p \in path(K, q)\ \forall\, i \in \mathbb{N} : p_i \models \varphi \\
q \models AG\varphi & \Longleftrightarrow_{def} & \forall\, p \in path(K, q)\ \forall\, i \in \mathbb{N} : p_i \models \varphi \\
q \models \varphi\ EU\ \psi & \Longleftrightarrow_{def} & \exists\, p \in path(K, q)\ \exists\, i \in \mathbb{N} : p_i \models \psi \wedge \forall\, j < i : p_j \models \varphi \\
q \models \varphi\ AU\ \psi & \Longleftrightarrow_{def} & \forall\, p \in path(K, q)\ \exists\, i \in \mathbb{N} : p_i \models \psi \wedge \forall\, j < i : p_j \models \varphi
\end{array}
$$

$$q \models \varphi\ EW\ \psi \iff_{def} \exists\, p \in path(K,q)\ \forall\, i \in \mathbb{N} : p_i \models \psi \Rightarrow \forall\, j < i : p_j \models \varphi$$
$$q \models \varphi\ AW\ \psi \iff_{def} \forall\, p \in path(K,q)\ \forall\, i \in \mathbb{N} : p_i \models \psi \Rightarrow \forall\, j < i : p_j \models \varphi$$
$$q \models \varphi\ ER\ \psi \iff_{def} \exists\, p \in path(K,q)\ \forall\, i \in \mathbb{N} : p_i \models \psi \vee \exists\, j < i : p_j \models \varphi$$
$$q \models \varphi\ AR\ \psi \iff_{def} \forall\, p \in path(K,q)\ \forall\, i \in \mathbb{N} : p_i \models \psi \vee \exists\, j < i : p_j \models \varphi$$

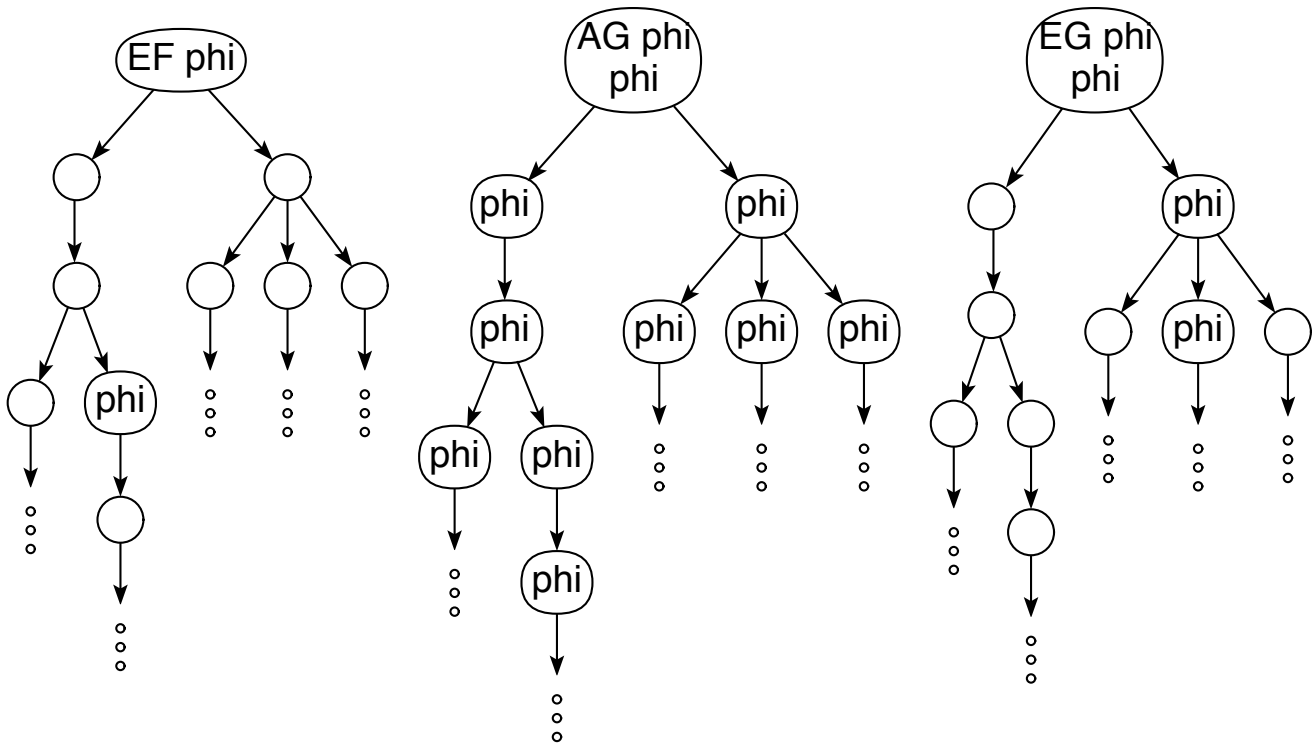


Fig. 1. *Transition graphs whose roots satisfy $EF\varphi$, $AG\varphi$ and $EG\varphi$, respectively*

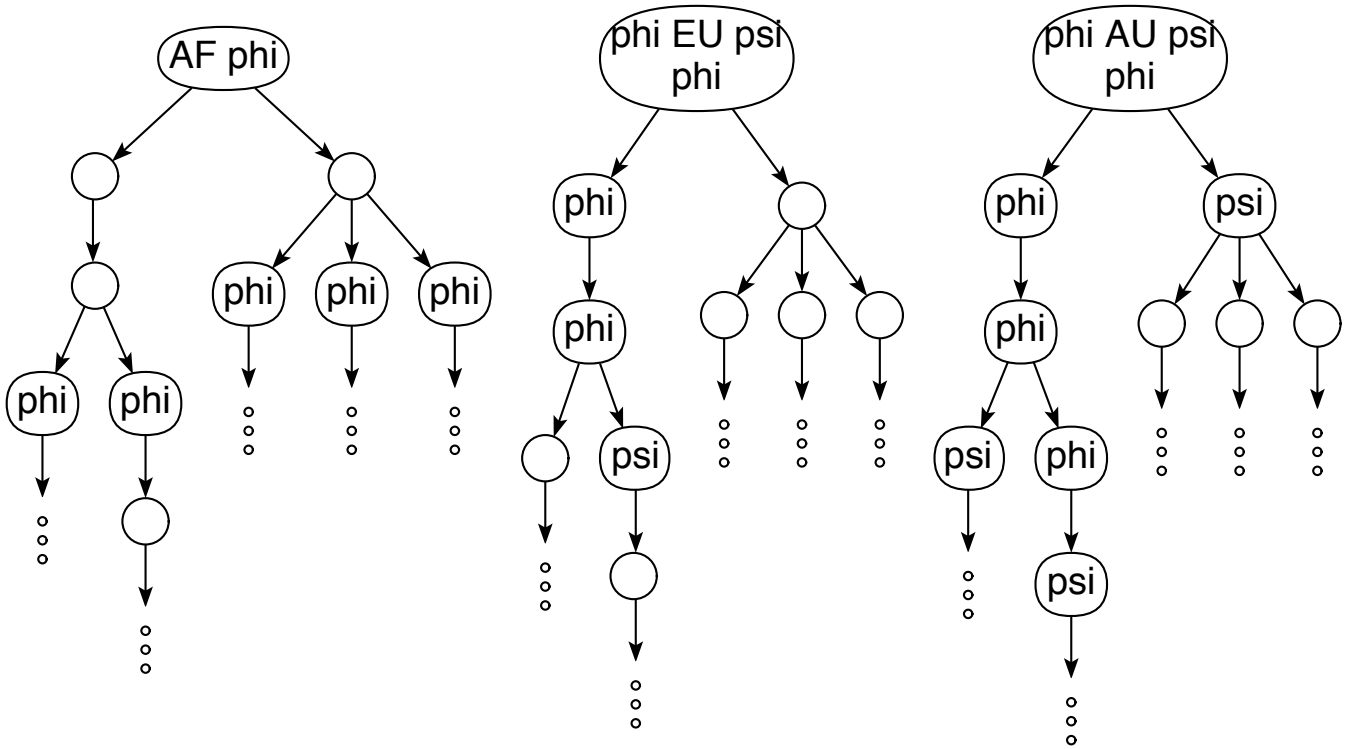


Fig. 2. *Transition graphs whose roots satisfy $AF\varphi$, $\varphi EU\psi$ and $\varphi AU\psi$, respectively*

Validity of path formulas

$$
\begin{array}{lll}
p \models at & \Longleftrightarrow_{def} & p_0 \in value(at) \\
p \models at(lab) & \Longleftrightarrow_{def} & p_0 \in valueL(at, lab) \\
p \models true & & \text{for all } p \in S \cup path(K) \\
p \models false & & \text{for no } p \in S \cup path(K) \\
p \models \neg\varphi & \Longleftrightarrow_{def} & p \not\models \varphi \\
p \models \varphi \vee \psi & \Longleftrightarrow_{def} & p \models \varphi \text{ oder } q \models \psi \\
p \models \varphi \wedge \psi & \Longleftrightarrow_{def} & p \models \varphi \text{ und } q \models \psi \\
p \models \varphi \Rightarrow \psi & \Longleftrightarrow_{def} & p \models \neg\varphi \vee \psi \\
p \models x & \Longleftrightarrow_{def} & s \in c(x) \\
p \models \mu x\varphi & \Longleftrightarrow_{def} & p \in \text{least solution in x of the equation } x = \varphi \\
p \models \nu x\varphi & \Longleftrightarrow_{def} & p \in \text{greatest solution in x of the equation } x = \varphi \\
p \models next\varphi & \Longleftrightarrow_{def} & \lambda i.p_{i+1} \models \varphi \\
p \models F\varphi & \Longleftrightarrow_{def} & \exists\, i \in \mathbb{N} : \lambda k.p_{i+k} \models \varphi \\
p \models G\varphi & \Longleftrightarrow_{def} & \forall\, i \in \mathbb{N} : \lambda k.p_{i+k} \models \varphi \\
p \models \varphi\ U\ \psi & \Longleftrightarrow_{def} & \exists\, i \in \mathbb{N} : \lambda k.p_{i+k} \models \psi \wedge \forall\, j < i : \lambda k.p_{j+k} \models \varphi \\
p \models \varphi\ W\ \psi & \Longleftrightarrow_{def} & \forall\, i \in \mathbb{N} : \lambda k.p_{i+k} \models \psi \Rightarrow \forall\, j < i : \lambda k.p_{j+k} \models \varphi \\
p \models \varphi\ R\ \psi & \Longleftrightarrow_{def} & \forall\, i \in \mathbb{N} : \lambda k.p_{i+k} \models \psi \vee \exists\, j < i : \lambda k.p_{j+k} \models \varphi \\
p \models \varphi \rightsquigarrow \psi & \Longleftrightarrow_{def} & p \models G(\varphi \Rightarrow F\psi)
\end{array}
$$

Global semantics

The abstract syntax of the above grammar generating state and path formulas – except rules (5) and (9) – is given by a constructive signature $\Sigma = (S, OP)$ with sort set $S = \{sf, pf\}$.

The sf- or pf-constructors of Σ are usually called **modal** or **temporal operators**, respectively.

Every state or path formula is represented by a Σ-term over V.

The **global semantics** of state and path formulas is given by the following Σ-Algebra $\mathcal{A} = (A, Op)$ (see [19]):

$$\begin{aligned} A_{sf} &= \mathcal{P}(Q) \\ A_{pf} &= \mathcal{P}(path(K)) \end{aligned}$$

For all $qs, qs' \subseteq Q$ and $f : Q \to \mathcal{P}(Q)$,

$$\begin{aligned} imgsShares(qs)(f)(qs') &=_{def} \{q \in qs \mid f(q) \cap qs' \neq \emptyset\}, \\ imgsSubset(qs)(f)(qs') &=_{def} \{q \in qs \mid f(q) \subseteq qs'\}. \end{aligned}$$

For all $at \in At$, $lab \in Lab$, $qs, qs' \subseteq Q$ and $ps, ps' \subseteq path(K)$,

$$\begin{aligned} at^{\mathcal{A}} &= value(at) \\ at(lab)^{\mathcal{A}} &= valueL(at, lab) \end{aligned}$$

$$
\begin{array}{lcl}
true^{\mathcal{A}} & = & Q \\
false^{\mathcal{A}} & = & \emptyset \\
\neg^{\mathcal{A}}(qs) & = & Q \setminus qs \\
\vee^{\mathcal{A}} & = & \cup \\
\wedge^{\mathcal{A}} & = & \cap \\
qs \Rightarrow^{\mathcal{A}} qs' & = & (Q \setminus qs) \cup qs' \\
EX^{\mathcal{A}} & = & imgsShares(Q)(trans) \\
AX^{\mathcal{A}} & = & imgsSubset(Q)(trans) \\
\langle lab \rangle^{\mathcal{A}} & = & imgsShares(Q)(flip(transL)(lab)) \\
[lab]^{\mathcal{A}} & = & imgsSubset(Q)(flip(transL)(lab)) \\
true^{\mathcal{A}} & = & path(K) \\
\neg^{\mathcal{A}}(ps) & = & path(K) \setminus ps \\
ps \Rightarrow^{\mathcal{A}} ps' & = & (path(K) \setminus ps) \cup ps' \\
next^{\mathcal{A}}(ps) & = & \{p \in path(K) \mid \lambda i.p_{i+1} \in ps\}
\end{array}
$$

flip $: (a \rightarrow b \rightarrow c) \rightarrow (b \rightarrow a \rightarrow c)$ denotes the Haskell function that exchanges the arguments of a cascaded binary function.

The remaining modal and temporal operators are reduced to μ- resp. ν-abstractions:

$$
\begin{array}{lcl}
EF\varphi & = & \mu x.\varphi \vee EX\ x \\
AF\varphi & = & \mu x.\varphi \vee (AX\ x \wedge EX\ true) \\
EG\varphi & = & \nu x.\varphi \wedge (EX\ x \vee AX\ false) \\
AG\varphi & = & \nu x.\varphi \wedge AX\ x \\
\varphi\ EU\ \psi & = & \mu x.\psi \vee (\varphi \wedge EX\ x) \\
\varphi\ AU\ \psi & = & \mu x.\psi \vee (\varphi \wedge AX\ x) \\
\varphi\ EW\ \psi & = & \nu x.\psi \vee (\varphi \wedge EX\ x) \\
\varphi\ AW\ \psi & = & \nu x.\psi \vee (\varphi \wedge AX\ x) \\
\varphi\ ER\ \psi & = & \nu x.\psi \wedge (\varphi \vee EX\ x) \\
\varphi\ AR\ \psi & = & \nu x.\psi \wedge (\varphi \vee AX\ x)
\end{array}
$$

$$
\begin{array}{lcl}
F\varphi & = & \mu x.\varphi \vee next\ x \\
G\varphi & = & \nu x.\varphi \wedge next\ x \\
\varphi\ U\ \psi & = & \mu x.\psi \vee (\varphi \wedge next\ x) \\
\varphi\ W\ \psi & = & \nu x.\psi \vee (\varphi \wedge next\ x) \\
\varphi\ R\ \psi & = & \nu x.\psi \wedge (\varphi \vee next\ x) \\
\varphi \rightsquigarrow \psi & = & G(\varphi \Rightarrow F\psi)
\end{array}
$$

Σ-terms representing μ- or ν-abstractions are evaluated in $\mathcal{A}$ as follows:

Since A_{sf} and A_{pf} are powersets and thus ω-bicomplete carrier sets – with partial order $\subseteq$, least element $\emptyset$ and greatest element Q resp. $path(K)$ (see [19]) –, **Kleene's Fixpoint Theorem** (see [13, 19]) provides us with the least fixpoint

$$lfp(f) = \bigsqcup_{n \in \mathbb{N}} f^n(false^{\mathcal{A}})$$

and the greatest fixpoint

$$gfp(f) = \bigsqcap_{n \in \mathbb{N}} f^n(true^{\mathcal{A}})$$

of every S-sorted ω-bicontinuous function $f : A \to A$.

If A_{sf} und A_{pf} are finite, then

$$lfp(f) = fixpt(\leq)(f)(false^{\mathcal{A}}) \quad \text{and} \quad gfp(f) = fixpt(\geq)(f)(true^{\mathcal{A}})$$

where

$$fixpt : (A \to A \to Bool) \to (A \to A) \to (A \to A)$$

is defined as follows:

$$fixpt(\leq)(f)(a) \;=\; if\ f(a) \leq a\ then\ a\ else\ fixpt(\leq)(f)(f(a))$$

Let φ be a modal formula satisfying the above assumptions and $g : V \to A$ be an S-sorted function.

The extension $g^* : T_\Sigma(V) \to \mathcal{A}$ of g to Σ-terms is inductively defined as follows:

For all $x \in V$, $s \in \{sf, pf\}$, $op : s^n \to s \in OP$, $\varphi_1, \ldots, \varphi_n \in T_\Sigma(V)_s$ and $\varphi \in T_\Sigma(V)$,

$$\begin{aligned}
g^*(x) &= g(x), \\
g^*(op(\varphi_1, \ldots, \varphi_n)) &= op^{\mathcal{A}}(g^*(\varphi_1), \ldots, g^*(\varphi_n)), \\
g^*(\mu x.\varphi) &= lfp(\lambda a.g[a/x]^*(\varphi)), && (1) \\
g^*(\nu x.\varphi) &= gfp(\lambda a.g[a/x]^*(\varphi)). && (2)
\end{aligned}$$

Under the above assumptions on formulas and transition relations, the function

$$\lambda a.g[a/x]^*(\varphi) : A \to A$$

is ω-bicontinuous and thus (1) is the least and (2) the greatest fixpoint of $\lambda a.g[a/x]^*(\varphi)$.

Two formulas φ and ψ are **$\mathcal{A}$-equivalent**, written: $\varphi \sim \psi$, if for all $g : V \to A$,

$$g^*(\varphi) = g^*(\psi).$$

For instance,

$$
\begin{array}{lcl}
\neg F\varphi & \sim & G(\neg\varphi) \\
\neg(\varphi\ R\ \psi) & \sim & \neg\varphi\ U\ \neg\psi \\
F\varphi & \sim & \mathit{true}\ U\ \varphi \\
G\varphi & \sim & \varphi\ W\ \mathit{false} \\
\varphi\ U\ \psi & \sim & (\varphi\ W\ \psi) \wedge F\psi \\
\varphi\ W\ \psi & \sim & (\varphi\ U\ \psi) \vee G\varphi
\end{array}
$$

In the following chapters 3 to 7 we only deal with state formulas. Path formulas (and temporal logic) are treated in chapter 22 in the context of the specification `stream`.

Kripke models in Expander2

The command *specification > build Kripke model* constructs a Kripke model

$$K = (Q, At, Lab, trans, transL, value, valueL)$$

from parts of the actual specification SP. The sets Q, At and Lab are stored in the constants

```
states, atoms, labels :: [Term String]
```

where `Term String` is the `String`-instance of the datatype

```
data Term a = V a | F a [Term a] | Hidden Special
type TermS = Term String
leaf a = F a []
```

SP should contain an axiom `states == inits` where `inits` evaluates to a list of initial states. *build Kripke model* adds to `states` all states that are reachable from `inits` via the transition axioms of SP, which have the following form:

φ `==> st -> st'`, φ `==> st -> branch$sts`,

φ `==> (st,lab) -> st'`, φ `==> (st,lab) -> branch$sts`

`st`, `lab`, `st'` and `sts` are supposed to evaluate to a state or atom, a label, a state or a list of states, respectively.

The transition axioms of SP are converted into the transition and value relations of K and stored as the lists

```
trans, value ::  [[Int]]

transL, valueL ::  [[[Int]]]
```

such that for all $i, j, k \in \mathbb{N}$,

$$
\begin{aligned}
i \in \texttt{trans}!!j \quad &\iff \texttt{states}!!i \in \mathit{trans}(\texttt{states}!!j),\\
i \in \texttt{transL}!!j!!k &\iff \texttt{states}!!i \in \mathit{transL}(\texttt{states}!!j)(\texttt{labels}!!k),\\
i \in \texttt{value}!!j \quad &\iff \texttt{atoms}!!i \in \mathit{value}(\texttt{atoms}!!j),\\
i \in \texttt{valueL}!!j!!k &\iff \texttt{atoms}!!i \in \mathit{valueL}(\texttt{atoms}!!j)(\texttt{labels}!!k).
\end{aligned}
$$

Signatures in Expander2 (see `Eterm.hs`)

Symbols, simplification axioms and transition axioms are stored in the signature of SP that is an object of the following O'Haskell type.

```
struct Sig = isPred,isCopred,isConstruct,isDefunct,isFovar,isHovar,blocked
                                :: String -> Bool
             hovarRel           :: String -> String -> Bool
             simpls,transitions :: [(Term String,[Term String],Term String)]
             states,atoms,labels :: [Term String]
             trans,value        :: [[Int]]
             transL,valueL      :: [[[Int]]]
             safeEqs            :: Bool

out,parents :: Sig -> [[Int]]
out sig     = invertRel sig.atoms sig.states sig.value
parents sig = invertRel sig.states sig.states sig.trans

invertRel :: [a] -> [b] -> [[Int]] -> [[Int]]
invertRel as bs iss = map f $ indices_ bs where f i = searchAll (i `elem`) iss

outL,parentsL :: Sig -> [[[Int]]]
outL sig     = invertRelL sig.labels sig.atoms sig.states sig.valueL
parentsL sig = invertRelL sig.labels sig.states sig.states sig.transL

invertRelL :: [a] -> [b] -> [c] -> [[[Int]]] -> [[[Int]]]
invertRelL as bs cs isss = map f $ indices_ cs
                           where f i = map g [0..length as-1] where
                                       where g j = searchAll h isss where
                                                   h iss = i `elem` iss!!j
```

Commands of the solver template (see `Ecom.hs`) access components of the current signature via the following command:

```
getSignature = do
         let (ps,cps,cs,ds,fs,hs) = symbols
             (sts,ats,labs,tr,trL,va,vaL) = kripke
             isPred       = (`elem` ps)  ||| projection
             isCopred     = (`elem` cps) ||| projection
             isConstruct  = (`elem` cs)  ||| just . parse int |||
                            just . parse real ||| just . parse quoted |||
                            just . parse (strNat "inj")
             isDefunct    = (`elem` ds) ||| projection
             isFovar      = shares [x,base x] fs
             isHovar      = shares [x,base x] $ map fst hs
             hovarRel x y = isHovar x &&
                            case lookup (base x) hs of
                                 Just es@(_:_) -> isHovar y || y `elem` es
                                 _ -> not $ isFovar y
             (block,xs) = constraints
             blocked x = if block then z `elem` xs else z `notElem` xs
                         where z = head $ words x
             simpls = simplRules; transitions = transRules
             states = sts; atoms = ats; labels = labs; trans = tr
             transL = trL; value = va; valueL = vaL; safeEqs = safe;
         return $ struct ..Sig
```

```
projection :: String -> Bool
projection = just . parse (strNat "get")
```

Embedding of trans, transL, out and outL into the simplifier

```
mkStates,mkLabels,mkAtoms :: Sig -> [Int] -> TermS
mkStates sig = mkList . map (sig.states!!)
mkLabels sig = mkList . map (sig.labels!!)
mkAtoms sig = mkList . map (sig.atoms!!)

simplifyS sig (F "trans" state@(_:_)) =
               if null sig.states then Just $ mkList $ simplReducts sig True st
                                  else do i <- search (== st) sig.states
                                          Just $ mkStates sig $ sig.trans!!i
               where st = mkTup state

simplifyS sig (F "$" [F "transL" state@(_:_),lab]) =
              if null sig.states then Just $ mkList $ simplReducts sig True
                                                    $ mkPair st lab
                                 else do i <- search (== st) sig.states
                                         k <- search (== lab) sig.labels
                                         Just $ mkStates sig $ sig.transL!!i!!k
              where st = mkTup state
```

```
simplifyS sig (F "out" state@(_:_)) =
                    if null sig.states then Just $ mkList $ filter f sig.atoms
                                       else do i <- search (== st) sig.states
                                               Just $ mkAtoms sig $ out sig!!i
                    where st = mkTup state
                          f = elem st . simplReducts sig True

simplifyS sig (F "$" [F "outL" state@(_:_),lab]) =
                 if null sig.states then Just $ mkList $ filter f sig.atoms
                                    else do i <- search (== st) sig.states
                                            k <- search (== lab) sig.labels
                                            Just $ mkAtoms sig $ outL sig!!i!!k
                 where st = mkTup state
                       f = elem (mkPair st lab) . simplReducts sig True
```

Further functions for analyzing the Kripke model stored in the actual signature

All successors of state

```
simplifyS sig (F "succs" state) = do i <- search (== st) sig.states
                                     Just $ mkStates sig $ fixpt subset f [i]
                                  where st = mkTup state
                                        f is = is `join` joinMap g is
                                        ks = indices_ sig.labels
                                        g i = sig.trans!!i `join`
                                              joinMap ((sig.transL!!i)!!) ks
```

Direct predessors of state

```
simplifyS sig (F "parents" state) = do i <- search (== st) sig.states
                                       Just $ mkStates sig $ parents sig!!i
                                    where st = mkTup state

simplifyS sig (F "$" [F "parentsL" state,lab])
                                  = do i <- search (== st) sig.states
                                       k <- search (== lab) sig.labels
                                       Just $ mkStates sig $ parentsL sig!!i!!k
                                    where st = mkTup state
```

All predecessors of state

```
simplifyS sig (F "preds" state)   = do i <- search (== st) sig.states
                                       Just $ mkStates sig $ fixpt subset f [i]
                                    where st = mkTup state
                                          f is = is `join` joinMap g is
                                          ks = indices_ sig.labels
                                          g i = parents sig !!i `join`
                                                joinMap ((parentsL sig!!i)!!) ks
```

All state/label sequences from state1 to state2

```
simplifyS sig (F "$" [F "traces" state,st']) =
                                  do i <- search (== st) sig.states
                                     j <- search (== st') sig.states
                                     jList $ map (mkStates sig) $ traces f i j
                                  where st = mkTup state
                                        ks = indices_ sig.labels
                                        f i = sig.trans!!i `join`
                                              joinMap ((sig.transL!!i)!!) ks

simplifyS sig (F "$" [F "tracesL" state,st']) =
                               do i <- search (== st) sig.states
                                  j <- search (== st') sig.states
                                  jList $ map (mkLabels sig) $ tracesL ks f i j
                               where st = mkTup state
                                     ks = indices_ sig.labels
                                     f i k = (sig.transL!!i)!!k

traces :: Eq a => (a -> [a]) -> a -> a -> [[a]]
traces f a = h [a] a where
             h visited a c = if a == c then [[a]]
                                       else do b <- f a`minus`visited
                                               trace <- h (b:visited) b c
                                               [a:trace]
```

```
tracesL :: Eq a => [lab] -> (a -> lab -> [a]) -> a -> a -> [[lab]]
tracesL labs f a = h [a] a where
                   h visited a c = if a == c then [[]]
                                   else do lab <- labs
                                           b <- f a lab`minus`visited
                                           trace <- h (b:visited) b c
                                           [lab:trace]
```

Model checking as simplification

The following specification provides

- simplification axioms for reducing modal operators such that they can be evaluated in *ctlAlg* with *eval/G* (see above),
- simplification axioms for reducing state or path formulas to *True* or *False*,
- (co-)Horn clauses for proving state or path formulas by (co)resolution.

The constant **noProcs** is set by pressing the *specification > set number of processes* button.

modal

```
defuncts: noProcs procs states labels atoms valid
preds:    X sat satL -> true false not \/ /\ `then` or and alloutL
          allanytrans allanytransL EX XE <> >< EF AF FE FA `EU` `AU`
copreds:  ~ eqstate AX XA # ## AG EG GE GA `EW` `AW` `ER` `AR`
fovars:   s s' st st' lab at
hovars:   X

axioms:
```

```
  procs == [0..noProcs-1]    -- used in many specifications that import modal

-- state equivalence relations

 & (eqstate <==> NU X.rel((st,st'),out$st=out$st' &
                                   alloutL(st,st') &
                                   allanytrans(X)(st,st') &
                                   allanytrans(X)(st',st) &
                                   allanytransL(X)(st,st') &
                                   allanytransL(X)(st',st)))

 & (st ~ st' ===> out$st=out$st' &
                  alloutL(st,st') &
                  allanytrans(~)(st,st') & allanytrans(~)(st',st) &
                  allanytransL(~)(st,st') & allanytransL(~)(st',st))

 & (alloutL(st,st') <==> all(rel(lab,outL(st)$lab=outL(st')$lab))$labels)

 & (allanytrans(P)(st,st')  <==> allany(P)(trans$st,trans$st'))
 & (allanytransL(P)(st,st') <==> all(rel(lab,allany(P)(transL(st)$lab,
                                                        transL(st')$lab)))
                                     $labels)

-- simplification of first-order state formulas
```

```
 & (sat(at)$st <==> at `in` out$st)
 & (satL(at,lab)$st <==> at `in` outL(st)$lab)

 & (EX(P)$st <==> any(P)$concat$trans(st):map(transL$st)$labels)
 & (AX(P)$st <==> all(P)$concat$trans(st):map(transL$st)$labels)
 & ((lab<>P)$st <==> any(P)$transL(st)$lab)
 & ((lab#P)$st <==> all(P)$transL(st)$lab)

 & (XE(P)$st <==> any(P)$parents$st)
 & (XA(P)$st <==> all(P)$parents$st)
 & ((lab><P)$st <==> any(P)$parentsL(st)$lab)
 & ((lab##P)$st <==> all(P)$parentsL(st)$lab)

 & (true$st <==> True)
 & (false$st <==> False)
 & (not(P)$st <==> Not(P$st))
 & ((P/\Q)$st <==> P$st & Q$st)
 & ((P\/Q)$st <==> P$st | Q$st)
 & ((P`then`Q)$st <==> (P$st ==> Q$st))

-- simplification of second-order state formulas

 & (or <==> foldl(\/)$false)
 & (and <==> foldl(/\)$true)
```

```
 & (EF$P <==> MU X.(P\/EX$X))                -- forward finally on some path
 & (FE$P <==> MU X.(P\/XE$X))                -- backwards finally on some path
 & (AF$P <==> MU X.(P\/(AX(X)/\EX$true)))    -- forward finally on all paths
 & (FA$P <==> MU X.(P\/(XA(X)/\XE$true)))    -- backwards finally on all paths
 & (EG$P <==> NU X.(P/\(EX(X)\/AX$false)))   -- forward generally on some path
 & (GE$P <==> NU X.(P/\(XE(X)\/XA$false)))   -- backwards generally on some path
 & (AG$P <==> NU X.(P/\AX$X))                -- forward generally on all paths
 & (GA$P <==> NU X.(P/\XA$X))                -- backwards generally on all paths

 & ((P`EU`Q) <==> MU X.(Q\/(P/\EX$X)))       -- until
 & ((P`AU`Q) <==> MU X.(Q\/(P/\AX$X)))
 & ((P`EW`Q) <==> NU X.(Q\/(P/\EX$X)))       -- weak until
 & ((P`AW`Q) <==> NU X.(Q\/(P/\AX$X)))
 & ((P`ER`Q) <==> NU X.(Q/\(P\/EX$X)))       -- release
 & ((P`AR`Q) <==> NU X.(Q/\(P\/AX$X)))

-- narrowing of first-order state formulas

 & (EF(P)$st <=== P$st | EX(EF$P)$st)        -- forward finally on some path
 & (AG(P)$st ===> P$st & AX(AG$P)$st)        -- forward generally on all paths
 & (AF(P)$st <=== P$st | AX(AF$P)$st & EX(true)$st)
                                             -- forward finally on all paths
 & (EG(P)$st ===> P$st & (EX(EG$P)$st | AX(false)$st))
                                             -- forward generally on some path
 & (FE(P)$st <=== P$st | XE(FE$P)$st)        -- backwards finally on some path
```

```
& (GA(P)$st ===> P$st & XA(GA$P)$st)        -- backwards generally on all paths
& (FA(P)$st <=== P$st | XA(FA$P)$st & XE(true)$st)
                                            -- backwards finally on all paths
& (GE(P)$st ===> P$st & (XE(GE$P)$st | XA(false)$st))
                                            -- backwards generally on some path

& ((P`EU`Q)$st <=== Q$st | P$st & EX(P`EU`Q)$st)       -- until
& ((P`AU`Q)$st <=== Q$st | P$st & AX(P`AU`Q)$st)
& ((P`EW`Q)$st ===> Q$st | P$st & EX(P`EW`Q)$st)       -- weak until
& ((P`AW`Q)$st ===> Q$st | P$st & AX(P`AW`Q)$st)
& ((P`ER`Q)$st ===> Q$st & (P$st | EX(P`ER`Q)$st))     -- release
& ((P`AR`Q)$st ===> Q$st & (P$st | AX(P`AR`Q)$st))
```

Model checking as term evaluation

Given the representation of a Kripke model K produced by *build Kripke model* (see chapte 2), the following Haskell code implements the global semantics of state formulas w.r.t. K (see chapter 1).

A signature for state formulas

```
struct Modal a label = true,false :: a
                       neg :: a -> a
                       or,and :: a -> a -> a
                       ex,ax,xe,xa :: a -> a
                       dia,box,aid,xob :: label -> a -> a
```

The `Modal`*-algebra of state sets*

```
ctlAlg :: Sig -> Modal [Int] Int
ctlAlg sig = struct true  = sts
                    false = []
                    neg   = minus sts
                    or    = join
                    and   = meet
                    ex    = imgsShares sts $ f sig.trans sig.transL
                    ax    = imgsSubset sts $ f sig.trans sig.transL
                    xe    = imgsShares sts $ f (parents sig) (parentsL sig)
                    xa    = imgsSubset sts $ f (parents sig) (parentsL sig)
                    dia   = imgsShares sts . flip (g sig.transL)
```

```
                box   = imgsSubset sts . flip (g sig.transL)
                aid   = imgsShares sts . flip (g $ parentsL sig)
                xob   = imgsSubset sts . flip (g $ parentsL sig)
          where sts = indices_ sig.states
                f trans transL i = join (trans!!i) $ joinMap (g transL i)
                                                   $ indices_ sig.labels
                g transL i k = transL!!i!!k
```

The interpreter of state formulas in `ctlAlg`

```
type States = Maybe [Int]

foldModal :: Sig -> TermS -> States
foldModal sig = f $ const Nothing where
          alg = ctlAlg sig
          f :: (String -> States) -> TermS -> States
          f g (F x []) | just a       = a where a = g x
          f _ (F "true" [])           = Just alg.true
          f _ (F "false" [])          = Just alg.false
          f g (F "not" [t])           = do a <- f g t; Just $ alg.neg a
          f g (F "\\/" [t,u])         = do a <- f g t; b <- f g u
                                           Just $ alg.or a b
          f g (F "/\\" [t,u])         = do a <- f g t; b <- f g u
                                           Just $ alg.and a b
          f g (F "`then`" [t,u])      = do a <- f g t; b <- f g u
                                           Just $ alg.or (alg.neg a) b
```

```
f g (F "EX" [t])            = do a <- f g t; Just $ alg.ex a
f g (F "AX" [t])            = do a <- f g t; Just $ alg.ax a
f g (F "XE" [t])            = do a <- f g t; Just $ alg.xe a
f g (F "XA" [t])            = do a <- f g t; Just $ alg.xa a
f g (F "<>" [lab,t])        = do a <- f g t; k <- searchL lab
                                 Just $ alg.dia k a
f g (F "#" [lab,t])         = do a <- f g t; k <- searchL lab
                                 Just $ alg.box k a
f g (F "><" [lab,t])        = do a <- f g t; k <- searchL lab
                                 Just $ alg.aid k a
f g (F "##" [lab,t])        = do a <- f g t; k <- searchL lab
                                 Just $ alg.xob k a
f g (F ('M':'U':' ':x) [t]) = fixptM subset (step g x t) alg.false
f g (F ('N':'U':' ':x) [t]) = fixptM supset (step g x t) alg.true
f _ (F "$" [at,lab])        = do i <- searchA at; k <- searchL lab
                                 Just $ sig.valueL!!i!!k
f _ at                      = do i <- searchA at; Just $ sig.value!!i

searchA,searchL :: Term String -> Maybe Int
searchA at  = search (== at) sig.atoms
searchL lab = search (== lab) sig.labels

step :: (String -> States) -> String -> TermS -> [Int] -> States
step g x t a = f (upd g x $ Just a) t
```

`f(g) :: Term String -> Maybe [Int]` implements the (modal part of the) extension $g^* : T_\Sigma(V) \to \mathcal{A}$ (see Kripke models in Expander2).

Embedding of foldModal(sig) into the simplifier

$$\frac{eval(\varphi)}{\varphi^{\mathcal{A}}} \qquad \frac{evalG(\varphi)}{\textit{transition graph of } K \textit{ with all } q \in \varphi^{\mathcal{A}} \textit{ colored green}}$$

```
simplifyS (F "eval" [phi]) sig = do sts <- foldModal sig phi
                                    Just $ mkStates sig sts

simplifyS (F "evalG" [phi]) sig = do sts <- foldModal sig phi
                                     let f st = if st `elem` map (strs!!) sts
                                                then "dark green$"++st else st
                                     Just $ mapT f $ eqsToGraph [] eqs
     where [strs,labs] = map (map showTerm0)[sig.states,sig.labels]
           (eqs,_) = if null labs then relToEqs 0 $ mkPairs strs strs sig.trans
                     else relLToEqs 0 $ mkTriples strs labs strs sig.transL
```

Specifications that import `modal`

Actions for building and verifying a specification involving a Kripke model

- enter the name of the specification into the entry field
- press the *return* key: the specification is parsed
- press the *specification* > *build Kripke model* button
  A Kripke model is constructed from axioms of the specification
- press *graph* > *show graph of Kripke model* > *here*, and a graph comprising the states, atoms and transitions of the Kripke model is displayed on the solver canvas
- press *paint*, and the graph is displayed on the painter canvas (see below)
- customize the graph structure by moving nodes and red arc support points
- press *combis*, and the red arc support points are removed

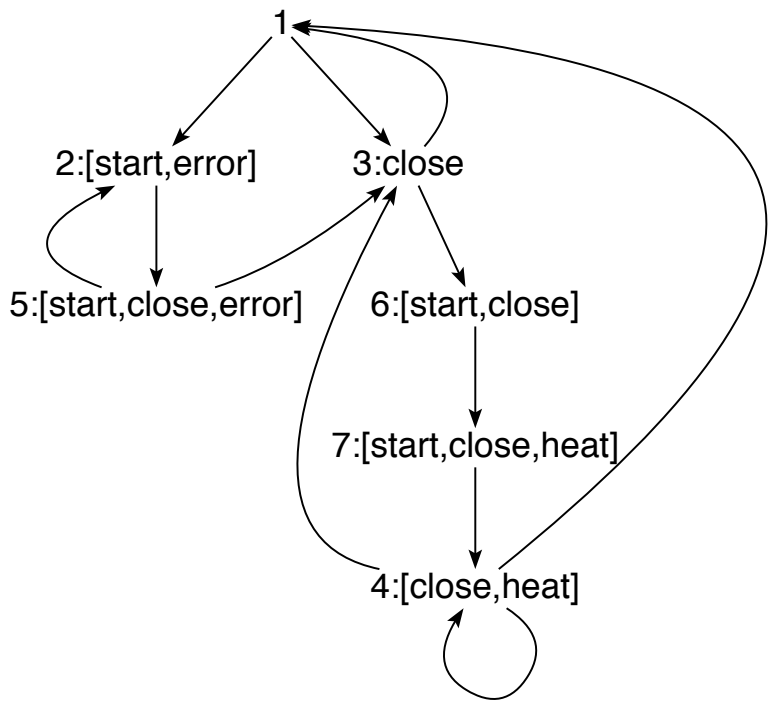
`micro` (see [4], section 4.1)

```
specs:          modal
constructs:     start close heat error

axioms:
```

```
  states == [1]             -- initial sates
& atoms == [start,close,heat,error]

& 1 -> branch[2,3] & 2 -> 5 & 3 -> branch[1,6]
& 4 -> branch[1,3,4] & 5 -> branch[2,3] & 6 -> 7 & 7 -> 4

& start -> branch[2,5,6,7]
& close -> branch[3..7]
& heat -> branch[4,7]
& error -> branch[2,5]

conjects:                   --          proof files

  EF(Sat$heat)$5            --> True    micro1P
& EG(Sat$heat)$7            --> True    micro2P, micro2cycleP
& AF(Sat$heat)$6            --> True    (16 parallel simplifications)
& all(EF$Sat$heat)$states   --> True    micro3P
& all(AF$Sat$heat)[4,6,7]   --> True    (26 parallel simplifications)

terms:

    eval$EF$heat                        --> states
<+> eval$EG$heat                        --> [4,7]
<+> eval$AF$heat                        --> [4,6,7]
<+> eval$start`then`heat                --> [1,3,4,7]
```

```
<+> eval$EX$start`then`heat              --> [1,3,4,5,6,7]
<+> eval$error`then`AG$not$heat          --> [1,3,4,6,7]
```

After the above actions the following graph appears on the painter canvas:



Four of the five conjectures of `micro` can be proved by applying simplification axioms and *selection* > *permute subtrees*. Due to the fact that the second conjecture involves the *greatest* fixpoint EG (see `modal`) is led into a cycle with simplification and subtree permutation (see `micro2cycleP`). Applying *selection* > *coinduction*, however, leads to a successful proof (see `micro2P`). It contains *two* coinduction steps because *EG(Sat$heat)$7* can only be shown along with *EG(Sat$heat)$4* (see chapter 9 below or [19], chapter 17). Here is the proof as recorded by Expander2 in `micro2P`:

```
0. Derivation of

EG(Sat(heat))$7

All simplifications are admitted.
Equation removal is safe.

1. Adding

  (EG0(z0)$z1 <=== z0 = Sat(heat) & z1 = 7)

to the axioms and applying COINDUCTION wrt

  (EG(P)$st ===> P(st) & (EX(EG(P))$st | AX(false)$st))

at positions [] of the preceding trees leads to
```

```
EG0(Sat(heat))$4

The reducts have been simplified in parallel.

2. Adding

  (EG0(z2)$z3 <=== z2 = Sat(heat) & z3 = 4)

to the axioms and applying COINDUCTION wrt

  (EG(P)$st ===> P(st) & (EX(EG(P))$st | AX(false)$st))

at positions [] of the preceding trees leads to

EG0(Sat(heat))$1 | EG0(Sat(heat))$3 | EG0(Sat(heat))$4

The reducts have been simplified in parallel.

3. NARROWING the preceding trees (one step) leads to

EG0(Sat(heat))$3 | EG0(Sat(heat))$4

The axioms have been MATCHED against their redices.
The reducts have been simplified in parallel.
```

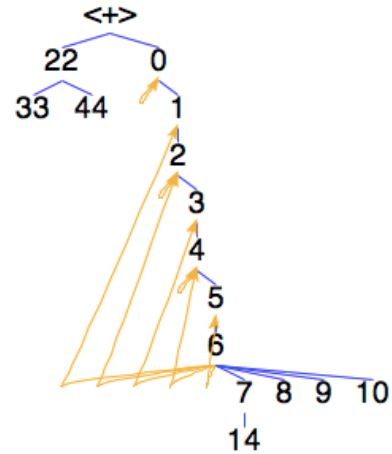
```
4. NARROWING the preceding trees (one step) leads to

EG0(Sat(heat))$4

The axioms have been MATCHED against their redices.
The reducts have been simplified in parallel.
The formula coincides with no. 1

5. NARROWING the preceding trees (one step) leads to

True

The axioms have been MATCHED against their redices.
The reducts have been simplified in parallel.
```

trans0

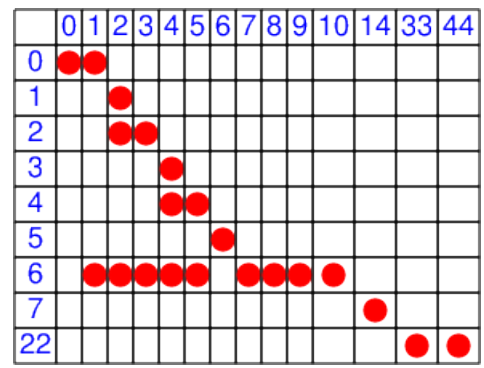
```
specs:          modal
constructs:     is less
defuncts:       drawI drawS

axioms:

  states == [0,22]
& atoms == map($)$prodL[[is,less],states]

& (st < 6 ==> st -> st+1<+>ite(st`mod`2 = 0,st,()))
& 6 -> branch$[1..5]++[7..10]
& 7 -> 14
& 22 -> 33<+>44

& is$st -> st
& less$st -> valid(<st)

-- widget interpreters:

& drawI == wtree $ fun(st,ite(st`in`states,
                              color(index(st,states),length$states)$circ$11,
                              frame(3)$blue$text$st))

& drawS$F == wtree $ fun(x,ite(F$x,blue$frame(5)$F$text$x,text$x))
```

```
conjects:

  (=14)$3                 --> False
& AG(<14)$9               --> True
& all(EX(=4))[3,4,6]      --> True
& sat(is$4)$3             --> False
& sat(is$4)$4             --> True
& x^4^y->st               -- narrow&simplify --> 5^x^y = st | 4^x^y = st

terms:

    eval$is$4             --> [4]
<+> eval$less$4           --> [0,1,2,3]
<+> eval(EX$less$4)       --> [0,1,2,6]
<+> eval(EF$less$4)       --> [0..6]
<+> eval(AF$less$4)       --> [0..3]
<+> eval(EG$less$4)       --> [0..2]
<+> eval(AG$less$4)       --> []
<+> eval(AG$less$14)      --> [8..10]
<+> eval(is(4) `then` EF$less$2)        --> [0..10,14,22,33,44]
<+> eval(is(4) `then` EF$is$0)          --> [0..3,5..10,14,22,33,44]
<+> eval(AG $ is(4) `then` EF$is$0)     --> [7..10,14,22,33,44]
<+> trans(6) `meet` eval(less$2)        --> [1]
<+> f(x^4^y)              -- rewrite&simplify --> f(5^x^y) <+> f(4^x^y)
```

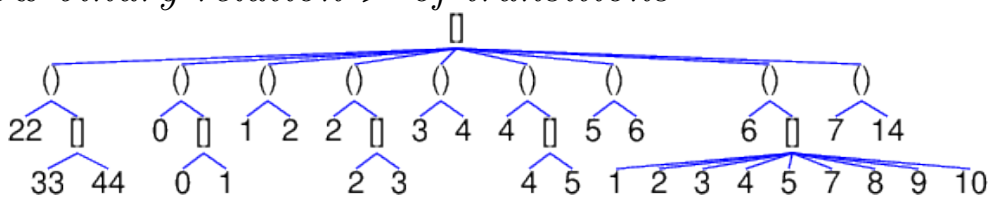
Graphical representations of the transitions of `trans0`



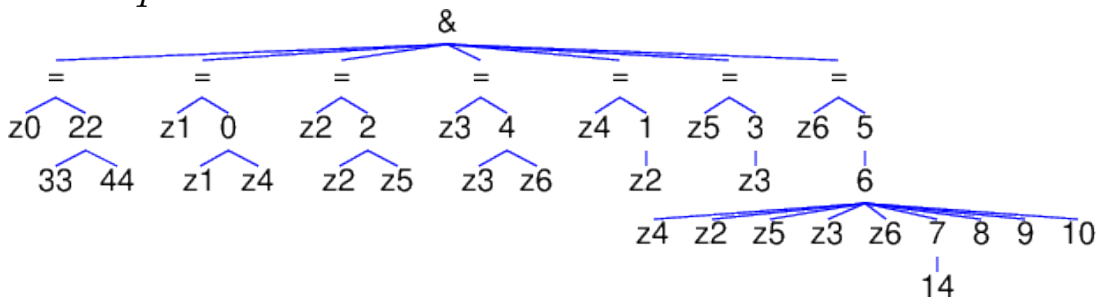
- press *graph* > *show graph of transitions* > *here*



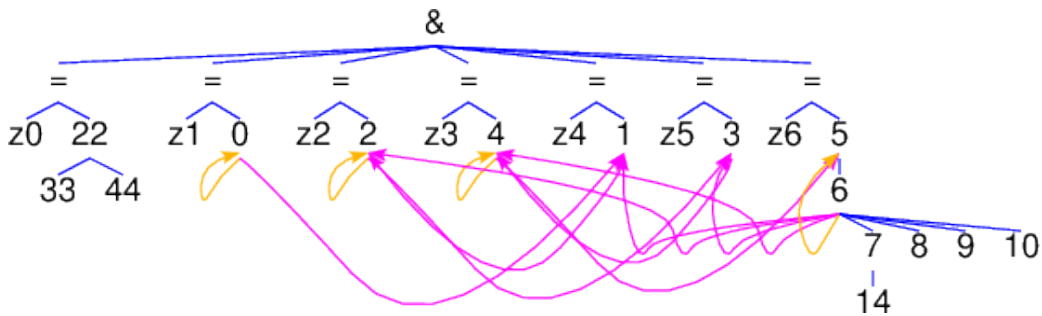
- press *graph* > *show Boolean matrix* > *of transitions*
- press *graph* > *show binary relation* > *of transitions*



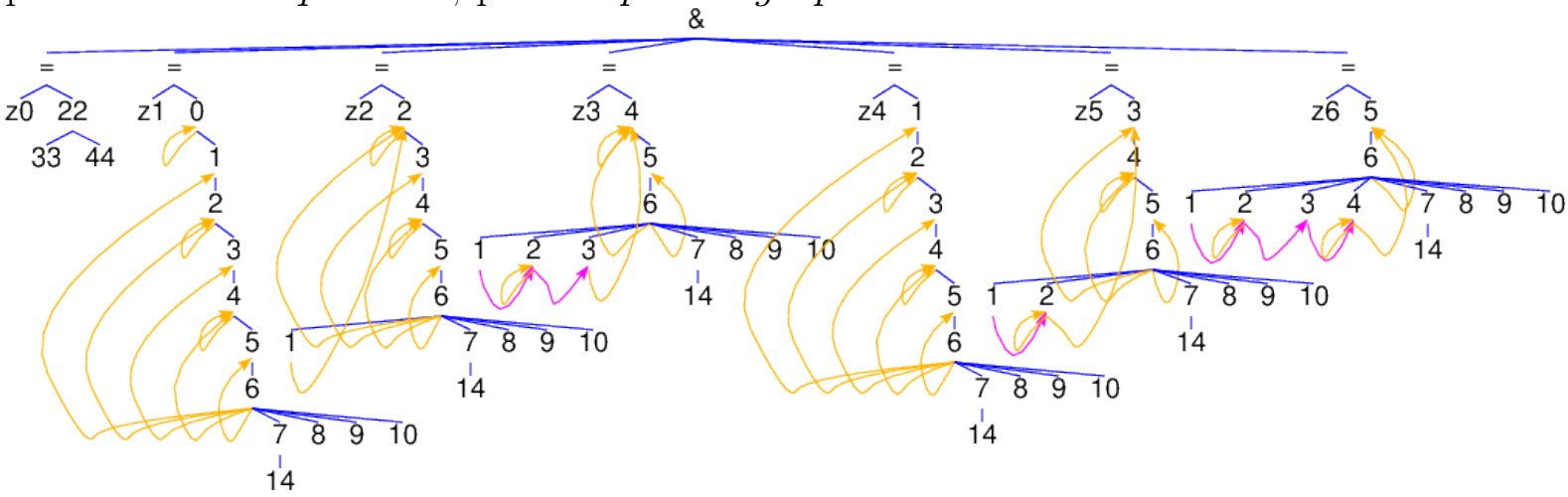
- press *graph* > *show graph of transitions* > *here*
  press *build iterative equations*



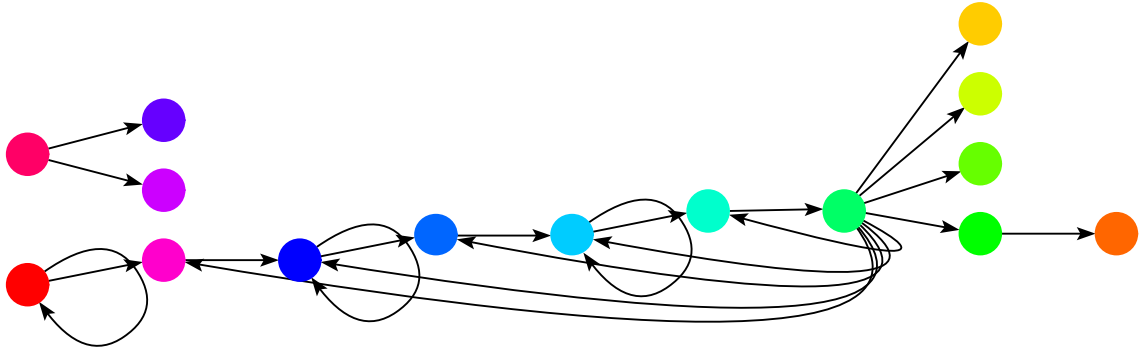
- press *graph* > *show graph of transitions* > *here*
  press *build iterative equations*
  press *connect equations*



- press *graph* > *show graph of transitions* > *here*; *build iterative equations*
  press *connect equations*; press *separate graphs*

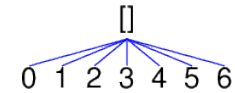


- press *graph* > *show graph of transitions* > *here*; enter *drawI* into the entry field
  press *tree* > *tree*; press *paint*; enter *t1-90* into the mode field
  press *arrange*; adapt red support points and press *combis* to remove them

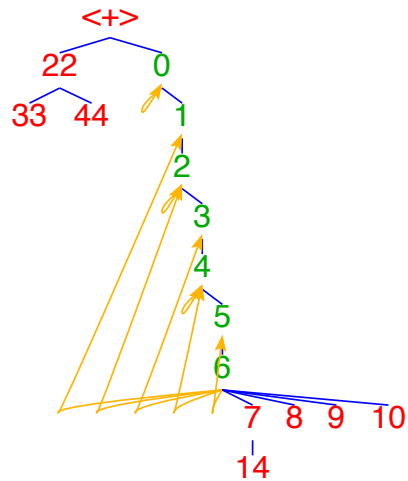


Graphical representations of the set of states satisfying the modal formula *EF(less(4))*

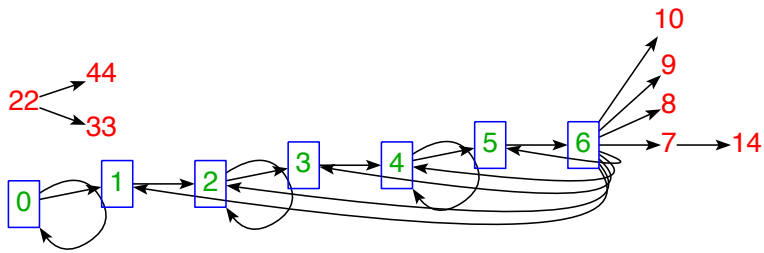
- enter *evalEFless$4* into the text field; press *parse up*; press *simplify*



- enter *evalGEFless$4* into the text field; press *parse up*; press *simplify*



- enter *evalGEFless$4* into the text field; press *parse up*; press *simplify*
  enter *drawS$dark green* into the entry field; press *tree > tree*; press *paint*
  enter *t1-90* into the mode field; press *arrange*; press *combis*



The following specifications use `procs` and `noProcs` (see modal). The value of `noProcs` is to be set by writing a positive natural number into the entry field and pressing *signature > set number of processes*.

*****

Model checking as data flow analysis

Transformation of a modal formula into a flow graph

```
simplifyS sig (F "stateflow" [t]) | just u = u where u = initStates sig t

simplifyS sig (F "stateflow" [t]) = initStates True sig $ mkCycles [] t

mkCycles :: [Int] -> TermS -> TermS
mkCycles p t@(F x _) | isFixF x = addToPoss p $ eqsToGraph [0] eqs
                                  where (_,eqs,_) = fixToEqs t 0
mkCycles p (F x ts)             = F x $ zipWithSucs mkCycles p ts

data RegEq = Equal String (Term String) | Diff String (Term String)

fixToEqs :: Term String -> Int -> (Term String,[RegEq],Int)
fixToEqs (F x [t]) n | isFixF x = (F z [],Equal z (F mu [u]):eqs,k)
                           where mu:y:_ = words x
                                 b = y `elem` foldT f t
                                 f x xss = if isFixF x then joinM xss `join1` y
                                                       else joinM xss
                                           where _:y:_ = words x
                                 z = if b then y++show n else y
                                 (u,eqs,k) = if b then fixToEqs (t>>>g) $ n+1
```

```
                                                  else fixToEqs t n
                                  g = F z [] `for` y
fixToEqs (F x ts) n = (F x us,eqs,k)
                      where (us,eqs,k) = f ts n
                            f (t:ts) n = (u:us,eqs++eqs',k')
                                         where (u,eqs,k) = fixToEqs t n
                                               (us,eqs',k') = f ts k
                            f _ n      = ([],[],n)
fixToEqs t n        = (t,[],n)

unit = leaf "()"

initStates :: Sig -> Term String -> Maybe (Term String)
initStates sig = f
     where sts = mkList sig.states
           f (t@(V x))          = do guard $ isPos x; Just t
           f (F "true" [])      = Just sts
           f (F "false" [])     = Just mkNil
           f (F "`then`" [t,u]) = f (F "\\/" [F "not" [t],u])
           f (F "MU" [t])       = do t <- f t; jOP "MU" "[]" [t]
           f (F "NU" [t])       = do t <- f t; jOP "NU" (showTerm0 sts) [t]
           f (F "EX" [t])       = do t <- f t; jOP "EX" "()" [t]
           f (F "AX" [t])       = do t <- f t; jOP "AX" "()" [t]
           f (F "<>" [lab,t])   = do p lab; t <- f t; jOP "<>" "()" [lab,t]
           f (F "#" [lab,t])    = do p lab; t <- f t; jOP "#" "()" [lab,t]
```

```
f (F "not" [t])      = do t <- f t; jOP "not" "()" $ [t]
f (F "\\/" ts)       = do ts <- mapM f ts; jOP "\\/" "()" ts
f (F "/\\" ts)       = do ts <- mapM f ts; jOP "/\\" "()" ts
f at                 = do i <- search (== at) sig.atoms
                          Just $ mkList $ map (sig.states!!)
                                          $ sig.value!!i
p lab = guard $ just $ search (== lab) sig.labels
jOP op val = Just . F (op++"::"++val)
```

Flow graph transformation step

```
simplifyS sig t | just flow = do guard changed; Just $ mkFlow True sig ft id
                  where flow = parseFlow True sig t $ parseVal sig
                        (ft,changed) = evalStates sig t $ get flow

data Flow a = Atom a | Neg (Flow a) a | Comb Bool [Flow a] a |
              Mop Bool (Term String) (Flow a) a | Fix Bool (Flow a) a |
              Pointer [Int]

parseFlow :: Sig -> Term String -> (Term String -> Maybe a) -> Maybe (Flow a)
parseFlow sig t parseVal = f t where
      f (F x [u]) | take 5 x == "not::"
                  = do g <- f u; val <- dropVal 5 x; Just $ Neg g val
     f (F x ts)  | take 4 x == "\\/::"
                  = do gs <- mapM f ts; val <- dropVal 4 x
                       Just $ Comb True gs val
```

```
f (F x ts)  | take 4 x == "/\\::"
              = do gs <- mapM f ts; val <- dropVal 4 x
                   Just $ Comb False gs val
f (F x [u]) | take 4 x == "EX::"
              = do g <- f u; val <- dropVal 4 x
                   Just $ Mop True (leaf "") g val
f (F x [u]) | take 4 x == "AX::"
              = do g <- f u; val <- dropVal 4 x
                   Just $ Mop False (leaf "") g val
f (F x [lab,u]) | take 4 x == "<>::"
              = do g <- f u;  guard $ lab `elem` sig.labels
                   val <- dropVal 4 x; Just $ Mop True lab g val
f (F x [lab,u]) | take 3 x == "#::"
              = do g <- f u;  guard $ lab `elem` sig.labels
                   val <- dropVal 3 x; Just $ Mop False lab g val
f (F x [u])       | take 4 x == "MU::"
              = do g <- f u; val <- dropVal 4 x; Just $ Fix True g val
f (F x [u])       | take 4 x == "NU::"
              = do g <- f u; val <- dropVal 4 x; Just $ Fix False g val
f (V x)           | isPos x
              = do guard $ q `elem` positions t
                   Just $ Pointer q where q = getPos x
f val         = do val <- parseVal val; Just $ Atom val
dropVal n x   = do val <- parse (term sig) $ drop n x; parseVal val
```

```
parseVal sig (t@(F "()" [])) = Just t
parseVal sig t = do F "[]" ts <- Just t; guard $ ts `subset` sig.states; Just t

evalStates :: Sig -> Term String -> Flow (Term String) -> (Flow (Term String),Bool)
evalStates sig t flow = up [] flow
 where ps = maxis (<<=) $ fixPositions t
       up p (Neg g val)
            | val1 == unit || any (p<<) ps = (Neg g1 val, b)
            | True = if null ps then (Atom val2, True)
                                else (Neg g1 val2, b || val /= val2)
                        where q = p++[0]; (g1,b) = up q g
                              val1 = fst $ getVal flow q
                              val2 = mkList $ minus sig.states $ subterms val1
       up p (Comb c gs val)
            | any (== unit) vals || any (p<<) ps = (Comb c gs1 val, or bs)
            | True = if null ps then (Atom val1, True)
                                else (Comb c gs1 val1, or bs || val /= val1)
                    where qs = succsInd p gs
                          (gs1,bs) = unzip $ zipWith up qs gs
                          vals = map (fst . getVal flow) qs
                          val1 = mkList $ foldl1 (if c then join else meet)
                                        $ map subterms vals
       up p (Mop c lab g val)
            | val1 == unit || any (p<<) ps = (Mop c lab g1 val, b)
            | True = if null ps then (Atom val2, True)
```

```
                        else (Mop c lab g1 val2, b || val /= val2)
              where q = p++[1]; (g1,b) = up q g
                    val1 = fst $ getVal flow q
                    f True = shares; f _ = flip subset
                    val2 = mkList $ filter (f c (subterms val1) . h)
                                    sig.states
                    h st = map (sig.states!!) $ tr i
                           where i = get $ search (== st) sig.states
                    tr i = if lab == leaf "" then sig.trans!!i
                           else sig.transL!!i!!j
                           where j = get $ search (== lab) sig.labels
up p (Fix c g val)
     | val1 == unit || any (p<<) ps = (Fix c g1 val, b)
     | True = if f c subset valL val1L
              then if not b && p `elem` ps then (Atom val, True)
                                           else (Fix c g1 val, b)
                   else (Fix c g1 $ mkList $ h c valL val1L, True)
              where f True = flip; f _ = id
                    h True = join; h _ = meet
                    q = p++[0]; (g1,b) = up q g
                    val1 = fst $ getVal flow q
                    valL = subterms val; val1L = subterms val1
up _ g = (g,False)
```

```
mkFlow :: Bool -> Sig -> Flow a -> (a -> TermS) -> TermS
mkFlow b sig flow mkVal = f flow
       where f (Atom val)                  = mkVal val
             f (Neg g val)                 = h "not" val [f g]
             f (Comb True gs val)          = h "\\/" val $ map f gs
             f (Comb _ gs val)             = h "/\\" val $ map f gs
             f (Mop True (F "" []) g val) = h "EX" val [f g]
             f (Mop _ (F "" []) g val)    = h "AX" val [f g]
             f (Mop True lab g val)        = h "<>" val [lab,f g]
             f (Mop _ lab g val)           = h "#" val [lab,f g]
             f (Fix True g val)            = h "MU" val [f g]
             f (Fix _ g val)               = h "NU" val [f g]
             f (Pointer p)                 = mkPos p
             h op val ts = if b then F (op ++ "::" ++ showTerm0 (mkVal val)) ts
                                else F op $ ts ++ [mkVal val]
```

Examples

```
-- trans1

specs:          modal
preds:          two TWO X Y
copreds:        one ONE
constructs:     a b
hovars:         X Y
```

```
axioms:

states == [2] & labels == [a,b] &

(2,b) -> 1<+>3 & (3,b) -> 3 & (3,a) -> 4 & (4,b) -> 3 &

(one <==> NU X.(two/\(b#X))) &
(two <==> MU Y.((a<>true)\/(b<>Y)))

(ONE(st) ===> (TWO/\(b#ONE))(st)) &
(TWO(st) <=== ((a<>true)\/(b<>TWO))(st))

conjects:

all(one)[1,2] &          --> False       simplify depthfirst
all(two)[2,3,4]          --> True        simplify depthfirst
all(ONE)[3,4]            --> True        coinduction + narrow match      (1)

terms:

eval$one <+>             --> [3,4]
eval$two <+>             --> [2,3,4]
```

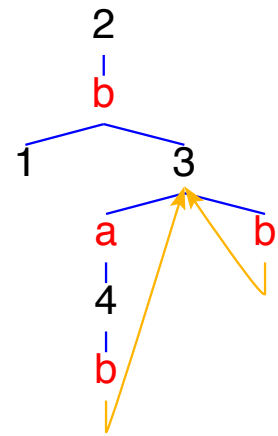


Fig. 3. *Transitions of* `trans1`

Proof of (1):

```
0. Derivation of all(ONE)[3,4]

All simplifications are admitted.
Equation removal is safe.

1. SIMPLIFYING the preceding trees (one step) leads to

ONE(3) & ONE(4)

2. Adding
```

```
  (ONE0(z0) <=== z0 = 3 | z0 = 4)

to the axioms and applying COINDUCTION wrt

  (ONE(st) ===> (TWO/\(b#ONE))$st)

at positions [] of the preceding trees leads to

TWO(3) & all(ONE0)[3] & TWO(4)

The reducts have been simplified.

3. NARROWING the preceding trees (one step) leads to

ONE0(3) & TWO(4)

The axioms have been MATCHED against their redices.
The reducts have been simplified.

4. NARROWING the preceding trees (one step) leads to

TWO(4)

The axioms have been MATCHED against their redices.
The reducts have been simplified.
```

```
The formula coincides with no. 4

5. NARROWING the preceding trees (one step) leads to

TWO(3)

The axioms have been MATCHED against their redices.
The reducts have been simplified.

6. NARROWING the preceding trees (one step) leads to

True

The axioms have been MATCHED against their redices.
The reducts have been simplified.
```

For the simplification steps of

$$\texttt{stateflow}(\texttt{one}),$$

see stateflow_one.html.

```
-- trans2

specs:          modal
preds:          X Y three six seven' Sat SatL
copreds:        NOTthree four five seven eight
constructs:     a A B
defuncts:       drawS
hovars:         X Y

axioms:

states == [1] & labels == [a] & atoms == [A,B] &

(1,a) -> 1<+>2 & (2,a) -> 3 & (3,a) -> 1<+>4 & (4,a) -> 4 &

A -> branch[2,3,4] & (B,a) -> 3 &

(Sat(at)$st <==> at `in` out$st) &
(SatL(at,lab)$st <==> at `in` outL(st)$lab) &

(three    <==> MU X.(a#(A\/X))) &
(four     <==> NU X.(a#(A/\X))) &
(five     <==> NU X.((a#X)/\six)) &
(six      <==> MU X.(A\/(a#X))) &
(seven    <==> NU X.(MU Y.(a#((A/\X)\/Y)))) &
```

```
(eight    <==> NU X.(a#X)) &

(THREE(st)    <=== (a#(Sat(A)\/THREE))(st)) &
(NOTTHREE(st) ===> (a<>(not(Sat(A))/\NOTTHREE))(st)) &
(four(st)     ===> (a#(Sat(A)/\four))(st)) &
(five(st)     ===> ((a#five)/\six)(st)) &
(six(st)      <=== (Sat(A)\/(a#six))(st)) &
(eight(st)    ===> (a#eight)(st)) &

(seven(st) ===> seven'(st)) &                          -- alternating fixpoints
(seven'(st) <=== (a#((A/\seven)\/seven'))(st)) &

drawS == wtree $ fun(dark green$x,green$text$x,
                     red$x,text$x,
                     [],text[],
                     x,red$text$x)

conjects:

all(THREE)[2,4] &                --> True        match&narrow
all(NOTTHREE)[1,3] &             --> True        match&narrow + coinduction    (1)
(THREE(st) & st `in` states ==> st = 2 | st = 4) &                             (2)
                                 --> True        match&narrow + induction
(NOTTHREE(st) & st `in` states ==> st = 1 | st = 3)
                                 --> True        match&narrow
```

```
map(eval)[three,not$three,four,five,six,seven,eight] &                    (3)
                                --> [[2,4],[1,3],[4],[4],[2,3,4],[4],[1,2,3,4]]

-- stateflow$seven returns [2,4] instead of [4] because seven is alternating

terms:   eval$not(three)/\six   --> [3]
```

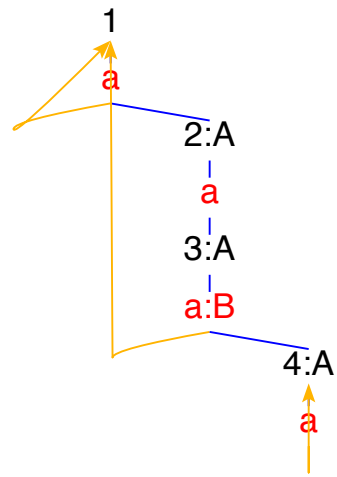


Fig. 4. *Kripke model of* `trans2`

Proof of (1) :

```
0. Derivation of all(NOTTHREE)[1,3]

All simplifications are admitted.
```

```
Equation removal is safe.

1. SIMPLIFYING the preceding trees (one step) leads to

NOTTHREE(1) & NOTTHREE(3)

2. Adding

  (NOTTHREE0(z0) <=== z0 = 1 | z0 = 3)

to the axioms and applying COINDUCTION wrt

  (NOTTHREE(st) ===> (a<>(not(Sat(A))/\NOTTHREE))$st)

at positions [] of the preceding trees leads to

A `NOTin` [A] & NOTTHREE0(4) & NOTTHREE0(2) | A `NOTin` [] & NOTTHREE0(1)

The reducts have been simplified.

3. NARROWING the preceding trees (one step) leads to

NOTTHREE0(1)

The axioms have been MATCHED against their redices.
```

```
The reducts have been simplified.

4. NARROWING the preceding trees (one step) leads to

True

The axioms have been MATCHED against their redices.
The reducts have been simplified.
```

Proof of (2):

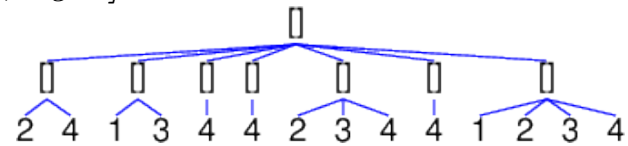
```
0. Derivation of

THREE(st) & st `in` states ==> st = 2 | st = 4

All simplifications are admitted.
Equation removal is safe.

1. SHIFTING SUBFORMULAS at positions [0,1] of the preceding trees leads to

THREE(st) ==> (st `in` states ==> st = 2 | st = 4)

2. Adding

  (THREE0(st) ===> (st `in` states ==> st = 2 | st = 4))
```

```
to the axioms and applying FIXPOINT INDUCTION wrt

  (THREE(st) <=== (a#(Sat(A)\/THREE))$st)

at positions [] of the preceding trees leads to

(A `in` [A] & A `in` [] ==> False) & (THREE0(2) & A `in` [] ==> False) &
(A `in` [A] & THREE0(1) ==> False) & (THREE0(2) & THREE0(1) ==> False) &
(THREE0(4) & A `in` [] ==> False) & (THREE0(4) & THREE0(1) ==> False)

The reducts have been simplified.

3. NARROWING the preceding trees (one step) leads to

THREE0(1) ==> False

The axioms have been MATCHED against their redices.
The reducts have been simplified.

4. NARROWING the preceding trees (one step) leads to

True

The axioms have been MATCHED against their redices.
The reducts have been simplified.
```

Graphical representations of the set of states satisfying (3)

- enter *map(eval)[three,not$three,four,five,six,seven,eight]* into the text field

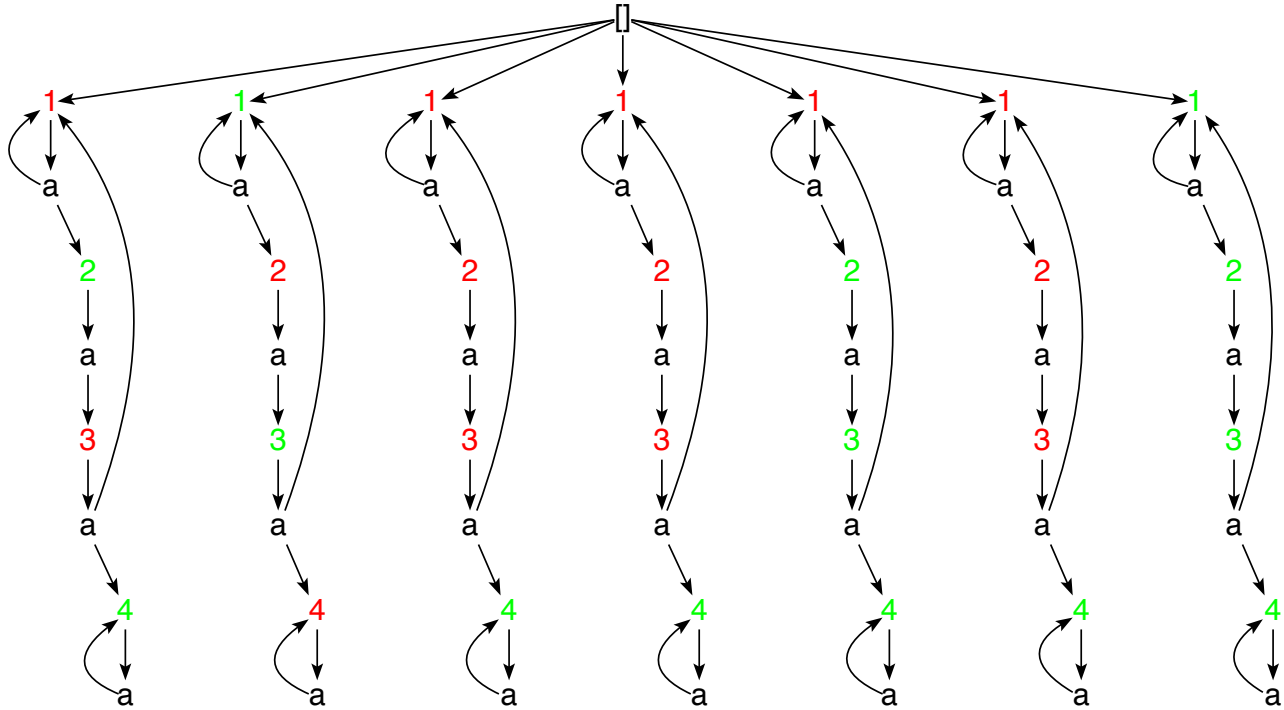
  press *parse up*; press *simplify*



- enter *map(evalG)[three,not$three,four,five,six,seven,eight]* into the text field

  press *parse up*; press *simplify*; enter *drawS* into the entry field

  press *tree* > *tree*; press *paint*; adapt red support points and press *combis* to remove them



For the simplification steps of

$$\texttt{map}(\texttt{stateflow})[\texttt{three}, \texttt{not\$three}, \texttt{four}, \texttt{five}, \texttt{six}, \texttt{seven}, \texttt{eight}]$$

and

$$\texttt{stateflow\$not}(\texttt{three})\texttt{/\backslash six},$$

see stateflow3.html and stateflow4.html, respectively.

Acceptors and regular expressions

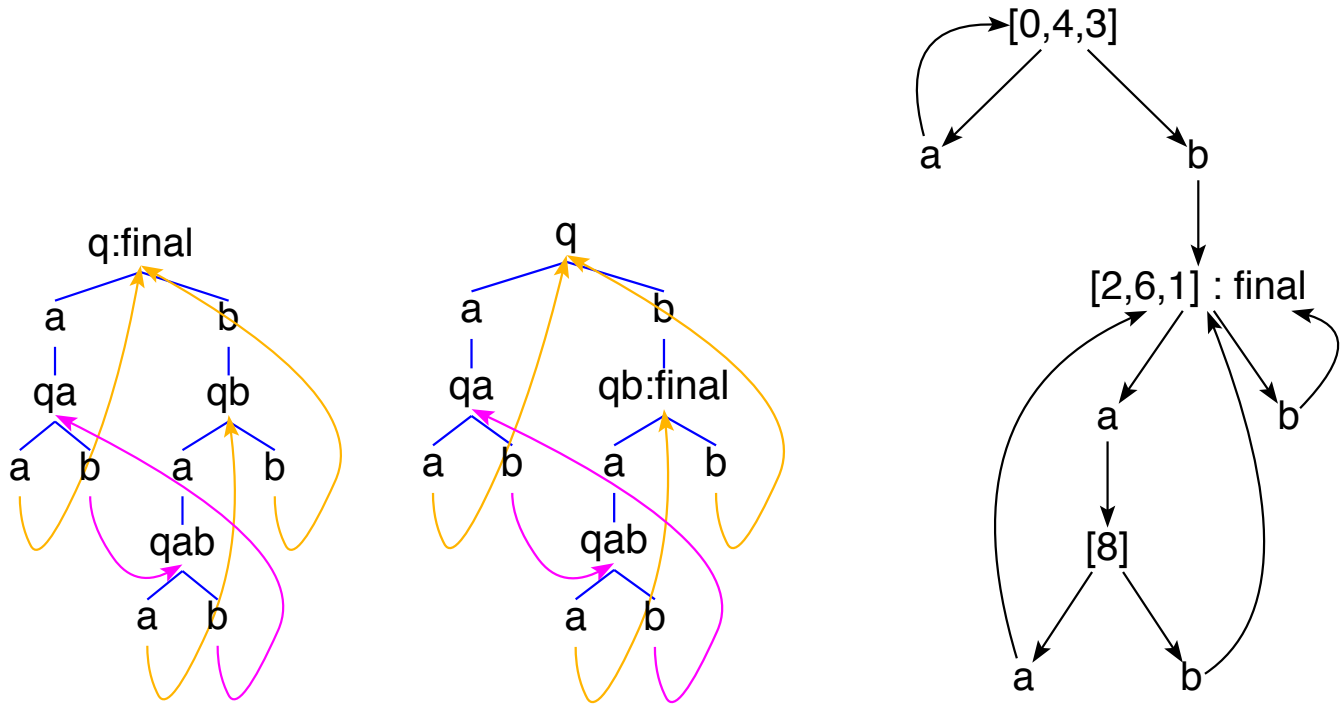
```
-- reglangs

constructs:     a b c d e h aa gg pp vv final q qa qb qab q1 q2
defuncts:       delta beta plus unfold unfoldND unfoldBro reg draw drawC
preds:          Unfold
fovars:         e st st' w k m

axioms:

states == [q] & labels == [a,b] & atoms == [final]

(q,a) -> qa     & (q,b) -> qb      &
(qa,a) -> q     & (qa,b) -> qab    &
(qb,a) -> qab   & (qb,b) -> q    &
(qab,a) -> qb   & (qab,b) -> qa  &

(q,a) -> q      & (q,b) -> q1      &
(q1,a) -> q2    & (q1,b) -> q1      &
(q2,a) -> q1    & (q2,b) -> q1      &

final -> q &    -- A1: acceptor of words with an even number of a's and an even
                -- number of b's if started in q
```

```
final -> qb &   -- A2: acceptor of words with an even number of a's and an odd
                -- number of b's if started in q

final -> q1 &   -- A3: acceptor of reg$11

(Unfold(st)[]    <=== final -> st) &
(Unfold(st)(x:w) <=== (st,x) -> st' & Unfold(st')$w) &

unfoldND$w == out <=< kfold(flip$transL)$w &    -- nondeterministic unfold
                                                -- (for kfold see base)
delta(st,x) == head$transL(st)$x &
beta$st == ite(null$out$st,0,1) &

unfold(st) == beta.foldl(delta)(st) &           -- deterministic unfold

-- regular expressions

plus$e == e*star(e) &

reg$1 == star$plus(a)+plus(b*c) &                   --> star(a+b*c)

reg$2 == a+a*a*star(a)*b+a*b*star(b)+b &            --> star(a)*b+a*star(b)

reg$3 == a+star(a)+eps+a &                          --> star(a)
```

```
reg$4 == (a+eps)*star(a)*(eps+a) &                  --> star(a)

reg$5 == (a+eps)*star(a+eps)*(a+eps)+a+eps &        --> star(a)

reg$6 == star(a)*b*star(b+eps)*(b+eps)+star(a)*b &  --> star(a)*b*star(b)

reg$7 == d*pp+c*h*e+a*b+c+a+a*b*c*d+star(a*a+b+a*b)+mt+c*aa+b*gg+eps+
         a+a*a+vv+a*a*8+d+b &
     --> (a*a*8)+a+(c*aa)+c+d+(a*b*c*d)+(c*h*e)+(b*gg)+(d*pp)+
     --  star((a*a)+(a*b)+b)+vv

reg$8 == a+((b+eps)*star(b+eps)*a)+
         ((a+((b+eps)*star(b+eps)*a))*star(a+b+eps+(mt*star(b+eps)*a))*
          (a+b+eps+(mt*star(b+eps)*a))) &           -- auto3
     --> star(b)*a*star(a+b)

reg$9 == a+c+eps+((a+c+eps)*star(a+c+eps)*(a+c+eps))+
         ((b+((a+c+eps)*star(a+c+eps)*b))*star(a+b+eps+(c*star(a+c+eps)*b))*
          (c+(c*star(a+c+eps)*(a+c+eps)))) &        -- auto4
     --> star(a+c)+(star(a+c)*b*star(a+b+(c*star(a+c)*b))*c*star(a+c))
     -- autoToReg.minimize.regToAuto --> star(a+c+b*star(a+b)*c)

reg$10 == star(a)+star(a)*b*star(c*star(a)*b)*c*star(a) &
```

```
reg$11 == star(a)*b*star(a*(a+b)+b) &

-- widget interpreters

draw(m) == wtree(m)$fun((eps,k,n),text$eps,
                        (st,k,n),ite(Int$st,
                                     color(k,n)$circ$11,
                                     frame$blue$text$st)) &

drawC == wtree $ fun(eps,gif(cat,16,14),x,text$x)

conjects:

Unfold(q)[] &                    --> False       -- unify & narrow
Unfold(q)[b] &                   --> True        -- derive & simplify & refute
Unfold(q)[b,b] &                 --> False
Unfold(q)[a,b,a,b,a,b,a] &       --> True                -- reglangs1
Unfold(q)[a,b,a,b,b,a,b,a] &     --> False
Unfold(q)[a,b,a,a,b,a] &         --> False
Unfold(q)[a,b,a,a,b,a,b]         --> True

terms:

unfoldND[q][] <+>                        --> []          -- simplify
unfoldND[q][b] <+>                       --> [final]
```

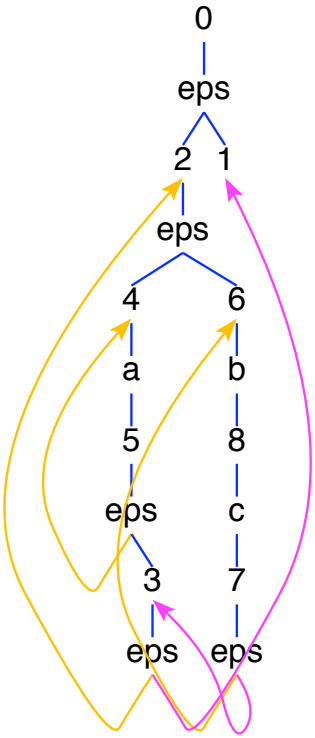
```
unfoldND[q][b,b] <+>                    --> []
unfoldND[q][a,b,a,b,a,b,a] <+>          --> [final]
unfoldND[q][a,b,a,b,b,a,b,a] <+>        --> []
unfoldND[q][a,b,a,a,b,a] <+>            --> []          -- 39 steps
unfoldND[q][a,b,a,a,b,a,b] <+>          --> [final]     -- 45 steps

unfold(q)[a,b,a,a,b,a] <+>              --> 0           -- 23 steps
unfold(q)[a,b,a,a,b,a,b] <+>            --> 1           -- 26 steps

unfold(qgg)[a,b,a,a,b] <+>              --> 0           -- 22 steps
unfold(qgg)[a,b,a,a,b,a] <+>            --> 1           -- 25 steps

unfoldBro(reg$2)[a,a,a,b] <+>           --> 1
unfoldBro(reg$2)[a,a,a,b,b] <+>         --> 0

auto$reg$1 <+>                  --> non-deterministic acceptor of reg$1
pauto$reg$1                     --> deterministic acceptor of reg$1
```



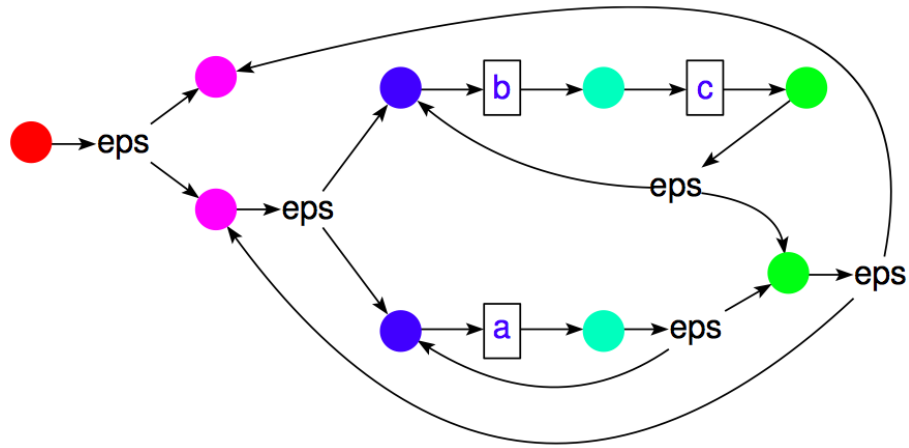
A1, A2 and the minimal acceptor of `reg$11` *that is isomorphic to A3 and whose state names were created by* `powerAuto` *(see below)*

The following nondeterministic automaton (with initial state 0, final state 1 and ϵ-transitions) accepts the language of the regular expression $reg1 = (a^+ + (bc)^+)^*$. It results from simplifying `auto$reg$1`.

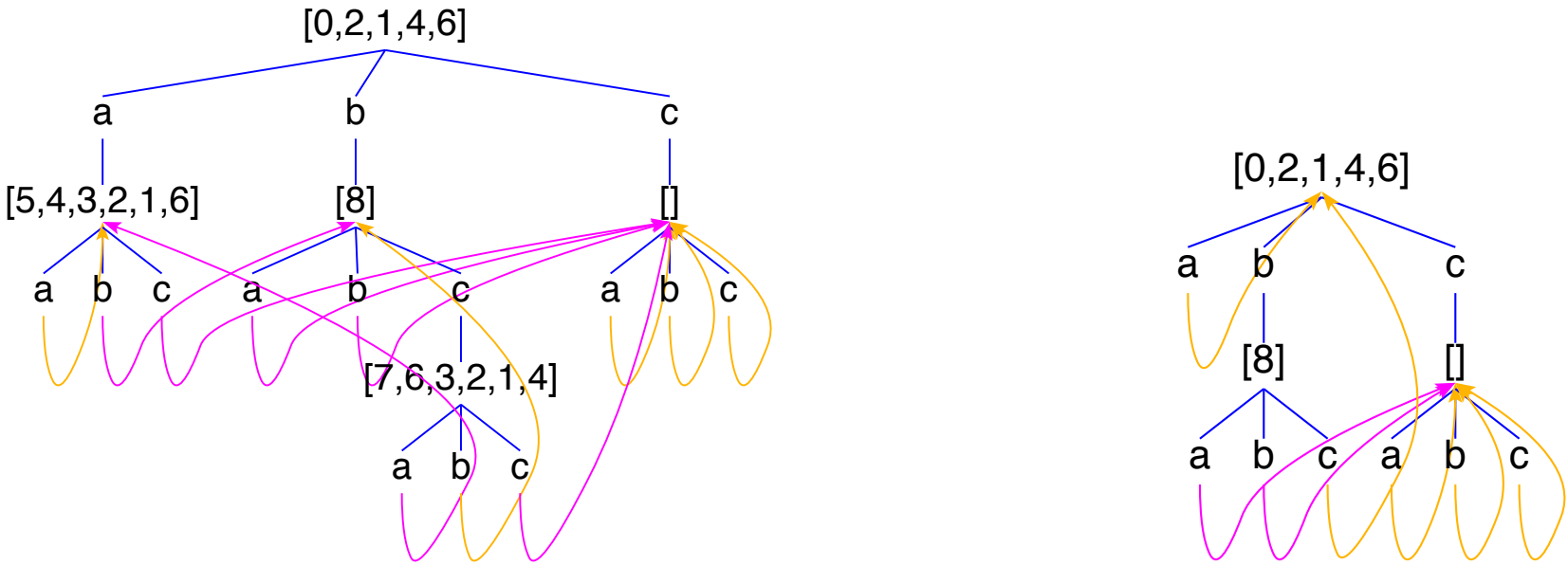
An application of `drawL` to the graph on the left leads to the graph on the right. The rotation by 90 degrees is achieved by drawing the graph in painter mode `t1-90`.



	eps	b	c	a
0	2 1			
2	4 6			
3	2 1			
4				5
5	4 3			
6		8		
7	6 3			
8			7	



The following equivalent deterministic automaton (with initial state [0,2,1,4,6] and final states [0,2,1,4,6], [5,4,3,2,1,6] and [7,6,3,2,1,4]) results from simplifying `pauto$reg$1` or entering `reg$1` into the solver text field and pressing *specification* > *build Kripke model from regular expression*.



Transitions of `pauto$reg$1` *- before and after minimization*

Again, the minimal automaton was obtained by pressing *specification* > *minimize* afterwards (see section 7).

Data type, parser and unparser for regular expressions

```
data RegExp = Mt | Eps | Const String | Sum_ RegExp RegExp |
              Prod RegExp RegExp | Star RegExp deriving Eq

parseRE :: Sig -> TermS -> Maybe (RegExp,[String])
                                      -- list of labels of acceptors
parseRE _ (F "mt" [])       = Just (Mt,[])
parseRE _ (F "eps" [])      = Just (Eps,["eps"])
parseRE sig (F a [])        = if sig.isDefunct a then Just (Var a,[a])
                                                 else Just (Const a,[a])
parseRE _ (F "+" [])        = Just (Mt,[])
parseRE sig (F "+" [t])     = parseRE sig t
parseRE sig (F "+" ts)      = do pairs <- mapM (parseRE sig) ts
                                 let (e:es,ass) = unzip pairs
                                 Just (foldl Sum_ e es,joinM ass)
parseRE _ (F "*" [])        = Just (Eps,["eps"])
parseRE sig (F "*" [t])     = parseRE sig t
parseRE sig (F "*" ts)      = do pairs <- mapM (parseRE sig) ts
                                 let (e:es,ass) = unzip pairs
                                 Just (foldl Prod e es,joinM ass)
parseRE sig (F "plus" [t]) = do (e,as) <- parseRE sig t
                                 Just (Prod e $ Star e,as)
parseRE sig (F "star" [t]) = do (e,as) <- parseRE sig t
                                 Just (Star e,as `join1` "eps")
```

```
parseRE sig (F "refl" [t]) = do (e,as) <- parseRE sig t
                                Just (Sum_ e Eps,as `join1` "eps")
parseRE _ (V x)            = Just (Var x,[x])
parseRE _ _                = Nothing

showRE :: RegExp -> TermS
showRE Mt           = leaf "mt"
showRE Eps          = leaf "eps"
showRE (Const a)    = leaf a
showRE e@(Sum_ _ _) = case summands e of []  -> leaf "mt"
                                         [e] -> showRE e
                                         es  -> F "+" $ map showRE es
showRE e@(Prod _ _) = case factors e of []  -> leaf "eps"
                                        [e] -> showRE e
                                        es  -> F "*" $ map showRE es
showRE (Star e)     = F "star" [showRE e]
showRE (Var x)      = V x
```

The following binary RegExp-relation are subrelations of regular language inclusion. In their defining equations, = means reverse implication.

```
instance Ord RegExp                              -- used by summands
         where Eps <= Star _                     = True
               e <= Star e' | e <= e'            = True
               Prod e1 e2 <= Star e | e1 <= e && e2 <= Star e = True
```

```
                Prod e1 e2 <= Star e | e1 <= Star e && e2 <= e = True
                e <= Prod e' (Star _) | e == e' = True
                e <= Prod (Star _) e' | e == e' = True
                Prod e1 e2 <= Prod e3 e4        = e1 <= e3 && e2 <= e4
                e <= Sum_ e1 e2                 = e <= e1 || e <= e2
                e <= e'                         = e == e'

(<*) :: RegExp -> RegExp -> Bool                        -- used by factors
Sum_ Eps e <* Star e' | e == e' = True
e <* e' = False

(<+) :: RegExp -> RegExp -> Bool       -- Eps <+ e, e1*e2 <+ e3*e4 iff e1 <+ e3
Eps <+ e                   = True      -- products are ordered by left factors
Const a <+ Const b         = a <= b
e@(Const _) <+ Prod e1 e2 = e <+ e1
e@(Star _)  <+ Prod e1 e2 = e <+ e1
Prod e _ <+ e'@(Const _)  = e <+ e'
Prod e _ <+ e'@(Star _)   = e <+ e'
Prod e _ <+ Prod e' _     = e <+ e'
e <+ e'                    = e == e'

(+<) :: RegExp -> RegExp -> Bool       -- Eps +< e, e1*e2 <+ e3*e4 iff e2 <+ e4
Eps +< e                   = True      -- products are ordered by right factors
Const a +< Const b         = a <= b
e@(Const _) +< Prod e1 e2 = e +< e2
```

```
e@(Star _)  +< Prod e1 e2 = e +< e2
Prod _ e +< e'@(Const _)  = e +< e'
Prod _ e +< e'@(Star _)   = e +< e'
Prod _ e +< Prod _ e'     = e +< e'
e +< e'                   = e == e'
```

Flattening sums and products

```
summands,factors :: RegExp -> [RegExp]
summands (Sum_ e e') = joinMapR (<=) summands [e,e']   -- e <= e' ==> e+e' = e'
summands Mt          = []                              -- mt+e = e
summands e           = [e]                             -- e+mt = e
factors (Prod e e')  = joinMapR (<*) factors [e,e']    -- e <* e' ==> e*e' = e'
factors Eps          = []                              -- eps*e = e
factors e            = [e]                             -- e*eps = e

joinMapR :: Ord b => (a -> [b]) -> [a] -> [b]
joinMapR f = foldl (foldl insertR) [] . map f

insertR :: Ord a => [a] -> a -> [a]
insertR s@(x:s') y | y <= x = s
                   | x <= y = insertR s' y
                   | True   = x:insertR s' y
insertR _ x = [x]
```

Sorting summands

```
sortRE :: (RegExp -> RegExp -> Bool) -> ([RegExp] -> RegExp) -> RegExp
                                                            -> RegExp
sortRE r prod e@(Sum_ _ _) = mkSumR $ map (sortRE r prod) $ qsort r
                                                          $ summands e
sortRE r prod e@(Prod _ _) = prod $ map (sortRE r prod) $ factors e
sortRE r prod (Star e)     = Star $ sortRE r prod e
sortRE _ _ e               = e

mkSumR,mkProdL,mkProdR :: [RegExp] -> RegExp
mkSumR []  = Mt
mkSumR es  = foldr1 Sum_ es
mkProdL [] = Eps
mkProdL es = foldl1 Prod es
mkProdR [] = Eps
mkProdR es = foldr1 Prod es
```

Distributing products over sums

```
-- e*(eps+e') = e+e*e', e*(e1+e2) = e*e1+e*e2
-- (eps+e')*e = e+e'*e, (e1+e2)*e = e1*e+e2*e

distribute :: RegExp -> RegExp
distribute = fixpt (==) f
  where f e = case e of Sum_ _ _   -> mkSumR $ map f $ concatMap g $ summands e
                        Prod e1 e2 -> case g e of [e1,e2] -> Sum_ (f e1) $ f e2
                                                  _       -> Prod (f e1) $ f e2
```

```
                    Star e     -> Star $ distribute e
                    _          -> e
            where g (Prod e (Sum_ Eps e')) = [e,Prod e e']
                  g (Prod e (Sum_ e1 e2))  = [Prod e e1,Prod e e2]
                  g (Prod (Sum_ Eps e') e) = [e,Prod e' e]
                  g (Prod (Sum_ e1 e2) e)  = [Prod e1 e,Prod e2 e]
                  g e                      = [e]
```

Reducing regular expressions iteratively

```
reduceLeft,reduceRight,reduceRE :: RegExp -> RegExp
reduceLeft  = fixpt (==) (reduceRE . sortRE (+<) mkProdL)
reduceRight = fixpt (==) (reduceRE . sortRE (<+) mkProdR)
reduceRE e  = case e of Sum_ e1 e2 -> if sizeRE e' < sizeRE e then e'
                                      else Sum_ (reduceRE e1) $ reduceRE e2
                                      where es = summands e
                                            e' = mkSumR $ f es
                                            f (Eps:es) = reduceS True es
                                            f es       = reduceS False es
                        Prod _ _ -> if Mt `elem` es then Mt else mkProdR es
                                    where es = map reduceRE $ factors e
                        Star Mt           -> Eps
                        Star Eps          -> Eps
                        Star (Star e)     -> Star e
                        Star (Sum_ Eps e) -> Star e
                        Star (Sum_ e Eps) -> Star e
```

```
                        -- star(star(e)+e') = star(e+e')
                        Star (Sum_ (Star e) e') -> Star $ Sum_ e e'
                        -- star(star(e)+e') = star(e+e')
                        Star (Sum_ e (Star e')) -> Star $ Sum_ e e'
                        -- star(e*star(e)+e') = star(e+e')
                        Star (Sum_ (Prod e1 (Star e2)) e3) | e1 == e2
                                                -> Star $ Sum_ e1 e3
                        -- star(star(e)*e+e') = star(e+e')
                        Star (Sum_ (Prod (Star e1) e2) e3) | e1 == e2
                                                -> Star $ Sum_ e1 e3
                        -- star(e'+e*star(e)) = star(e+e')
                        Star (Sum_ e1 (Prod e2 (Star e3))) | e2 == e3
                                                -> Star $ Sum_ e1 e2
                        -- star(e'+star(e)*e) = star(e+e')
                        Star (Sum_ e1 (Prod (Star e2) e3)) | e2 == e3
                                                -> Star $ Sum_ e1 e2
                        Star e -> Star $ reduceRE e
                        _      -> e

sizeRE :: RegExp -> Int
sizeRE Eps          = 0
sizeRE (Sum_ e e') = sizeRE e+sizeRE e'
sizeRE (Prod e e') = sizeRE e+sizeRE e'+1
sizeRE (Star e)    = sizeRE e+1
sizeRE _           = 1
```

```
 reduceS :: Bool -> [RegExp] -> [RegExp]

-- eps+e*star(e)   = star(e),   eps+star(e)*e   = star(e)       (1)
-- eps+e*e*star(e) = e+star(e), eps+star(e)*e*e = e+star(e)     (2)

 reduceS True (Prod e (Star e'):es) | e == e' = reduceS False $ Star e:es
 reduceS True (Prod (Star e) e':es) | e == e' = reduceS False $ Star e:es
 reduceS True (Prod e (Prod e1 (Star e2)):es)
                          | e == e1 && e == e2 = reduceS False $ e:Star e:es
 reduceS True (Prod (Prod e e1) (Star e2):es)
                          | e == e1 && e == e2 = reduceS False $ e:Star e:es
 reduceS True (Prod (Star e1) (Prod e2 e):es)
                          | e == e1 && e == e2 = reduceS False $ e:Star e:es
 reduceS True (Prod (Prod (Star e1) e2) e:es)
                          | e == e1 && e == e2 = reduceS False $ e:Star e:es

-- e+e*e'    = e*(eps+e'), e+e'*e    = (eps+e')*e        prepares (1) and (2)
-- e*e1+e*e2 = e*(e1+e2),  e1*e+e2*e = (e1+e2)*e

 reduceS b (e:Prod e1 e2:es) | e == e1  = reduceS b $ Prod e (Sum_ Eps e2):es
 reduceS b (Prod e1 e2:e:es) | e == e2  = reduceS b $ Prod (Sum_ Eps e1) e:es
 reduceS b (Prod e1 e2:Prod e3 e4:es)
                             | e1 == e3 = reduceS b $ Prod e1 (Sum_ e2 e4):es
 reduceS b (Prod e1 e2:Prod e3 e4:es)
                             | e2 == e4 = reduceS b $ Prod (Sum_ e1 e3) e2:es
```

```
reduceS b (e:es) = e:reduceS b es
reduceS True _   = [Eps]
reduceS _ _      = []
```

A regular expression **reg** entered into the text field is reduced by simplifying **reduce$reg**.

A regular expression displayed on the canvas (resp. its highlighted subexpression) is reduced by pressing *specification* > *reduce regular expression*.

Nondeterministic acceptor with ϵ-transitions of a regular language

```
type NDA = Int -> String -> [Int]

regToAuto :: RegExp -> ([Int],NDA)
regToAuto e = ([0..nextq-1],delta)
   where (delta,nextq) = eval e 0 1 (const2 []) 2
         eval :: RegExp -> Int -> Int -> NDA -> Int -> (NDA,Int)
         eval Mt _ _ delta nextq             = (delta,nextq)
         eval Eps q q' delta nextq           = (upd2L delta q "eps" q',nextq)
         eval (Const a) q q' delta nextq     = (upd2L delta q a q',nextq)
         eval (Sum_ e e') q q' delta nextq   = eval e' q q' delta' nextq'
                       where (delta',nextq') = eval e q q' delta nextq
         eval (Prod e e') q q' delta nextq   = eval e' nextq q' delta' nextq'
                       where (delta',nextq') = eval e q nextq delta $ nextq+1
```

```
        eval (Star e) q q' delta nextq      = (delta2,nextq')
                      where q1 = nextq+1
                            (delta1,nextq') = eval e nextq q1 delta $ q1+1
                            delta2 = fold2 f delta1 [q,q1,q1,q] [nextq,nextq,q',q']
                            f delta q = upd2L delta q "eps"

upd2L :: (Eq a,Eq b,Eq c) => (a -> b -> [c]) -> a -> b -> c -> a -> b -> [c]
upd2L f a b = upd f a . upd (f a) b . join1 (f a b)
```

Deterministic acceptor constructed from a nondeterministic one with ϵ-transitions

```
type PDA = [Int] -> String -> [Int]

powerAuto :: NDA -> [String] -> ([[Int]],PDA)
powerAuto nda labs = (reachables,deltaH)
     where reachables = fixpt subset deltaM [epsHull [0]]
           deltaM qss = qss `join` [deltaH qs a | qs <- qss, a <- labs]
           deltaH qs  = epsHull . delta qs
           delta qs a = joinMap (flip nda a) qs
           epsHull :: [Int] -> [Int]
           epsHull qs = if qs' `subset` qs then qs else epsHull $ qs `join` qs'
                        where qs' = delta qs "eps"
```

regToAuto and *powerAuto* are applied successively to the regular expression on the canvas by pressing *specification* > *build Kripke model from regular expression*.

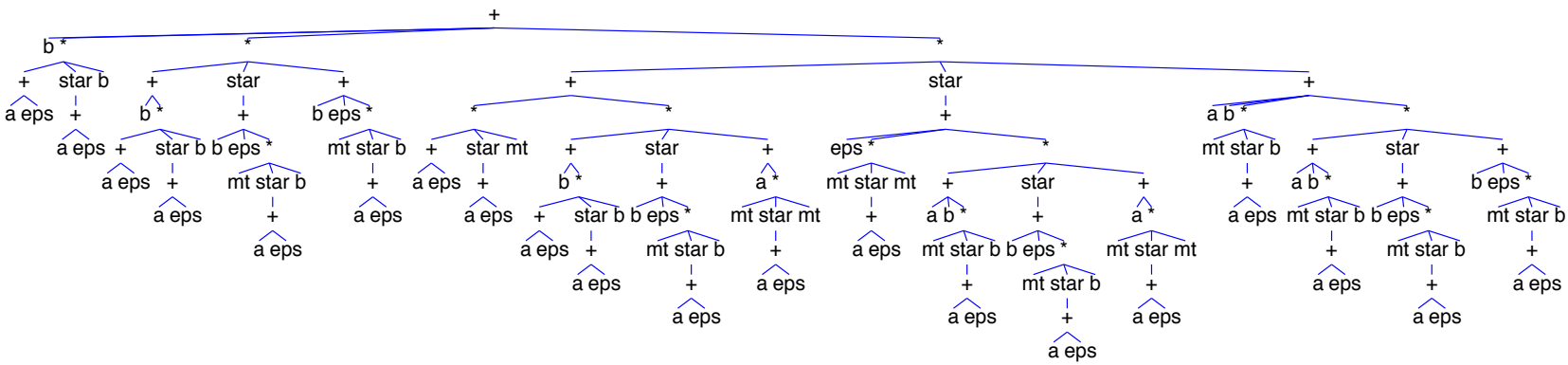
Two ways of building a regular expression from an acceptor

Kleene's algorithm

```
autoToReg :: Sig -> TermS -> RegExp
autoToReg sig start = if null finals then Const "no final states"
                                     else foldl1 Sum_ $ map (f lg i) finals
         where finals = sig.value!!0
               lg = length sig.states-1
               Just i = search (== start) sig.states
               f (-1) i j = if i == j then Sum_ (delta i i) Eps else delta i j
               f k i j    = Sum_ (f' i j) $
                                 Prod (f' i k) $ Prod (Star $ f' k k) $ f' k j
                            where f' = f $ k-1
               delta :: Int -> Int -> RegExp
               delta i j = if null labs then Mt else foldl1 Sum_ labs
                           where labs = [Const $ showTerm0 $ (sig.labels)!!k |
                                                 k <- indices_ sig.labels,
                                                 sig.transL!!i!!k == [j]]
```

If an initial state has been entered into the entry field and the current Kripke model is a deterministic acceptor, *autoToReg* is applied to it by pressing *specification* > *build regular expression*.

Regular expressions produced by Kleene's algorithm are often rather space-consuming. For instance, acceptor A3 of **reglangs** is transformed into:



Applying *reduceLeft* (see above; press *specification* > *reduce left-folded regular expression*) leads to:

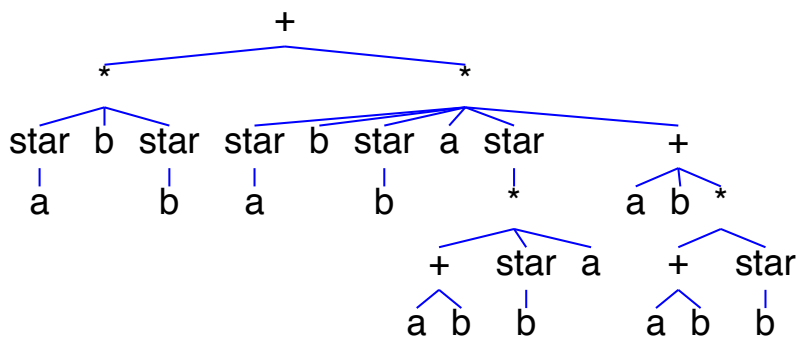
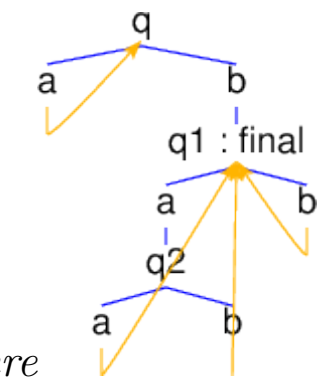


Fig. 5.

Kain's algorithm

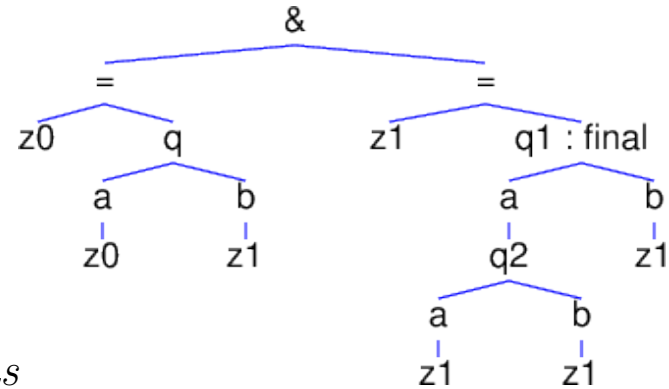
An alternative, more algebraic method to build regular expressions (REs) from acceptors has been described in [6], section 2.3, and recently been generalized to polynomial coalgebras [2, 3]. It resembles Gaussian elimination insofar as the acceptor (graph) is turned into a set of equations with variables (here for REs), which is then solved via a sequence of replacements of variables by solving REs.

For instance, the following sequence of Expander2 commands leads from the transition graph of A3 to a solution of variable z_0 that is equivalent to the RE of Fig. 5. For proving the equivalence with Expander2, see chapter 9.

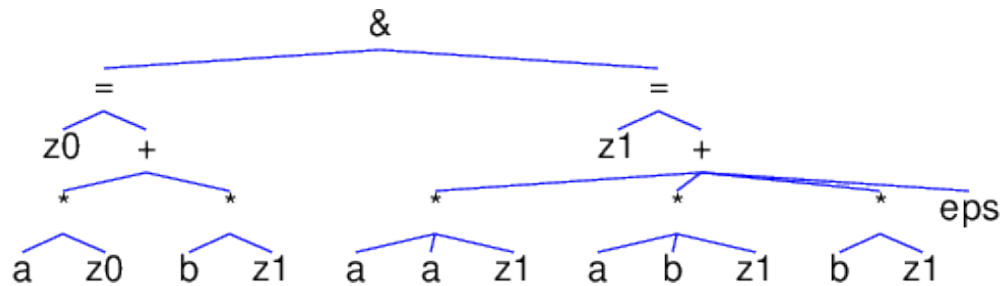


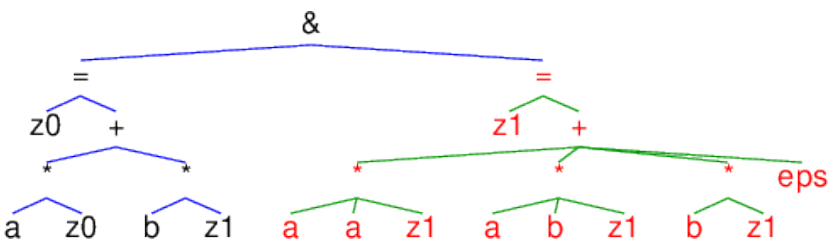
graph > show graph of labelled Kripke model here

graph > build iterative equations

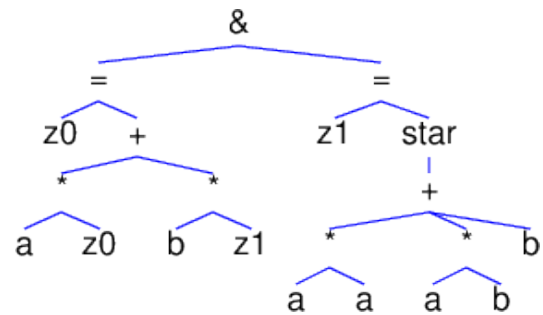


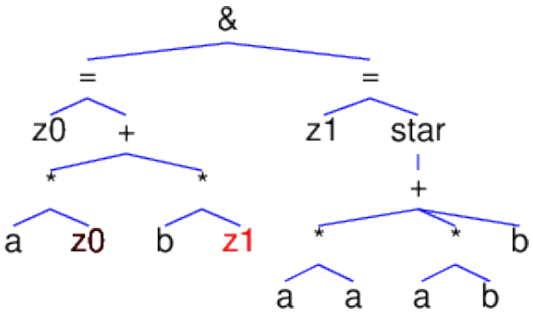
specification > regular equations (see below)



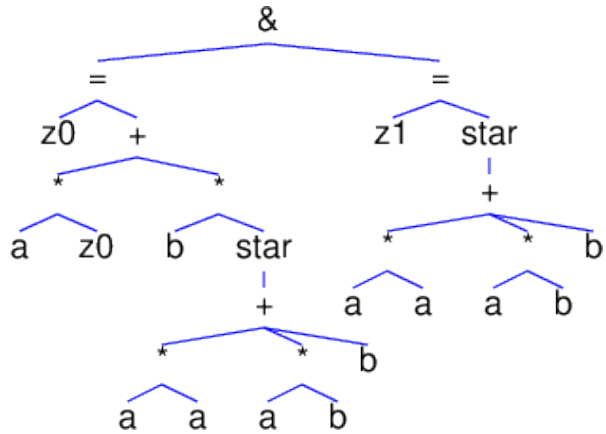


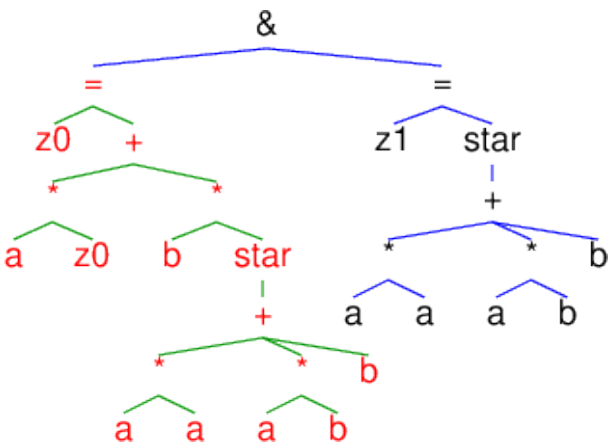
specification > solve regular equation (see below)



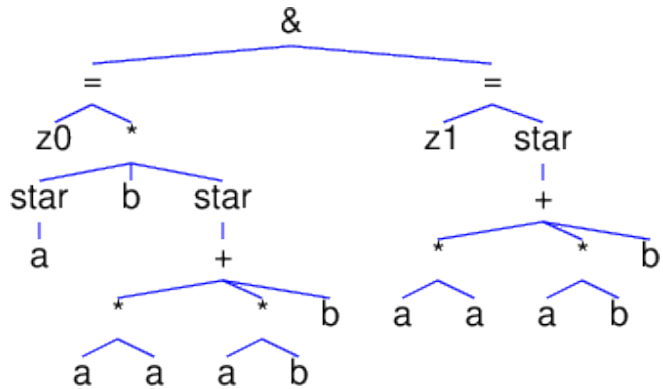


specification >
substitute variables
(see below)





specification >
solve regular
equation




```
autoEqsToRegEqs :: Sig -> [IterEq] -> [IterEq]
autoEqsToRegEqs sig = map f
       where f (Equal x t) = case parseRE sig $ g t of
                                 Just (e,_) -> Equal x $ showRE $ distribute e
```

```
                          _ -> Equal x $ leaf "OOPS"
g (F q ts)     = F "+" $ if n >= 0 && drop n q == "final"
                         then leaf "eps":map h ts else map h ts
                 where n = length q-5
g t            = t
h (F lab [t]) = F "*" [F lab [],g t]
h t            = t
```

The following function turns a *regular equation* $x = t_1 \cdot x + \cdots + t_n \cdot x + t$ into its least solution $(t_1 + \cdots + t_n)^* \cdot t$, which is unique if $\epsilon \notin \{t_1, \ldots, t_n\}$ (see [6], pp. 53 f.).

```
solveRegEq :: Sig -> IterEq -> Maybe TermS
solveRegEq sig (Equal x t) = do (e,_) <- parseRE sig $ f $ case t of F "+" ts -> ts
                                                                     _  -> [t]
                                Just $ mkEq (V x) $ showRE e
          where f ts = case us of []  -> leaf "mt"
                                  [u] -> F "*" [star,u]
                                  _   -> F "*" [star,F "+" us]
                     where us = ts `minus` recs
                           recs = [t | t@(F "*" us) <- ts, last us == V x]
                           star = case recs of
                                      []    -> leaf "eps"
                                      [rec] -> F "star" [g rec]
                                      _     -> F "star" [F "+" $ map g recs]
                g (F "*" [t,_]) = t
                g (F "*" ts)    = F "*" $ init ts
                g _             = leaf "OOPS"

substituteVars :: TermS -> [IterEq] -> [[Int]] -> Maybe TermS
substituteVars t eqs ps = do guard $ all isV ts
                             Just $ fold2 replace1 t ps $ map (subst . root) ts
                       where ts = map (getSubterm1 t) ps
                             subst x = if nothing t || x `isin` u then V x else u
                                       where t = lookup x $ eqPairs eqs
                                             u = get t
```

```
****

-- eosum

constructs:     esum osum elab olab final unfold
defuncts:       parity
preds:          even odd

axioms:

parity$n == ite(even$n,elab,olab) &

(even$0 <==> True) &
(even$suc$x <==> odd$x) &
(odd$0 <==> False) &
(odd$suc$x <==> even$x) &

-- acceptor of number sequences whose sum is even

states == [esum] & labels == [elab,olab] & atoms == [final] &

(esum,elab) -> esum &
(esum,olab) -> osum &
(osum,olab) -> esum &
(osum,elab) -> osum &
```

```
final -> esum &

unfold$s == out <=< (kfold(flip$transL)$s) . map(parity)

terms:

unfold[esum][1..4] <+>  --> [final]
unfold[esum][1..5] <+>  --> []
unfold[osum][1..4] <+>  --> []
unfold[osum][1..5] <+>  --> [final]

sum[1..4] <+>           --> 10
sum[1..5]               --> 15
```

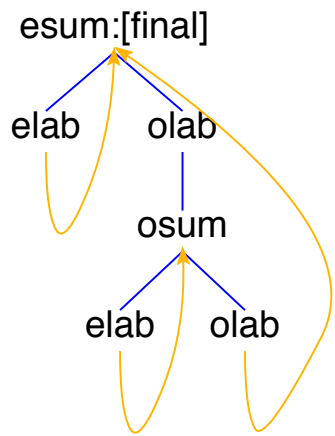


Fig. 6. *Kripke model of* `eosum`

State equivalence and minimization

The command *specification > state equivalence* computes the equivalence of states, $\sim \subseteq Q^2$, of the actual Kripke model $K = (Q, Lab, At, trans, transL, out, outL)$.

$\sim$ is also called *K-bisimilarity* and defined as the greatest binary relation on Q such that for all $q, q' \in Q$,

$$q \sim q' \quad \Rightarrow \quad \begin{cases} out(q) = out(q') \wedge \\ \forall x \in Lab : outL(q)(x) = outL(q')(x) \wedge \\ (trans(q), trans(q')) \in lift(\sim) \wedge \\ \forall x \in Lab : (transL(q)(x), transL(q')(x)) \in lift(\sim) \end{cases}$$

where

$$\begin{aligned} lift(\sim) \;=\; \{(qs, qs') \in \mathcal{P}(Q)^2 \mid\; & \forall\, q \in qs\ \exists\, q' \in qs' : q \sim q', \\ & \forall\, q' \in qs'\ \exists\, q \in qs : q \sim q'\}. \end{aligned}$$

Hence $\sim$ can be constructed stepwise as follows:

$$\begin{aligned} \sim \;&=\; \textstyle\bigcap_{n \in \mathbb{N}} \sim_n, \\ \sim_0 \;&=\; \{(q, q') \in Q^2 \mid\; out(q) = out(q'), \\ & \qquad \forall x \in Lab : outL(q)(x) = outL(q')(x)\}, \end{aligned}$$

$$\sim_{n+1} \;=\; \{(q,q') \in Q^2 \mid \; (trans(q), trans(q')) \in \mathit{lift}(\sim_n),$$
$$\forall x \in Lab : (transL(q)(x), transL(q')(x)) \in \mathit{lift}(\sim_n)\}.$$

Let $A = \mathcal{P}(|Q|^2 \times \mathcal{P}(\mathcal{P}(|Q|^2)))$ and $\chi(\supseteq)$ denote the characteristic function of inverse set inclusion.

Expander2 computes $\sim$ with the Paull-Unger method:

$$\sim \;=\; \{(states!!i, states!!j) \mid \exists\, ps : (i,j,ps) \in \mathit{fixpt}(\chi(\supseteq))(bisimStep)(bisim_0)\}$$

where

$$bisim_0 \;=\; \{(i,j,\{(trans!!i, trans!!j)\}$$
$$\cup\; \{(trans!!i!!k, trans!!j!!k) \mid 1 \le k \le |labels|\})$$
$$\mid i < j,\; states!!i \sim_0 states!!j\}$$

and for all $R \in A$,

$$bisimStep(R) \;=\; \{(k,l,ps) \in R \mid \forall\, (is,js) \in ps\; \forall\, i \in is\; \exists\, j \in js :$$
$$i = j \vee \exists\, ps' : (i,j,ps') \in R\}.$$

For the fixpoint generator $\mathit{fixpt} : (A \to A \to Bool) \to (A \to A) \to (A \to A)$, see section Global Semantics.

Haskell code for state equivalence

```
bisim :: Sig -> [(Int,Int)]
bisim = map (\(i,j,_) -> (i,j)) . fixpt supset bisimStep . bisim0

fixpt :: (a -> a -> Bool) -> (a -> a) -> a -> a
fixpt rel step = iter where
                 iter a = if b `rel` a then a else iter b where b = step a

type Triples a b = [(a,a,[b])]
type TriplesL a  = Triples a ([a],[a])

bisim0 :: Sig -> TriplesL Int
bisim0 sig is ks = [(i,j,f i j) | i <- is, j <- is, i < j,
                       eqSet (==) (out sig!!i) $ out sig!!j,
                       and $ zipWith (eqSet (==)) (outL sig!!i) $ outL sig!!j]
           where is = indices_ sig.states; ks = indices_ sig.labels
                 b = null sig.trans
                 f i j = if b then s else s `join1` (sig.trans!!i,sig.trans!!j)
                         where s = mkSet $ map h ks
                               h k = (sig.transL!!i!!k,sig.transL!!j!!k)
```

```
bisimStep :: Eq a => TriplesL a -> TriplesL a
bisimStep trips = [trip | trip@(_,_,bs) <- trips, all r bs] where
          r (is,js) = forallThereis r' is js && forallThereis r' js is
          r' i j = i == j || just (lookupL i j trips) || just (lookupL j i trips)

lookupL :: (Eq a,Eq b) => a -> b -> [(a,b,c)] -> Maybe c
lookupL a b ((x,y,z):s) = if a == x && b == y then Just z else lookupL a b s
lookupL _ _ _           = Nothing

simplifyS (F "~" [t,u]) sig | notnull sig.states
                            = do i <- search (== t) sig.states
                                 j <- search (== u) sig.states
                                 let cli = search (elem i) part
                                     clj = search (elem j) part
                                 Just $ mkConst $ cli == clj
                    where part = mkPartition (indices_ sig.states) $ bisim sig

mkPartition :: Eq a => [a] -> [(a,a)] -> [[a]]
mkPartition = foldl f . map single where
              f part (a,b) = if eqSet (==) cla clb
                             then part else (cla++clb):minus part s
                             where s@[cla,clb] = map g [a,b]
                                   g a = head $ filter (elem a) part
```

Hence a goal of the form $t \sim u$ is simplified by applying *bisim* to (t, u).

The equivalence of t and u can also be checked by applying the axioms for *eqState* (*eqStateL* in the case of labelled system) or `` `~` `` (`` `~L` `` in the case of labelled system) of `modal` to (t, u) (see chapter 4). The value of *eqState(t,u)* (*eqStateL(t,u)*) is obtained by simplification; the value of `` t`~`u `` (`` t`~L`u ``) is obtained by a coinduction step, followed by simplification steps.

Stepwise construction of $\sim$

```
simplifyS (F x@"equivStep" []) sig | notnull sig.states = Just $ F x [Hidden mat]
                      where sts = map showTerm0 sig.states
                            mat = ListMatL sts [(f i,f j,map (apply2 $ map f) ps)
                                                | (i,j,ps) <- bisim0 sig]
                            f = showTerm0 . (sig.states!!)

simplifyS (F x@"equivStep" [Hidden mat]) _ =
                             do ListMatL sts s <- Just mat; let new = bisimStep s
                                guard $ s /= new
                                Just $ F x [Hidden $ ListMatL sts new]
```

If *matrices* has been selected in the *interpreter menu* below the *paint* button, a simplification step of `equivStep` returns a matrix representation of $bisim_0$ in the painter. Further simplification steps in the painter return matrix representations of $bisimStep(bisim_0)$, $bisimStep^2(bisim_0)$, etc.

The command *specification* > *minimize* minimizes the actual Kripke model K by building the quotient of K with respect to $\sim$:

```
mkQuotient :: Sig -> ([TermS],[[Int]],[[[Int]]],[[Int]],[[[Int]]])
mkQuotient sig = (states,tr,trL,va,vaL)
                 where part = mkPartition (bisim sig) $ indices_ sig.states
                       states = map ((sig.states!!) . minimum) part
                       oldPos i = get $ search (== (states!!i)) sig.states
                       newPos i = get $ search (elem i) part
                       h = mkSet . map newPos
                       [is,js,ks] = map indices_ [states,labels,atoms]
                       tr = map f is  where f i = h $ sig.trans!!oldPos i
                       trL = map f is where f i = map (g i) js
                                           g i x = h $ sig.transL!!oldPos i!!x
                       va = map f ks  where f i = h $ sig.value!!i
                       vaL = map f ks where f i = map (g i) js
                                           g i x = h $ sig.valueL!!i!!x
```

Two Kripke models $\mathcal{K}$ and $\mathcal{K}'$ with initial states st resp. st' can be checked for equivalence simply by proving $st \sim st'$ with respect to the sum $\mathcal{K} \uplus \mathcal{K}'$.

Examples

```
-- auto1

constructs: D E dot plus minus final
defuncts:   states labels atoms

axioms:

  states == [1]
& labels == [D,E,dot,plus,minus]
& atoms  == [final]

& (1,D) -> 2 & (2,D) -> 3 & (2,dot) -> 4 & (3,D) -> 5 & (3,dot) -> 4
& (4,D) -> 6 & (4,E) -> 7 & (5,D) -> 5 & (5,dot) -> 4 & (6,D) -> 6 & (6,E) -> 7
& (7,D) -> 9 & (7,plus) -> 8 & (7,minus) -> 8 & (7,dot) -> 10 & (7,E) -> 10
& (8,D) -> 9 & (9,D) -> 9
& (x `in` [1,8,9] & y `in` labels-[D] |
   x `in` [2,3,5] & y `in` labels-[D,dot] |
   x `in` [4,6] & y `in` labels-[D,E] ==> (x,y) -> 10)
& final -> branch[2,3,4,5,6,9]

filter(5~)$states                      -->  [2,3,5]

-- specification > state equivalence  -->  [(2,3),(2,5),(3,5),(4,6)]
```

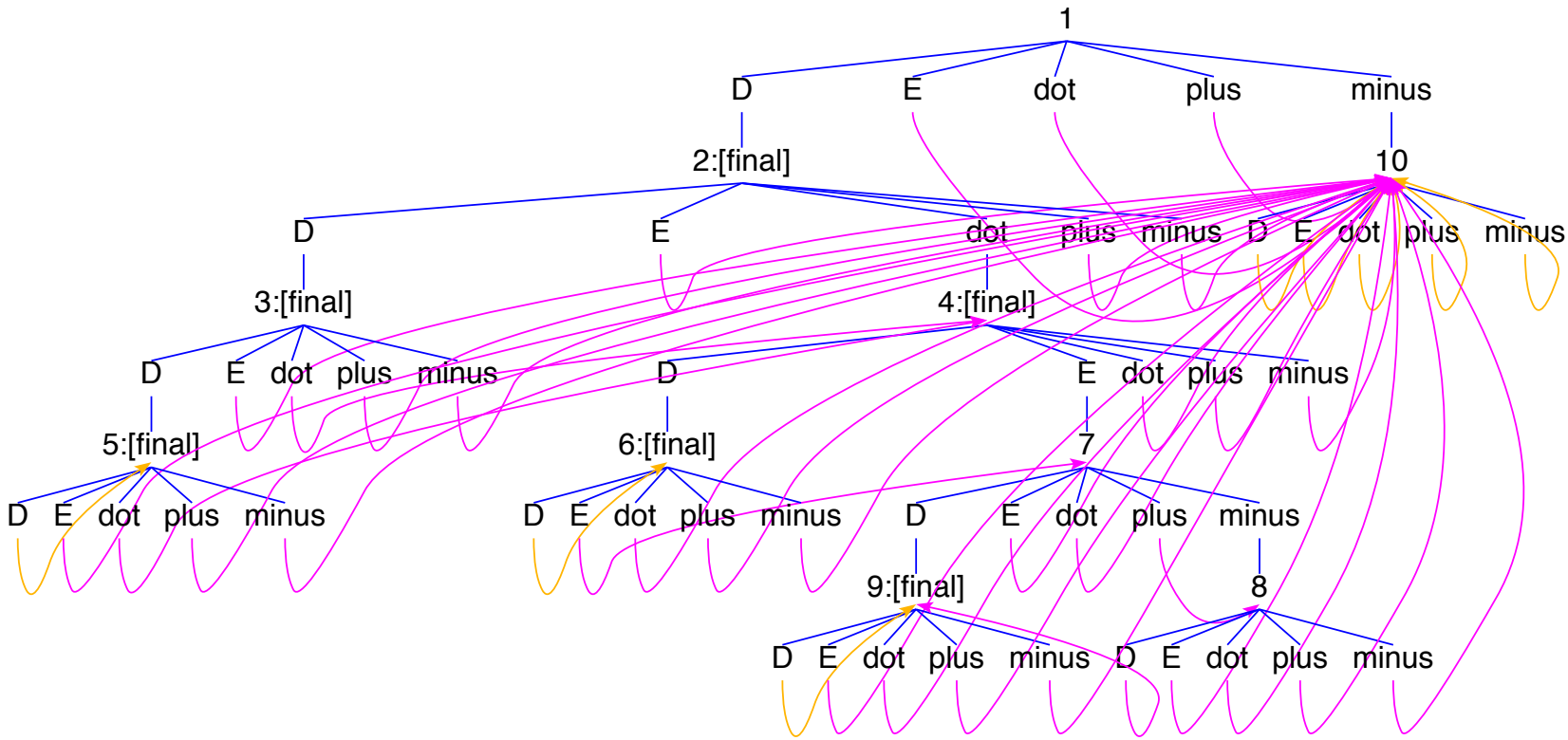


Fig. 7. *Kripke model of* `auto1`

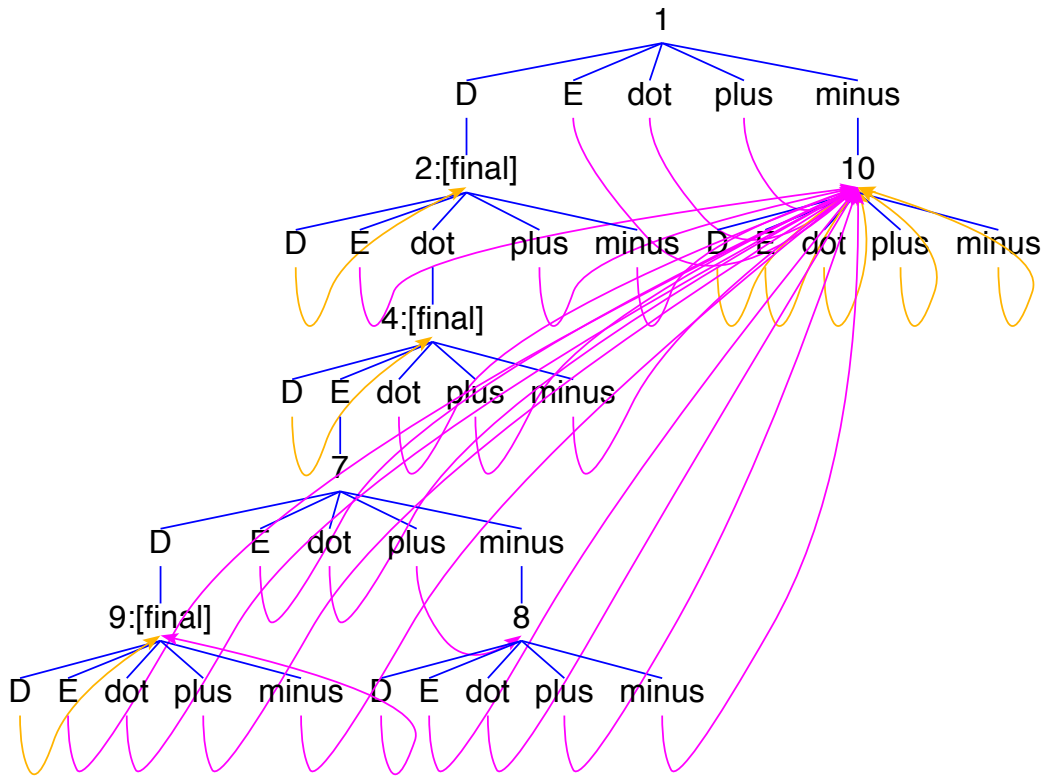


Fig. 8. *Kripke model of* `auto1` *after minimization*

The values of $bisim_0$, $bisimStep(bisim_0)$ and $bisimStep^2(bisim_0)$ (obtained by stepwise simplifying `equivStep`; see above) are listed here: auto1equiv.html.

```
-- auto2

constructs: a b c final

axioms:

  states == [1]
& labels == [a,b,c]
& atoms  == [final]

& (1,a) -> 3 & (1,b) -> 4 & (1,c) -> 2 & (2,a) -> 5 & (2,b) -> 5 & (2,c) -> 4
& (3,a) -> 6 & (3,b) -> 2 & (3,c) -> 3 & (4,a) -> 3 & (4,b) -> 1 & (4,c) -> 6
& (5,a) -> 5 & (5,b) -> 2 & (5,c) -> 3 & (6,a) -> 5 & (6,b) -> 5 & (6,c) -> 1

& (final,a) -> branch[2,3,5,6] & (final,b) -> branch[1..6]
& (final,c) -> branch[1,4]

--  specification > state equivalence  -->  [(1,4),(2,6)]
```

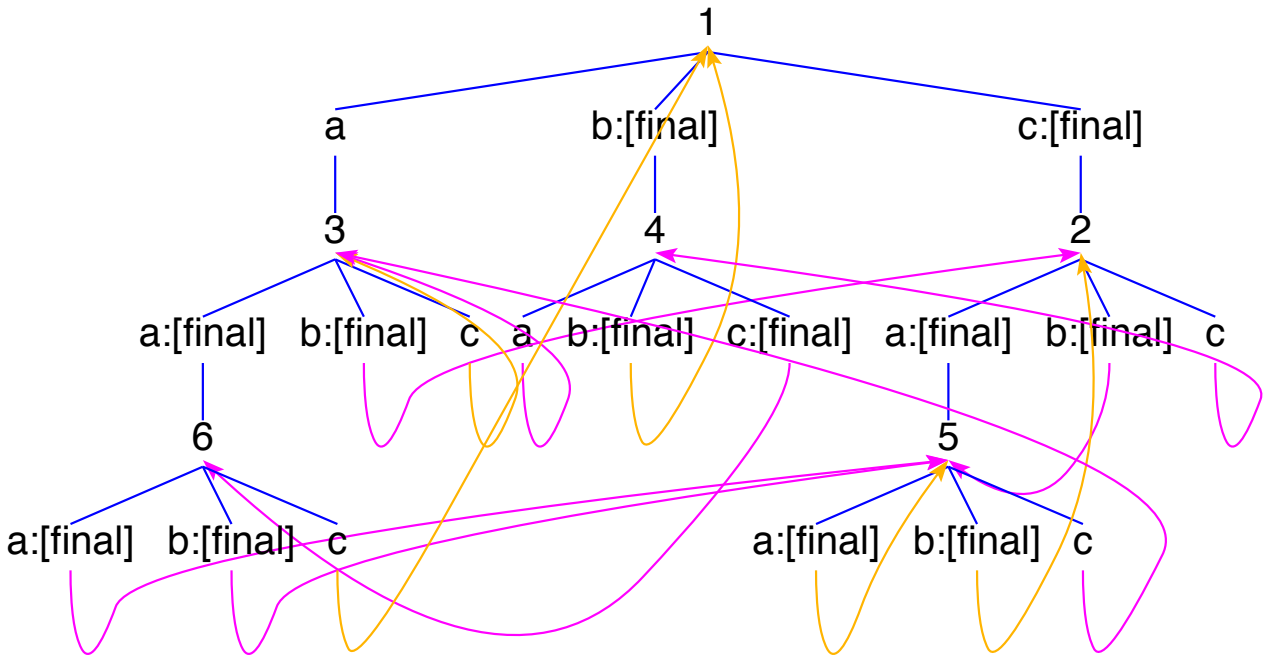


Fig. 9. *Kripke model of* `auto2`

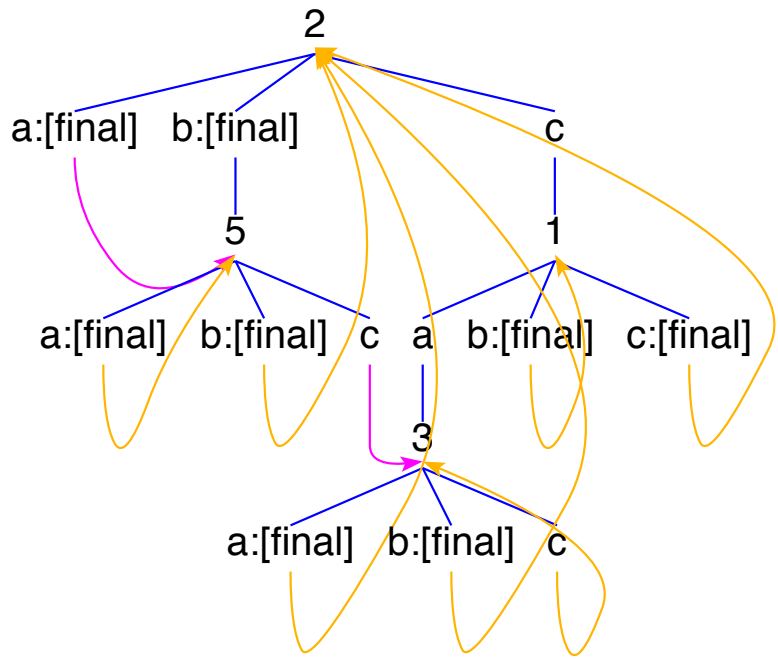
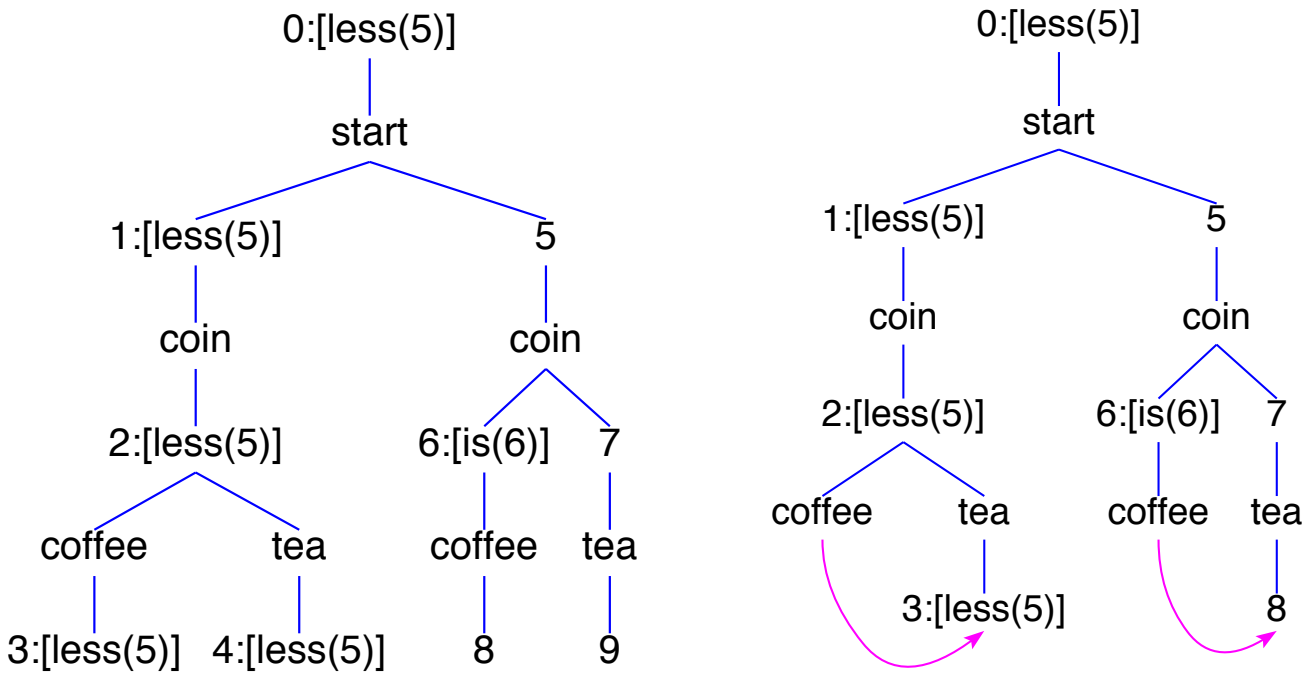


Fig. 10. *Kripke model of* `auto2` *after minimization*

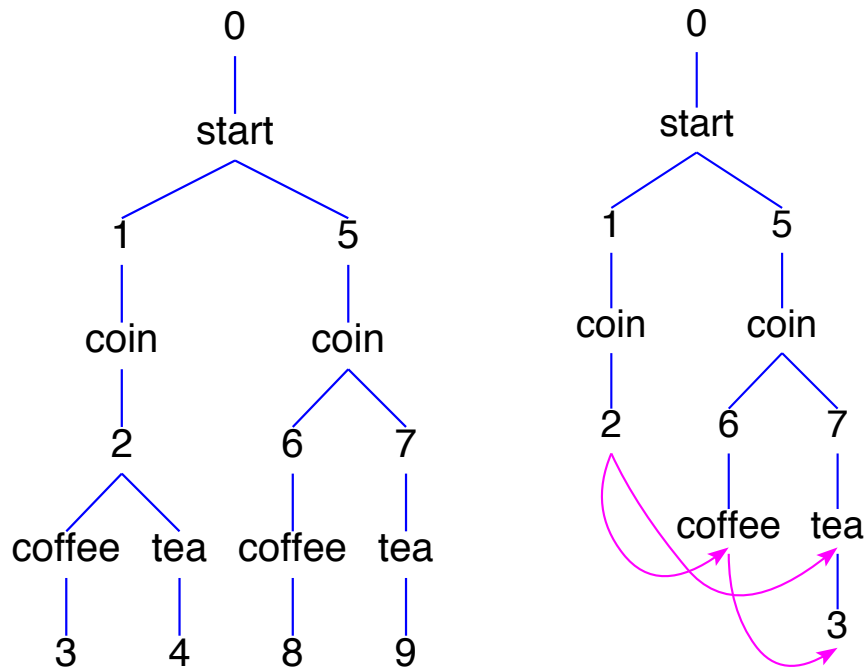
The values of $bisim_0$, $bisimStep(bisim_0)$ and $bisimStep^2(bisim_0)$ (obtained by stepwise simplifying **`equivStep`**; see above) are listed here: auto2equiv.html.

```
-- coin1

specs:      modal
constructs: start coin coffee tea odd is less
```

```
axioms:

states == [0] &
labels == [start,coin,coffee,tea] &
atoms == [less(5),is(6)] &              -- (*)

(0,start) -> branch[1,5] & (2,coffee) -> 3 &
(1,coin) -> 2 & (2,tea) -> 4 &
(5,coin) -> branch[6,7] & (6,coffee) -> 8 &
(7,tea) -> 9

is$st -> st &
less$st -> valid(<st)

-- specification > state equivalence
               --> [(3,4),(3,8),(3,9),(4,8),(4,9),(8,9)]
               --> with (*): [(3,4),(8,9)]
terms:

    eval(AF$less(5)\/is(6))  -->  [0,1,2,3,4,6]
<+> eval(FE$less(5)\/is(6))  -->  [0..9]
```



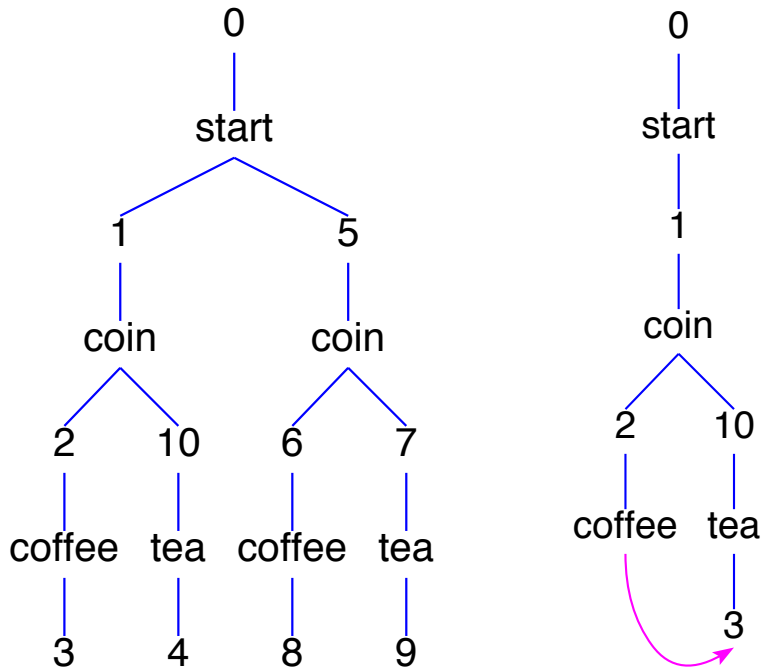
Kripke model of `coin1` *with* $(*)$ *before and after minimization*



Kripke model of `coin1` *without* $(*)$ *before and after minimization*

```
-- coin2

constructs: start coin coffee tea odd
defuncts:   states labels

axioms:
```

```
states == [0] & labels == [start,coin,coffee,tea] &

(0,start) -> branch[1,5] & (2,coffee) -> 3 &
(1,coin) -> branch[2,10] & (10,tea) -> 4 &
(5,coin) -> branch[6,7] & (6,coffee) -> 8 & (7,tea) -> 9

-- specification > state equivalence
   -->  [(1,5),(2,6),(10,7),(3,4),(3,8),(3,9),(4,8),(4,9),(8,9)]
```



Kripke model of `coin2` *before and after minimization*

Simplifications

(siehe auch [15], Kapitel 4 und 5, oder [16]) Simplifikationen führen zu normalisierten Formeln, die von folgender Grammatik erzeugt werden:

$$
\begin{array}{lcl}
NF & \longrightarrow & Impl \mid EC \mid AD \\
Impl & \longrightarrow & \forall X(CA \Rightarrow DE) \\
CA & \longrightarrow & \forall X\, NF \wedge \cdots \wedge \forall X\, NF \\
DE & \longrightarrow & \exists X\, NF \vee \cdots \vee \exists X\, NF \\
EC & \longrightarrow & \exists X(Ate \wedge \cdots \wedge Ate) \\
AD & \longrightarrow & \forall X(Ati \vee \cdots \vee Ati) \\
Ate & \longrightarrow & V = T \mid T = V \mid F(T) = F(T) \mid F(T) = C(T) \mid C(T) = F(T) \mid P(T) \\
Ati & \longrightarrow & V \neq T \mid T \neq V \mid F(T) \neq F(T) \mid F(T) \neq C(T) \mid C(T) \neq F(T) \mid P(T) \\
C & \longrightarrow & \textit{constructors} \\
F & \longrightarrow & \textit{defined functions} \\
P & \longrightarrow & \textit{relations} \\
T & \longrightarrow & \textit{terms and term tuples} \\
X & \longrightarrow & \textit{lists of bound variables} \\
V & \longrightarrow & \textit{variables}
\end{array}
$$

Jeder Simplifikationsschritt eines Termgraphen t besteht in der Tiefen- oder Breitensuche nach einem Redex, also einem Teilterm u von t, und dessen Ersetzung durch einen Term v, der sich wie folgt aus u ergibt:

(1) Lässt sich u mit der u.g. Funktion *evaluate* zu einem von u verschiedenen Term v bzgl. eingebauter Algebren partiell auswerten, dann ist v das Redukt.

(2) Scheitert (1) und ist eine der durch *simplifyGraph* definierten Graphersetzungsregeln auf u anwendbar, dann liefert deren Konklusion das Redukt.

(3) Scheitert (2), dann wird versucht, ein Simplifikationsaxiom, also eine evtl. bedingte Gleichung

$$\varphi_1 \wedge \cdots \wedge \varphi_n \;\; ==> \;\; f(t_1, \ldots, t_n) \;\; == \;\; t,$$

eine evtl. bedingte Äquivalenz

$$\varphi_1 \wedge \cdots \wedge \varphi_n \;\; ==> \;\; p(t_1, \ldots, t_n) \;\; <==> \;\; \varphi$$

oder ein evtl. bedingtes Transitionsaxiom

$$\varphi_1 \wedge \cdots \wedge \varphi_n \;\; ==> \;\; f(t_1, \ldots, t_n) \; -> \; t,$$

von links nach rechts auf u anzuwenden. Gelingt dies, dann liefert die Konklusion des Axioms das Redukt.
Ist das erste anwendbare Axiom eine Gleichung oder Äquivalenz, dann wird nur diese angewendet.

Ist es eine Transition, dann werden alle Transitionen unter den Axiomen der jeweiligen Spezifikation angewendet und deren Redukte mit `<+>` zu einem Term verknüpft. Transitionsaxiome werden auch beim *Narrowing* und *Rewriting* eingesetzt (siehe Kapitel 11). Als Simplifikationen verwendet sie Expander2 nur beim Aufbau einer Kripke-Struktur, die mit dem Befehl *specification* > *build Kripke model* erfolgt.

(4) Scheitert (3), dann wird der Termgraph u zu einem Baum v expandiert. Befindet sich u an einer Position von t mit positiver Polarität und ist eine der durch *simplCoInd* definierten (Co)Induktionsregeln auf u anwendbar, dann liefert deren Konklusion das Redukt.

(5) Scheitert (4), dann wird versucht, eine durch *simplifyF* definierte eingebaute Simplifikationsregel von links nach rechts auf v anzuwenden. Gelingt dies, dann liefert die Konklusion der Regel das Redukt. Die Menge der zugelassenen Signatursymbole kann mit entsprechenden Buttons im Menü *signature* verändert werden.

(6) Scheitert auch (5), dann wird die Suche nach einem Redex abgebrochen und, wieder bei t beginnend, ein Teilterm gesucht, auf dessen Expansion zu einem Baum v eine durch *expandFix* definierte Expansionsregel für Fixpunktausdrücke (s.u.) anwendbar ist. Gelingt dies, dann liefert die Konklusion der Regel das Redukt.

Bis auf die Schritte des Typs (4) transformieren Simplifikationen jede Formel φ in eine – bzgl. des jeweils zugrundeliegenden Modells – zu φ äquivalente Formel.

Simplifikationen des Typs (1): **Die Funktion `evaluate`**

```
evaluate :: Sig -> Term String -> Term String
evaluate sig = eval
   where eval (F "<==>" [t,u]) | eqTerm t u = mkTrue
         eval (F "===>" [t,u])              = mkImpl (eval t) $ eval u
         eval (F "<===" [t,u])              = mkImpl (eval u) $ eval t
         eval (F "&" ts)                    = mkConjunct $ map eval ts
         eval (F "|" ts)                    = mkDisjunct $ map eval ts
         eval (F ('A':'n':'y':x) [t])       = evalQ mkAny x $ eval t
         eval (F ('A':'l':'l':x) [t])       = evalQ mkAll x $ eval t
         eval (F "Not" [t])                 = mkNot sig $ eval t
         eval (F "id" [t])                  = eval t
         eval (F "head" [F ":" [t,_]])      = eval t
         eval (F "tail" [F ":" [_,t]])      = eval t
         eval (F "head" [t])    = case eval t of F "[]" (t:_) -> t
                                                 t -> F "head" [t]
         eval (F "tail" [t])    = case eval t of F "[]" (_:ts) -> mkList ts
                                                 t -> F "tail" [t]
         eval (F "bool" [t])    = case eval t of F "True" [] -> mkConst 1
                                                 F "False" [] -> mkConst 0
                                                 t -> F "bool" [t]
         eval (F "ite" [t,u,v]) = case eval t of F "True" [] -> eval u
                                                 F "False" [] -> eval v
                                                 t -> F "ite" [t,u,v]
```

```
eval (F ":" [t,u])     = case eval u of F "[]" ts -> mkList $ t':ts
                                        u -> F ":" [t',u]
                         where t' = eval t
eval (F "!!" [t,u])    = case eval t of
                          F "[]" ts | just i && k < length ts -> ts!!k
                          t -> F "!!" [t,eval u]
                         where i = foldArith intAlg u
                               k = get i
eval (F "length" [t])  = case eval t of
                              F "[]" ts -> mkConst $ length ts
                              t ->  F "length" [t]
eval (F "range" [t,u]) | just i && just k
                       = mkList $ map mkConst [get i..get k]
                         where i = foldArith intAlg t
                               k = foldArith intAlg u
eval (F "$" [F "lsec" [t,F x []],u]) = F x [eval t,eval u]
eval (F "$" [F "rsec" [F x [],t],u]) = F x [eval u,eval t]
eval (F "$" [F x [],F "()" ts])      = F x $ map eval ts
eval (F "$" [F x [],t])              = F x [eval t]
eval (F "suc" [t]) | just i          = mkConst $ get i+1
                                       where i = foldArith intAlg t
eval (F x p@[_,_]) | isOrd x && just ip = mkRel x $ get ip
                   | isOrd x && just rp = mkRel x $ get rp
                   | isOrd x && just fp = mkRel x $ get fp
                   | isOrd x && just sp = mkRel x $ get sp
```

```
                                  where ip = mapM intAlg.parseA p
                                        rp = mapM realAlg.parseA p
                                        fp = mapM linAlg.parseA p
                                        sp = mapM (relAlg sig).parseA p
eval (F "=" [t,u])   = evalEq t u
eval (F "=/=" [t,u]) = evalNeq t u
eval (F "<+>" ts)    = mkSum $ map eval ts
eval t | just i = mkConst $ get i
       | just r = mkConst $ get r
       | just f = mkLin $ get f
       | just s = mkList $ map (uncurry mkPair) $ sort (<) $ get s
                  where i = foldArith intAlg t
                        r = foldArith realAlg t
                        f = foldArith linAlg t
                        s = foldArith (relAlg sig) t
eval (F x ts)                  = F x $ map eval ts
eval t                         = t

evalQ f x t = f (words x `meet` frees sig t) $ eval t

evalEq t u | eqTerm t u = mkTrue
evalEq (F x ts) (F y us)
                       | x == y && f x && f y && all g ts && all g us
                                = mkConst $ ts `eqset` us
                                  where f x = x `elem` ["{}","<+>"]
```

```
                                        g = isValue sig
evalEq t (F "suc" [u]) | just n = evalEq (mkConst $ get n-1) u
                                  where n = parsePnat t
evalEq (F "suc" [u]) t | just n = evalEq u (mkConst $ get n-1)
                                  where n = parsePnat t
evalEq (F "[]" (t:ts)) (F ":" [u,v]) = mkConjunct
                                       [evalEq t u,evalEq (mkList ts) v]
evalEq (F ":" [u,v]) (F "[]" (t:ts)) = mkConjunct
                                       [evalEq u t,evalEq v $ mkList ts]
evalEq (F x ts) (F y us) | all sig.isConstruct [x,y] =
                              if x == y && length ts == length us
                              then mkConjunct $ zipWith (evalEq) ts us
                              else mkFalse
evalEq t u = mkEq (eval t) $ eval u

evalNeq t u | eqTerm t u = mkFalse
evalNeq (F "{}" ts) (F "{}" us)
          | just qs && just rs = mkConst $ not $ get qs `eqset` get rs
                                 where qs = mapM parseReal ts
                                       rs = mapM parseReal us
evalNeq t (F "suc" [u]) | just n = evalNeq (mkConst $ get n-1) u
                                   where n = parsePnat t
evalNeq (F "suc" [u]) t | just n = evalNeq u (mkConst $ get n-1)
                                   where n = parsePnat t
evalNeq (F "[]" (t:ts)) (F ":" [u,v]) =
```

```
                                mkDisjunct [evalNeq t u,evalNeq (mkList ts) v]
        evalNeq (F ":" [u,v]) (F "[]" (t:ts)) =
                                mkDisjunct [evalNeq u t,evalNeq v $ mkList ts]
        evalNeq (F x ts) (F y us) | all sig.isConstruct [x,y] =
                                      if x == y && length ts == length us
                                      then mkDisjunct $ zipWith (evalNeq) ts us
                                      else mkTrue
        evalNeq t u = mkNeq (eval t) $ eval u

mkAny xs (F ('A':'n':'y':y) [t]) = mkAny (xs++words y) t
mkAny xs (F "==>" [t,u])         = F "==>" [mkAll xs t,mkAny xs u]
mkAny xs (F "|" ts)              = F "|" $ map (mkAny xs) ts
mkAny [] t                       = t
mkAny xs t                       = mkBinder "Any" xs [t]

mkAll xs (F ('A':'l':'l':y) [t]) = mkAll (xs++words y) t
mkAll xs (F "&" ts)              = F "&" $ map (mkAll xs) ts
mkAll [] t                       = t
mkAll xs t                       = mkBinder "All" xs [t]
```

Anwendung der Fixpunktinduktion auf eine Ungleichung

Sei A ein vollständiger Verband und t^A monoton oder A ein ω-CPO und t^A ω-stetig (siehe [19], §3).

$$\frac{\mu x_1 \ldots x_n.t \leq u}{t[\pi_i(u)/x_i \mid 1 \leq i \leq n] \leq u} \Uparrow$$

```
simplifyF (F "<=" [F x [t],u]) _
    | leader x "mu" = Just $ F "<=" [t>>>forL us xs,u]
                            where _:xs = words x; us = mkGets xs u
```

$$\frac{(\mu x_1 \ldots x_n.t)(\vec{x}) \leq u}{\forall \vec{x}\ (t[\pi_i(\lambda \vec{x}.u)/x_i \mid 1 \leq i \leq n](\vec{x}) \leq u)} \Uparrow$$

```
simplifyF (F "<=" [F "$" [F x [t],arg],u]) _
    | leader x "mu" && null (subterms arg) =
                              Just $ mkAll [root arg]
                                   $ F "<=" [apply (t>>>forL us xs) arg,u]
                              where _:xs = words x
                                    us = mkGets xs $ F "fun" [arg,u]
```

Anwendung der Fixpunktinduktion auf eine Implikation

$$\frac{\mu x_1 \ldots x_n.\varphi \Rightarrow \psi}{\varphi[\pi_i(\psi)/x_i \mid 1 \leq i \leq n] \Rightarrow \psi} \Uparrow$$

```
simplifyF (F "`then`" [F x [t],u]) sig
    | leader x "MU" && monotone sig xs t =
                              Just $ F "`then`" [t>>>forL us xs,u]
                              where _:xs = words x; us = mkGets xs u

monotone :: Sig -> [String] -> Term String -> Bool
monotone sig xs = f True
   where f pol (F "`then`" [t,u]) = f (not pol) t && f pol u
         f pol (F "not" [t])      = f (not pol) t
         f pol (F x ts)           = if x `elem` xs then pol else all (f pol) ts
         f _ _                    = True
```

$$\frac{(\mu x_1 \ldots x_n.\varphi)(\vec{x}) \Rightarrow \psi}{\forall \vec{x}\ (\varphi[\pi_i(\lambda \vec{x}.\psi)/x_i \mid 1 \leq i \leq n](\vec{x}) \Rightarrow \psi)} \Uparrow$$

```
simplifyF (F "==>" [F "$" [F x [t],arg],u]) sig
    | leader x "MU" && null (subterms arg) && monotone sig xs t =
                              Just $ mkAll [root arg]
                                   $ mkImpl (apply (t>>>forL us xs) arg) u
                              where _:xs = words x; us = mkGets xs $ F "rel" [arg,u]
```

Anwendung der Coinduktion auf eine Ungleichung

Sei A ein vollständiger Verband und t^A monoton oder A ein ω-coCPO und t^A ω-costetig (siehe Fixpoints, Categories, and (Co)Algebraic Modeling, Kap. 3).

$$\frac{u \leq \nu x_1 \ldots x_n.t}{u \leq t[\pi_i(u)/x_i \mid 1 \leq i \leq n]} \Uparrow$$

```
simplifyF (F "<=" [u,F x [t]]) _
    | leader x "nu" = Just $ F "<=" [t>>>forL us xs,u]
                            where _:xs = words x; us = mkGets xs u
```

$$\frac{u \leq (\nu x_1 \ldots x_n.t)(\vec{x})}{\forall \vec{x}\ (u \leq t[\pi_i(\lambda \vec{x}.u)/x_i \mid 1 \leq i \leq n](\vec{x}))} \Uparrow$$

```
simplifyF (F "<=" [u,F "$" [F x [t],arg]]) _
    | leader x "nu" && null (subterms arg) =
                              Just $ mkAll [root arg]
                                   $ F "<=" [u,apply (t>>>forL us xs) arg]
                              where _:xs = words x
                                    us = mkGets xs $ F "fun" [arg,u]
```

Anwendung der Coinduktion auf eine Implikation

$$\frac{\psi \Rightarrow \nu x_1 \ldots x_n.\varphi}{\psi \Rightarrow \varphi[\pi_i(\psi)/x_i \mid 1 \leq i \leq n]} \Uparrow$$

```
simplifyF (F "`then`" [u,F x [t]]) sig
    | leader x "NU" && monotone sig xs t =
                                Just $ F "`then`" [u,t>>>forL us xs]
                                where _:xs = words x; us = mkGets xs u
```

$$\frac{\psi \Rightarrow (\nu x_1 \ldots x_n.\varphi)(\vec{x})}{\forall \vec{x}\ (\psi \Rightarrow \varphi[\pi_i(\lambda \vec{x}.\psi)/x_i \mid 1 \leq i \leq n](\vec{x}))} \Uparrow$$

```
simplifyF (F "==>" [u,F "$" [F x [t],arg]]) sig
    | leader x "NU" && null (subterms arg) && monotone sig xs t =
                                Just $ mkAll [root arg]
                                     $ mkImpl u $ apply (t>>>forL us xs) arg
                                where _:xs = words x
                                      us = mkGets xs $ F "rel" [arg,u]
```

Simplifikationen des Typs (5): **Die Spezifikation base**

```
-- base

defuncts: init last take drop <=< kfold
preds:  P Q R
fovars: x y z s s' m n i j k
hovars: F G P Q R

axioms:

[x]++s == x:s &
[]++s == s &
s++[] == s &
(x:s)++s' == x:s++s' &

init[x] == [] &
init$x:y:s == x:init$y:s &
init(s)++[last$s] == s &

last[x] == x &
last$x:y:s == last$y:s &

take(0)$s == [] &
take(n)$x:s == x:take(n-1)$s &
```

```
drop(0)$s == s &
drop(n)$x:s == drop(n-1)$s &

length$x:s == length(s)+1 &
length$init$x:s == length$s &

map(F)$x:s == F(x):map(F)$s &

G<=<F == concat.map(G).F &                        -- Kleisli composition for [ ]

kfold(F) == foldl$fun((s,x),concat$map(F$x)$s) &       -- Kleisli fold for [ ]

x+0 == x & x+0.0 == x &
0+x == x & 0.0+x == x &

x-0 == x & x-0.0 == x &
0-x == -x & 0.0-x == -x &

x*0 == 0 & x*0.0 == 0 &
0*x == 0 & 0.0*x == 0 &
x*1 == x & x*1.0 == x &
1*x == x & 1.0*x == x &

0/x == 0 & 0.0/x == 0 &
x/1 == x & x/1.0 == x &
```

```
x**0 == 1 & x**0.0 == 1 &
0**x == 0 & 0.0**x == 0 &
x**1 == x & x**1.0 == x &
1**x == 1 & 1.0**x == 1 &

([] = x:s <==> False) &
(x:s = [] <==> False) &

([] =/= x:s <==> True) &
(x:s =/= [] <==> True) &

(x <= x <==> True) &
(x >= x <==> True) &

(x < x <==> False) &
(x < suc$x <==> True) &

(x > x <==> False) &
(suc$x > x <==> True) &
(suc(x)-y > x-y <==> True) &
(x-y > x-suc(y) <==> True) &

(x `in` [] <==> False) &
(x `NOTin` [] <==> True)
```

Simplifikationen des Typs (5): **Subsumption**

Subsumption ist eine binäre, induktiv definierte Relation zwischen Formeln. In ihr sind alle syntaktisch – ohne Anwendung nichtlogischer Axiome – überprüfbaren Implikationen zusammengefasst.

$\sim$ bezeichne syntaktische Gleichheit modulo Umordnung von Argumenten permutativer Verknüpfungen sowie Umbenennung von Variablen.

$$
\begin{aligned}
\varphi \sim \psi &\implies \varphi \text{ subsumes } \psi \\
\varphi \text{ subsumes } \psi &\implies \neg\psi \text{ subsumes } \neg\varphi \\
\varphi' \text{ subsumes } \varphi \text{ and } \psi \text{ subsumes } \psi' &\implies \varphi \Rightarrow \psi \text{ subsumes } \varphi' \Rightarrow \psi' \\
\exists\, 1 \leq i \leq n : \varphi \text{ subsumes } \psi_i &\implies \varphi \text{ subsumes } \psi_1 \vee \cdots \vee \psi_n \\
\forall\, 1 \leq i \leq n : \varphi_i \text{ subsumes } \psi &\implies \varphi_1 \vee \cdots \vee \varphi_n \text{ subsumes } \psi \\
\forall\, 1 \leq i \leq n : \varphi \text{ subsumes } \psi_i &\implies \varphi \text{ subsumes } \psi_1 \wedge \cdots \wedge \psi_n \\
\exists\, 1 \leq i \leq n : \varphi_i \text{ subsumes } \psi &\implies \varphi_1 \wedge \cdots \wedge \varphi_n \text{ subsumes } \psi \\
\varphi(\vec{x}) \text{ subsumes } \psi(\vec{x}) &\implies \exists\vec{x}\varphi(\vec{x}) \text{ subsumes } \exists\vec{y}\psi(\vec{y}) \\
\varphi(\vec{x}) \text{ subsumes } \psi(\vec{x}) &\implies \forall\vec{x}\varphi(\vec{x}) \text{ subsumes } \forall\vec{y}\psi(\vec{y}) \\
\varphi(\vec{x}) \text{ subsumes } \psi(\vec{x}) \text{ and } \vec{x} \cap \mathit{freeVars}(\psi) = \emptyset &\implies \exists\vec{x}\varphi(\vec{x}) \text{ subsumes } \psi \\
\varphi(\vec{x}) \text{ subsumes } \psi(\vec{x}) \text{ and } \vec{x} \cap \mathit{freeVars}(\psi) = \emptyset &\implies \varphi \text{ subsumes } \forall\vec{y}\psi(\vec{y})
\end{aligned}
$$

$$\exists \vec{t} : \varphi \sim \psi(\vec{t}) \implies \varphi \text{ subsumes } \exists \vec{x} \psi(\vec{x})$$
$$\exists \vec{t} : \varphi(\vec{t}) \sim \psi \implies \forall \vec{x} \varphi(\vec{x}) \text{ subsumes } \psi$$
$$\exists \vec{t}, 1 \leq i_1, \ldots, i_k \leq n : \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \sim \psi(\vec{t}) \implies \varphi_1 \wedge \cdots \wedge \varphi_n \text{ subsumes } \exists \vec{x} \psi(\vec{x})$$
$$\exists \vec{t}, 1 \leq i_1, \ldots, i_k \leq n : \varphi(\vec{t}) \sim \psi_{i_1} \vee \cdots \vee \psi_{i_k} \implies \forall \vec{x} \varphi(\vec{x}) \text{ subsumes } \psi_1 \vee \cdots \vee \psi_n$$

Subsumptionsregeln φ subsumiere ψ.

$$\frac{\varphi \Rightarrow \psi}{\mathit{True}} \qquad \frac{\varphi \Leftrightarrow \psi}{\psi \Rightarrow \varphi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \neg\psi}{\mathit{False}} \qquad \frac{\varphi \vee \psi}{\psi} \qquad \frac{\varphi \vee \neg\psi}{\mathit{True}}$$

$$\frac{\varphi \wedge (\psi \Rightarrow \delta)}{\varphi \wedge \delta} \qquad \frac{\varphi \vee (\psi \Rightarrow \delta)}{\mathit{True}}$$

Move quantifiers upwards

Sei $\vec{x} = \theta(\vec{x_1} \cup \cdots \cup \vec{x_n})$ und θ eine Umbenennung von Variablen derart, dass für alle $1 \leq i \leq n$ keine Variable von $\theta(\vec{x_i})$ in einer Formel $\theta(\varphi_j)$ mit $1 \leq j \leq n$, $j \neq i$, frei vorkommt.

$$\frac{\exists \vec{x_1}\varphi_1 \wedge \cdots \wedge \exists \vec{x_n}\varphi_n}{\exists \vec{x}\theta(\varphi_1 \wedge \cdots \wedge \varphi_n)} \qquad \frac{\forall \vec{x_1}\varphi_1 \vee \cdots \vee \forall \vec{x_n}\varphi_n}{\forall \vec{x}\theta(\varphi_1 \vee \cdots \vee \varphi_n)} \qquad \frac{\exists \vec{x_1}\varphi_1 \Rightarrow \forall \vec{x_2}\varphi_2}{\forall \vec{x}\theta(\varphi_1 \Rightarrow \varphi_2)}$$

```
simplifyF c@(F "&" ts) sig = move ts [] [] False
         where move (F ('A':'n':'y':x) [t]:ts) us zs b =
                                  move ts (renameAll g t:us) (map g xs++zs) True
                                  where xs = words x
                                        g = renameAwayFrom sig xs (join ts us) c
               move (t:ts) us zs b = move ts (t:us) zs b
               move _ us zs b      = do guard b; Just $ mkAny (reverse zs)
                                                $ mkConjunct $ reverse us
```

General term replacement

Sei $root(t)$ ein Konstruktor, $root(u)$ kein Konstruktor und u kein Teilterm von t.

$$\frac{t = u \wedge \varphi(t)}{t = u \wedge \varphi(u)} \qquad \frac{t \neq u \vee \varphi(t)}{t \neq u \vee \varphi(u)}$$

$$\frac{t = u \wedge \varphi(t) \quad \Rightarrow \quad \psi(t)}{t = u \wedge \varphi(u) \quad \Rightarrow \quad \psi(u)} \qquad \frac{\varphi(t) \quad \Rightarrow \quad t \neq u \vee \psi(t)}{\varphi(u) \quad \Rightarrow \quad t \neq u \vee \psi(u)}$$

Zerlegung von Memberships, Gleichungen und Ungleichungen

$$\frac{t \in [t_1, \ldots, t_n]}{t = t_1 \vee \cdots \vee t = t_n} \qquad \frac{t \in [t_1, \ldots, t_n]}{t \neq t_1 \wedge \cdots \wedge t \neq t_n}$$

t contains a variable,
$t_1, \ldots, t_n$ consist of constructors

Let c and d be different constructors.

$$\frac{c(t_1, \ldots, t_n) = c(u_1, \ldots, u_n)}{t_1 = u_1 \wedge \cdots \wedge t_n = u_n} \qquad \frac{c(t_1, \ldots, t_n) = d(u_1, \ldots, u_n)}{\mathit{False}}$$

$$\frac{c(t_1, \ldots, t_n) \neq c(u_1, \ldots, u_n)}{t_1 \neq u_1 \vee \cdots \vee t_n \neq u_n} \qquad \frac{c(t_1, \ldots, t_n) \neq d(u_1, \ldots, u_n)}{\mathit{True}}$$

Remove equations and inequations

Let $x \in \vec{x} \setminus var(t)$.

$$\frac{\exists \vec{x}(x = t \wedge \varphi)}{\exists \vec{x}\varphi[t/x]} \qquad \frac{\forall \vec{x}(x \neq t \vee \varphi)}{\forall \vec{x}\varphi[t/x]}$$

$$\frac{\forall \vec{x}((x = t \wedge \varphi) \Rightarrow \psi)}{\forall \vec{x}(\varphi \Rightarrow \psi)[t/x]} \qquad \frac{\forall \vec{x}(\varphi \Rightarrow (x \neq t \vee \psi))}{\forall \vec{x}(\varphi \Rightarrow \psi)[t/x]}$$

The following rules also remove equations or inequations. However, in contrast to the preceding rules, they are—like induction and coinduction—proper expansion rules (see [19], chapter 10).

Let $\sigma : V \to T_\Sigma(V)$ be a substitution of variables, $dom(\sigma) = \{x \in V \mid \sigma(x) \neq x\}$ and $t, u \in T_\Sigma(V) \setminus dom(\sigma)$ such that $t\sigma = u\sigma$.

$$\frac{\exists\ dom(\sigma)(t = u \wedge \varphi)}{\varphi\sigma} \Uparrow \qquad \frac{\forall\ dom(\sigma)(t \neq u \vee \varphi)}{\varphi\sigma} \Uparrow$$

$$\frac{\forall\ dom(\sigma)((t = u \wedge \varphi) \Rightarrow \psi)}{(\varphi \Rightarrow \psi)\sigma} \Uparrow \qquad \frac{\forall\ dom(\sigma)(\varphi \Rightarrow (t \neq u \vee \psi)}{(\varphi \Rightarrow \psi)\sigma)} \Uparrow$$

Diese Regeln werden wie die obigen (Co)Induktionsregeln nur angewendet, wenn der erste Button im *signature*-Menü auf *simplification is unsafe* gesetzt ist.

λ-applications

Let $x, y, x_1, \ldots, x_n, xs$ be variables, $p, p_1, \ldots, p_n$ be patterns, $b, b_1, \ldots, b_n$ be formulas, $t, u, t_1, \ldots, t_n$ be arbitrary terms, $\sigma, \sigma_1, \ldots, \sigma_n$ be substitutions, es be a list of expressions of the form $b \to t$ and eqs be a list of equations of the form $p = t$.

A **guarded pattern** is an expression of the form $p|b$ ($p||b$ in Expander2). An unguarded pattern p is identified with the guarded pattern $p|\,True$.

A **lazy pattern** is a pattern preceeded by the symbol $\sim$.

$$\lambda(\sim p.t)(u) \to t\sigma \text{ if } p\sigma = u \tag{1}$$

$$\lambda(\sim p.t)(u) \to t[\lambda(p.x)(u)/x \mid x \in var(p)] \text{ if } p \not\leq u \tag{2}$$

$$\lambda(p.t)(u) \to t\sigma \text{ if } p\sigma = u \tag{3}$$

$$\lambda[p_1|b_1.t_1, \ldots, p_n|b_n.t_n](t) \to case[b_{i_1}\sigma_1 \to t_{i_1}\sigma_1, \ldots, b_{i_k}\sigma_1 \to t_{i_k}\sigma_k] \tag{4}$$

if for all $1 \leq i \leq k$, $p_{i_j}\sigma_j = t_{i_j}$, and $p_{i_1}, \ldots, p_{i_k}$ are the patterns among $p_1, \ldots, p_n$ that match t

$$case((True \to t) : es) \to t \tag{5}$$

$$case((False \to t) : es) \to case(es) \tag{6}$$

$$case[] \to undefined \tag{7}$$

$$\lambda((p|b.t \ where \ eqs +\!\!+ [p' = u]) : es) \to \lambda((p|(\lambda p'.b)(u).(\lambda p'.t)(u) \ where \ eqs) : es) \tag{8}$$

$$\lambda((t \ where \ []) : es) \to \lambda(t : es) \tag{9}$$

Note that rule (2) reduces a λ-application $\lambda(\sim p.t)(u)$ to (an instance of) t even if u does not match the pattern p.

If for all $1 \leq j \leq k$, $b_{i_j}\sigma_j$ can be reduced to *True* or *False* via other rules, these rules together with (4)-(7) provide a complete rewrite system for evaluating λ-expressions with at least two non-lazy patterns.

```
simplifyA sig app@(F "$" [F x [F "~" [pat],body],arg])
   | lambda x && just tsub = Just $ t>>>sub
                             where tsub = bodyAndSub sig True app pat 1 arg
                                   (t,sub) = get tsub

simplifyA sig app@(F "$" [F x [pat,body],arg])
   | lambda x && just tsub = Just $ t>>>sub
                             where tsub = bodyAndSub sig False app pat 1 arg
                                   (t,sub) = get tsub

simplifyA sig app@(F "$" [F x ts,arg])
   | lambda x && lg > 2 && even lg = Just $ F "CASE" $ pr1
                                          $ fold2 f ([],0,1) pats bodies where
     lg = length ts; pats = evens ts; bodies = odds ts
     f (cases,i,k) pat body = if just t then (cases++[get t],i+1,k+2)
                                        else (cases,i,k+2)
      where t = concat [do F "||" [pat,b] <- Just pat; result pat b,
                           result pat mkTrue]
```

```
          result pat b = do (u,sub) <- bodyAndSub sig False app pat k arg
                            Just $ F "->" [b>>>sub,addToPoss [i,1] $ u>>>sub]

bodyAndSub sig lazy app pat k arg =
              if lazy then Just (u,forL (map f xs) xs)
                      else do sub <- match sig xs arg' pat; Just (u,sub)
              where xs = sigVars sig pat
                    arg' = dropFromPoss [1] arg
                    f x = apply (F "fun" [pat,mkVar sig x]) arg'
                    bodyPos = [0,k]
                    body = getSubterm1 app bodyPos
                    h = getSubAwayFrom body (bounded sig body) $ frees sig arg'
                    t = collapseVars sig xs $ renameBound sig h body
                    u = addChar 't' t $ targets t `minus` markPoss 't' t

simplifyT (F "CASE" (F "->" [t,u]:_))  | isTrue t  = Just u
simplifyT (F "CASE" (F "->" [t,_]:ts)) | isFalse t = Just $ F "CASE" ts
```

Expansion eines Fixpunktterms oder einer Fixpunktformel

Let $\mathit{fix} \in \{\mu, \nu\}$.

$$\frac{\mathit{fix}\ x.t}{t[(\mathit{fix}\ x.t)/x]}$$

```
expandFix :: Sig -> Bool -> [Int] -> Term String -> [Int] -> Maybe (Term String)
expandFix sig firstCall muPos t p =
                  do guard $ not $ sig.blocked "fixes"
                     F mu [body] <- Just redex
                     guard $ isFix mu
                     let [_,x] = words mu
                         u = collapseVars sig [x] body
                         v = if firstCall then redex else V $ 'e':mkPos0 muPos
                         f = if x `elem` frees sig u then v `for` x else V
                         reduct = instantiate (const id) u f []
                     Just $ replace1 t p $ reduct
                  where redex = addTargets (getSubterm t p) p
```

Let $\mathit{fix} \in \{\mu, \nu\}$.

$$\frac{\pi_i(\mathit{fix}\ x.(t_1, \ldots, t_n))}{t_i}\ \ 1 \leq i \leq n,\ x \notin var(t_i)$$

```
simplifyT (F x [F mu [F "()" ts]]) | just i && isFix mu
                          = do guard $ k < length ts && foldT f t; Just t
                            where i = parse (strNat "get") x; k = get i; t = ts!!k
                                  [_,z] = words mu; f x bs = x /= z && and bs
```

For the auxiliary functions used here, see the modules *Eterm.hs* and *Esolve.hs* in the Expander2 package.

(Co)resolution and (co)induction on (co)predicates

(see also [19], chapters 10 and 11)

In Expander2, (co)resolution steps are performed by pressing the *narrow* button. Unification of axioms with the actual goal is restricted to matching if the *match/unify* button left of the *narrow* button is set to *match*. The intervening *all/random* button admits to switch between the application of all applicable axioms in parallel (see the respective rule succedents) and the random selection of a single applicable rule.

Resolution upon a least predicate p

$$\frac{p(t)}{\bigvee_{i=1}^{k} \exists Z_i : (\varphi_i\sigma_i \wedge \vec{x} = \vec{x}\sigma_i) \vee \bigvee_{i=k+1}^{m} \vec{x} = \vec{x}\sigma_i} \quad \Updownarrow$$

where $\gamma_1 \Rightarrow (p(t_1) \Longleftarrow \varphi_1), \ldots, \gamma_n \Rightarrow (p(t_n) \Longleftarrow \varphi_n)$ are the axioms for p and

$(*)$ $\vec{x}$ is a list of the variables of t,
for all $1 \leq i \leq n$, t_i consists of $\mathcal{C}$-constructors and variables,
there are $1 \leq k, m \leq n$ such that
for all $1 \leq i \leq k$, $t_i\sigma_i = t\sigma_i$, $\gamma_i\sigma_i \vdash \mathit{True}$ and $Z_i = \mathit{var}(t_i, \varphi_i)$,

$k < i \leq m$, $t'_i\sigma_i = t\sigma_i$ for some $t'_i \leq t_i$ with $t'_i \notin V$ (σ_i *partially unifies* t_i and t), for all $m < i \leq n$, t is not partially unifiable with t_i.

Coresolution upon a greatest predicate p

$$\frac{p(t)}{\bigwedge_{i=1}^{k} \forall Z_i : (\varphi_i\sigma_i \vee \vec{x} \neq \vec{x}\sigma_i) \wedge \bigwedge_{i=k+1}^{m} \vec{x} \neq \vec{x}\sigma_i}$$

where $\gamma_1 \Rightarrow (p(t_1) \Longrightarrow \varphi_1), \ldots, \gamma_n \Rightarrow (p(t_n) \Longrightarrow \varphi_n)$ are the axioms for p and $(*)$ holds true.

Narrowing upon a function f

$$\frac{r(\ldots, f(t), \ldots)}{\begin{array}{c}\bigvee_{i=1}^{k} \exists Z_i : (r(\ldots, u_i, \ldots)\sigma_i \wedge \varphi_i\sigma_i \wedge \vec{x} = \vec{x}\sigma_i) \vee \\ \bigvee_{i=k+1}^{m} (r(\ldots, f(t), \ldots)\sigma_i \wedge \vec{x} = \vec{x}\sigma_i)\end{array}} \quad \Updownarrow$$

where r is a predicate, $\gamma_1 \Rightarrow (f(t_1) = u_1 \Longleftarrow \varphi_1), \ldots, \gamma_n \Rightarrow (f(t_n) = u_n \Longleftarrow \varphi_n)$ are the axioms for f and $(*)$ holds true.

As pointed out in [1, 10, 11], partial unification is needed for ensuring the completeness of equational narrowing if the redex $f(t)$ is selected according to outermost ("lazy") strategies, which — as in the case of rewriting — are the only ones that guarantee termination and optimality.

Narrowing upon a transition relation $\rightarrow$

$$\frac{t \,{}^\wedge v \rightarrow t'}{\begin{array}{c}\bigvee_{i=1}^{k} \exists Z_i : ((u_i \,{}^\wedge v)\sigma_i = t'\sigma_i \wedge \varphi_i\sigma_i \wedge \vec{x} = \vec{x}\sigma_i) \vee \\ \bigvee_{i=k+1}^{l} ((t \,{}^\wedge v)\sigma_i \rightarrow t'\sigma_i \wedge \vec{x} = \vec{x}\sigma_i)\end{array}}$$

where $\gamma_1 \Rightarrow (t_1 \rightarrow u_1 \Longleftarrow \varphi_1), \ldots, \gamma_n \Rightarrow (t_n \rightarrow u_n \Longleftarrow \varphi_n)$ are the axioms for $\rightarrow$, σ_i is a unifier modulo associativity and commutativity of ${}^\wedge$ and $(*)$ holds true.

Fixpoint induction over a least predicate p

$$(1) \qquad \frac{p(x) \;\Rightarrow\; \psi(x)}{\bigwedge_{p(t) \Leftarrow \varphi \in AX} (\varphi[q/p] \Rightarrow \psi(t))} \Uparrow$$

$$(2) \qquad \frac{q(x) \;\Rightarrow\; \delta(x)}{\bigwedge_{p(t) \Leftarrow \varphi \in AX} (\varphi[q/p] \Rightarrow \delta(t))}$$

After applying (1), the predicate q and the axiom $q(x) \Rightarrow \psi(x)$ are added to the current specification.

After applying (2), the axiom $q(x) \Rightarrow \delta(x)$ is added to the current specification.

The correctness of (1) follows from the interpretation of p as the least relation satisfying AX_p. For the correctness of (2) see chapter 11 of [19].

Fixpoint coinduction over a greatest predicate p

$$(1) \qquad \frac{\psi(x) \;\Rightarrow\; p(x)}{\bigwedge_{p(t) \Rightarrow \varphi \in AX} (\psi(t) \Rightarrow \varphi[q/p])} \Uparrow$$

$$(2) \qquad \frac{\delta(x) \;\Rightarrow\; q(x)}{\bigwedge_{p(t) \Rightarrow \varphi \in AX} (\delta(t) \Rightarrow \varphi[q/p])} \quad p \notin \delta$$

After applying (1), the predicate q and the axiom $q(x) \Leftarrow \psi(x)$ are added to the current specification. If (the interpretation of) p is the greatest bisimulation on the underlying model, then proof correctness is not violated by adding congruence axioms for q, which can be done in Expander2 as follows:

- Move the cursor on the canvas to an occurrence of q;
- write words of the set $\{refl, symm, tran\} \cup \{fn \mid f$ is a function symbol with arity $n\}$ into the entry field;
- press the *axioms* > *add* > *from entry field* button. The reflexivity, symmetry, transitivity and/or f-compatibility axioms for q are added to AX.

After applying (2), the axiom $q(x) \Leftarrow \delta(x)$ is added to the current specification.

The correctness of (1) follows from the interpretation of p as the least relation satisfying AX_p. For the correctness of (2) see chapter 11 of [19].

(Co)Induction steps are performed by pressing the *selection* > *co/induction* button.

Expander2 kann (Co)Induktion auch auf Formeln anwenden, die – durch *Prämissen-* bzw. *Konklusionsdehnung* – in eine der Prämissen von (1) oder (2) transformierbar sind (siehe [15, 16]).

Außerdem lassen sich (1) und (2) auf mehrere – z.B. wechselseitig rekursiv definierte – (Co)Prädikate gleichzeitig anwenden.

Die in [16] und [17] erwähnte Möglichkeit, anstelle von AX_p die *n-fache Iteration* AX_p^n *von* AX_p mit

$$AX_p^n = \begin{cases} \{p(t) \Leftarrow \varphi[\textit{result of } n \textit{ resolution steps upon } p(u)/p(u)] \mid p(t) \Leftarrow \varphi \in AX_p\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{bei Induktion} \\ \{p(t) \Rightarrow \varphi[\textit{result of } n \textit{ coresolution steps upon } p(u)/p(u)] \mid p(t) \Rightarrow \varphi \in AX_p\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{bei Coinduktion} \end{cases}$$

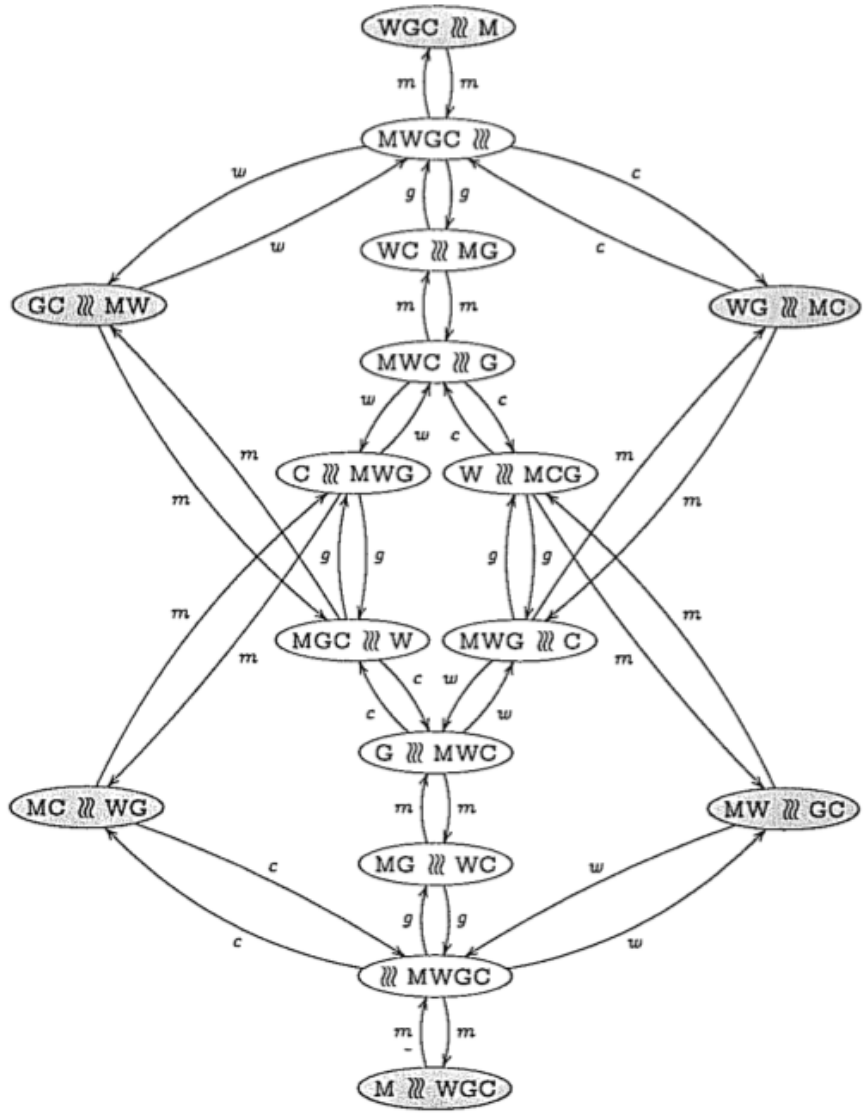
in den obigen Regeln zu verwenden, wurde inzwischen aus Expander2 entfernt, weil es wohl kaum Anwendungen dafür gibt.

Man-wolf-goat-cabbage problem

```
-- mwgc

specs:          modal
constructs:     m w g c final
defuncts:       goods draw1 draw2 draw3 pic river triple save'
preds:          Final

axioms:

goods  == [w,g,c] &
labels == m:goods &
states == [labels] &
atoms  == [final] &

final -> valid(Final) &
(Final$st <==> null$st | any(`in`[[m],[w,g],[g,c]])[st,labels-st]) &

(m `in` st & Not(Final$st) ==> (st,m) -> st-[m]) &

(m `NOTin` st & Not(Final$st) ==> (st,m) -> m:st) &

(x `in` goods & [m,x] `subset` st & Not(Final$st) ==> (st,x) -> st-[m,x]) &
```

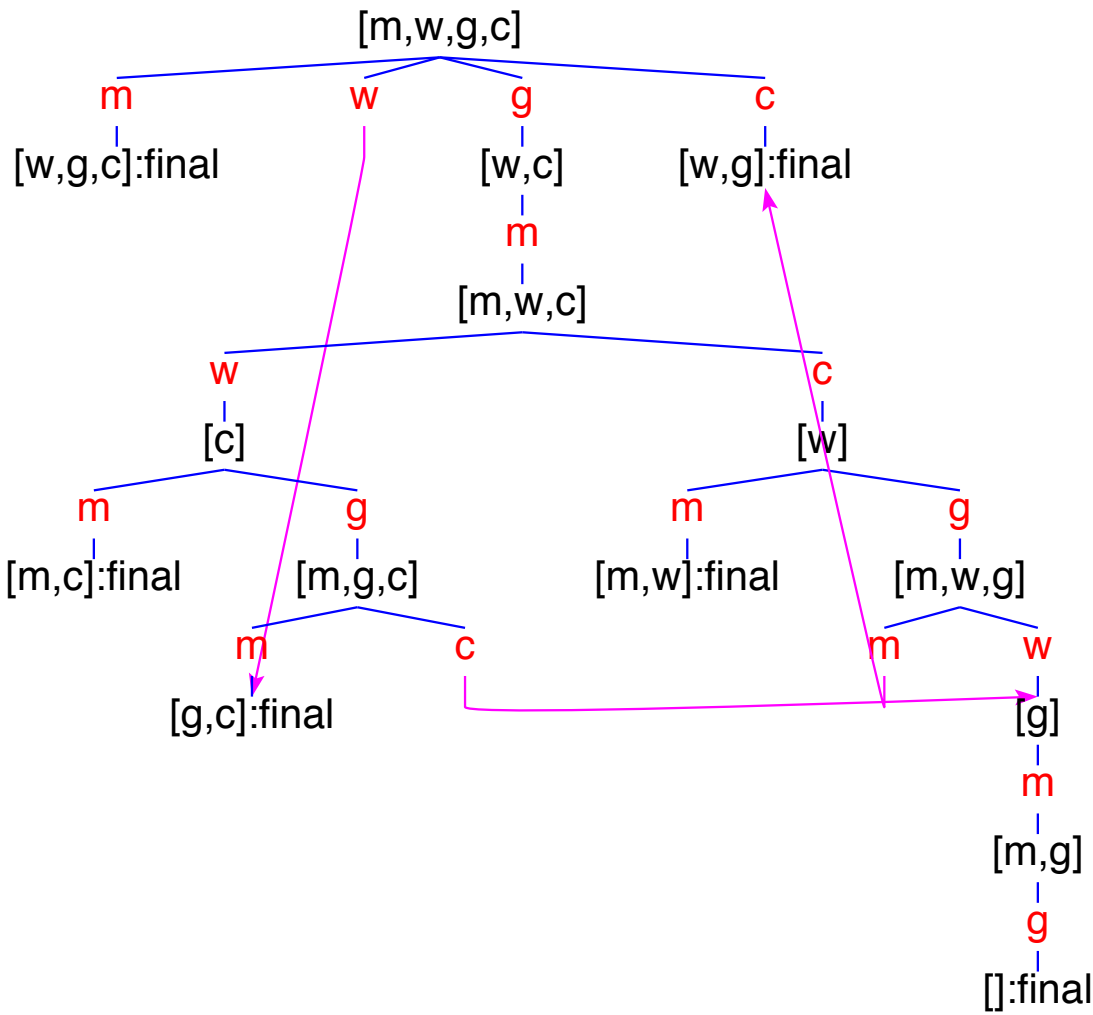
```
(x `in` goods & [m,x] `NOTshares` st & Not(Final$st)
                                    ==> (st,x) -> m:insertL(x)(st)(goods)) &

-- widget interpreters:

pic$m == gif(man,8,15) &
pic$w == gif(wolf,30,25) &
pic$g == gif(goat,28,25) &
pic$c == gif(cabbage,22,20) &
river == gif(wave,70,10) &
      -- blue$waveSF(5,20,45)

triple$s == center$shelf(1,3)[shelf(4)$map(pic)$s,river,
                              shelf(4)$map(pic)$labels-s] &

save'$s == save(triple$s,s) &

draw1 == wtree $ fun(red$x,text$x,x,triple$x) & -- paints states

draw2 == wtree $ fun(red$x,text$x,x,load$x) &   -- loads and paints states

draw3 == wtree $ fun(red$m,pic$m,               -- paints labels
                     red$x,center$shelf(2)[pic$m,pic$x],
                     x,text$x)
```

```
conjects:

(x `in` goods & x `in` [m,w,c]) & --> x = w | x = c

EF(null)$labels --> True
                --  (189 depthfirst simplification steps;
                --   79 parallel simplification steps + 3 subtree permutations)

terms:

insertL(c)[w,g]$goods <+> --> [w,g,c]

eval$EF$success <+>

triple[m,g,c] <+>
map(save')[[m,w,g,c],[w,c]] <+>
map(load.string)[[m,w,g,c],[w,c]] <+>

gggg(widg(tt(zz,widg(tt(pos 0 0,kk),load[w,c])),load[m,w,g,c]))
```

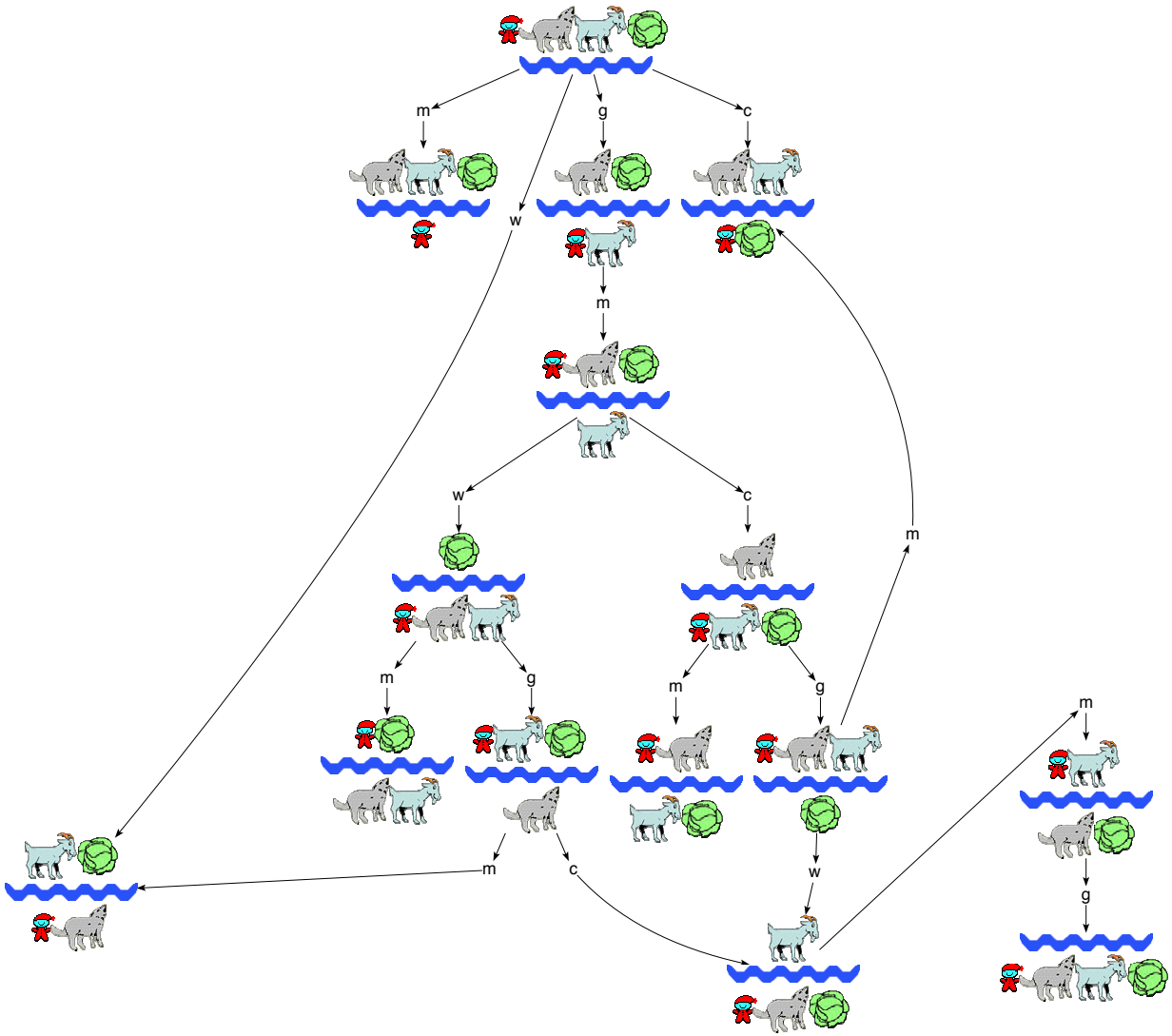


Transitions of `mwgc` *as presented in [7], Example 11.3*

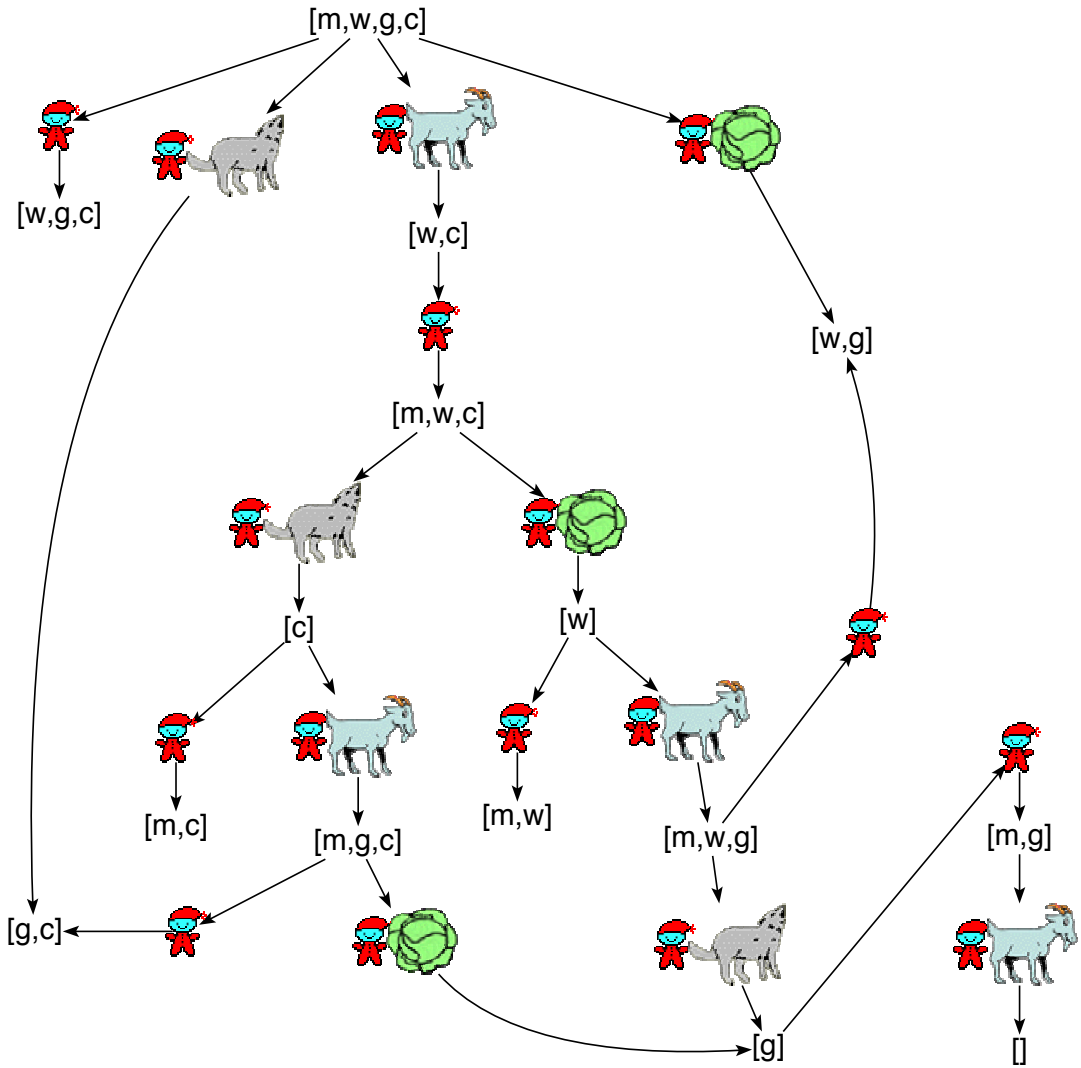
press *specification* > *build Kripke model .. cycle-free*
press *graph* > *show graph of Kripke model* > *here*



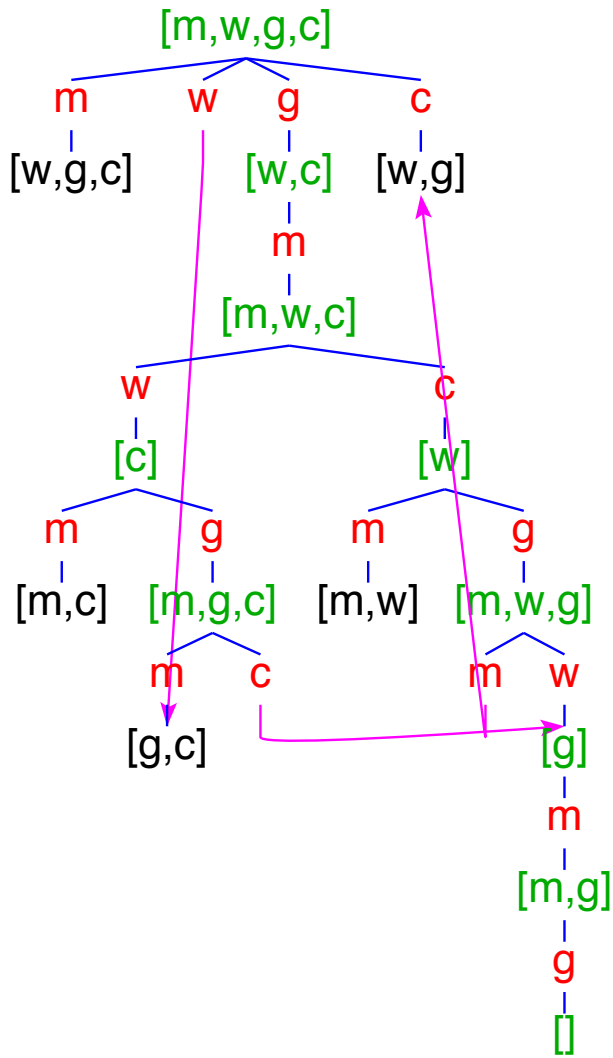
press *graph* > *show graph of labelled transitions* > *here*; enter *draw1* into the entry field
press *tree* > *tree*; press *paint*; adapt red support points and press *combis* to remove them



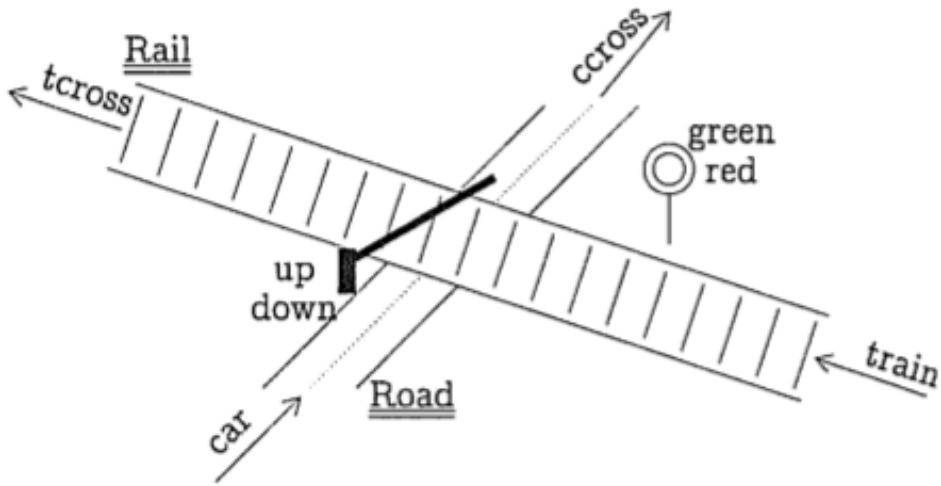
press *graph* > *show graph of labelled transitions* > *here*; enter *draw3* into the entry field
press *tree* > *tree*; press *paint*; adapt red support points and press *combis* to remove them



enter *evalGEFsuccess* into the text field; press *parse up*; press *simplify*



Railway crossing (see [7], section 14.3)



A railway crossing ([7], Fig. 14.1)

Process constructors as defined in [7], sections 11.3 and 14.1

Name	Syntax	Semantics
PROCESS VARIABLE	X	If $E \xrightarrow{a} E'$ and $X \stackrel{\text{def}}{=} E$ then $X \xrightarrow{a} E'$
ACTION PREFIX	$a.E$	$a.E \xrightarrow{a} E$
CHOICE (1)	$E + F$	If $E \xrightarrow{a} E'$ then $E + F \xrightarrow{a} E'$ If $F \xrightarrow{a} F'$ then $E + F \xrightarrow{a} F'$
CHOICE (2)	$\sum\{E_i : i \in I\}$	If $E_j \xrightarrow{a} E'$ with $j \in I$ then $\sum\{E_i : i \in I\} \xrightarrow{a} E'$
NIL	0	no transitions ($X \stackrel{\text{def}}{=} \sum \emptyset$)
Parallel Composition	E || F	If $E \xrightarrow{a} E'$ and $a \notin \mathrm{Sort}(F)$ then $E \parallel F \xrightarrow{a} E' \parallel F$. If $F \xrightarrow{a} F'$ and $a \notin \mathrm{Sort}(E)$ then $E \parallel F \xrightarrow{a} E \parallel F'$. If $E \xrightarrow{a} E'$ and $F \xrightarrow{a} F'$ and $a \in \mathrm{Sort}(E) \cup \mathrm{Sort}(F)$ then $E \parallel F \xrightarrow{a} E' \parallel F'$.

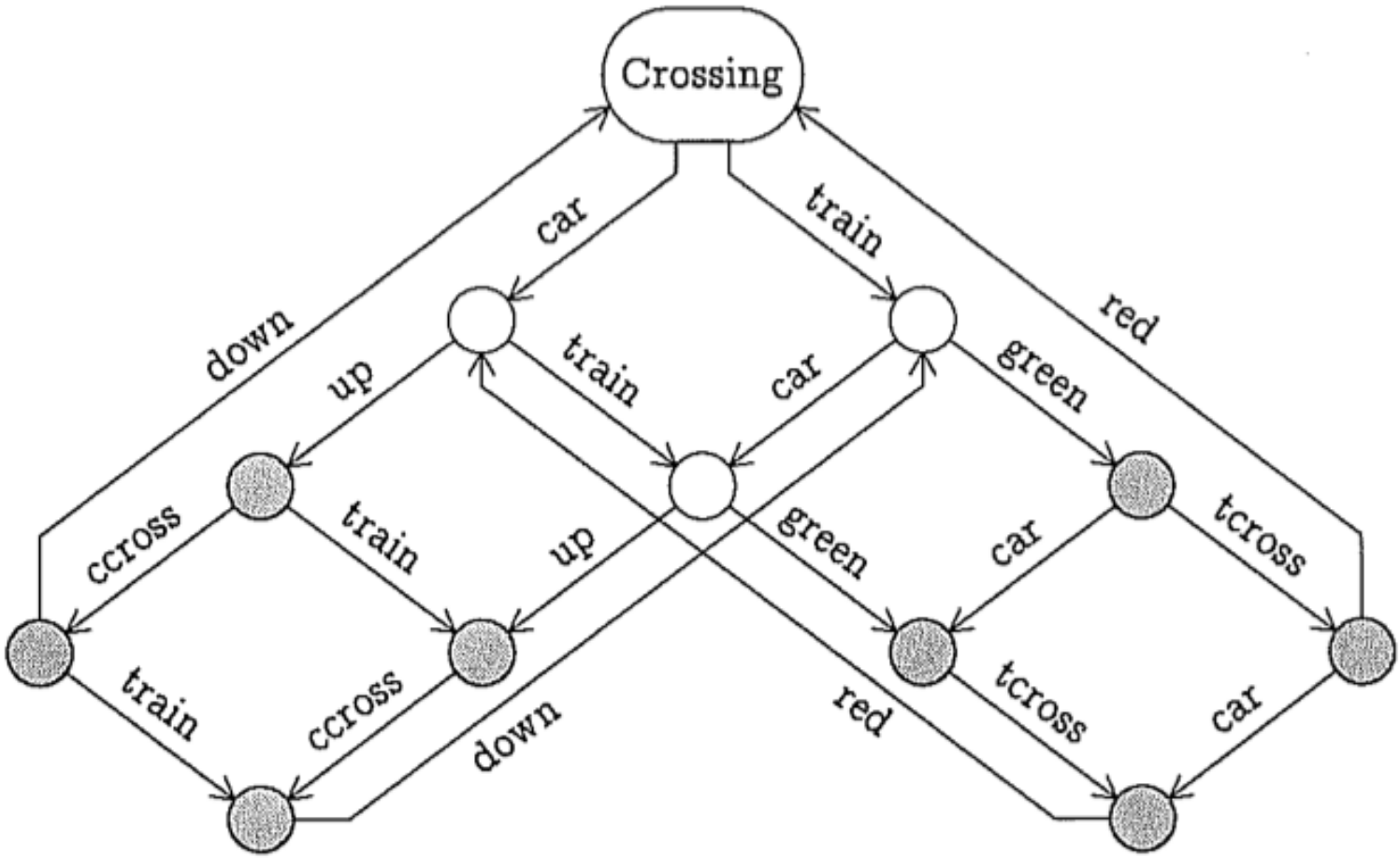
Railway crossing processes

			actions to be synchronized
$Road$	$=$	$car.up.ccross.down.Road$	$\{up, down\}$
$Rail$	$=$	$train.green.tcross.red.Rail$	$\{green, red\}$
$Controller$	$=$	$green.red.Controller + up.down.Controller$	$\{up, down, green, red\}$

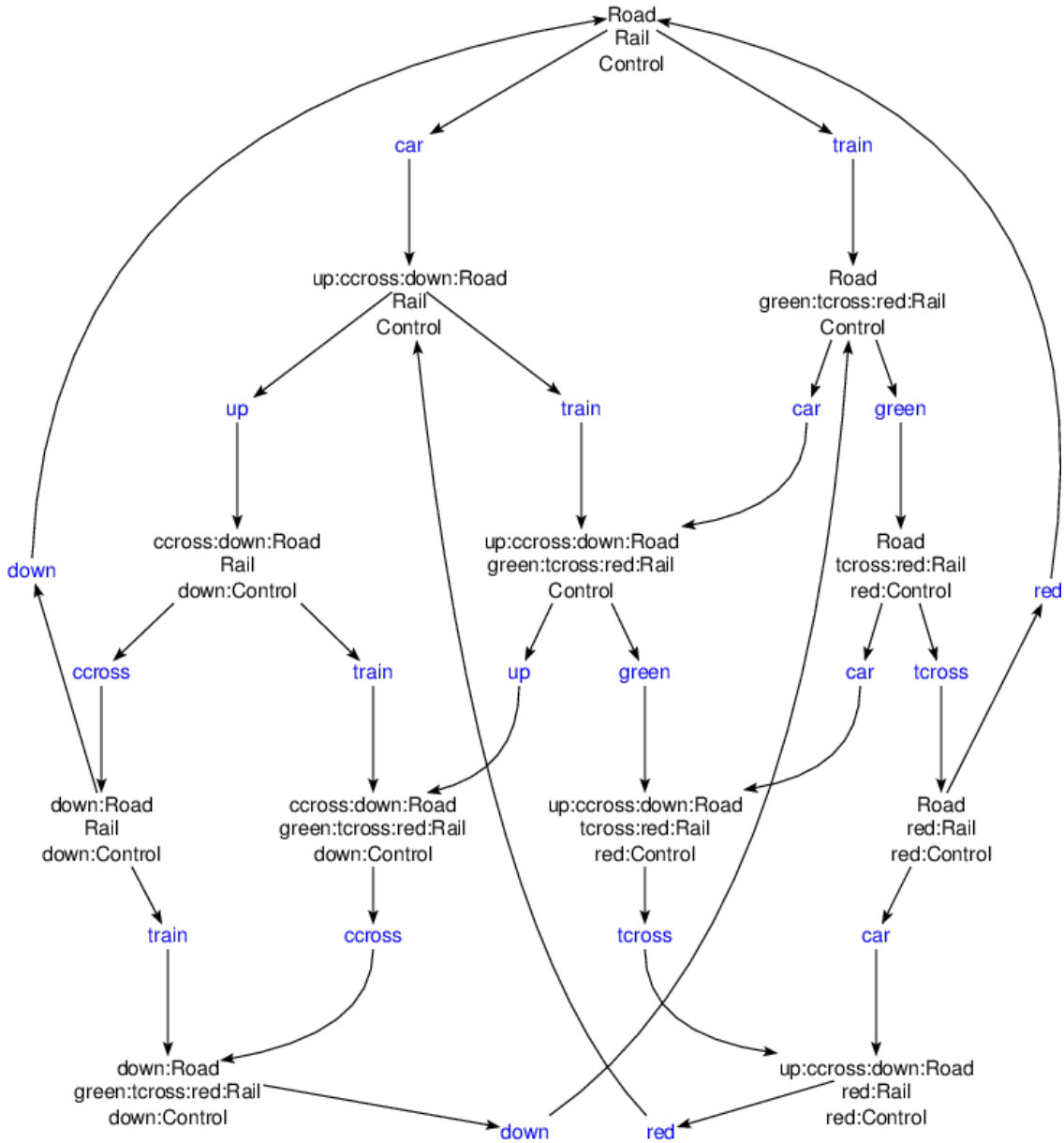
Equivalent processes must have the same sets of actions to be synchronized.

Process equivalence is the greatest relation $\sim \subseteq Proc^2$ such that for all $E, F \in Proc$,

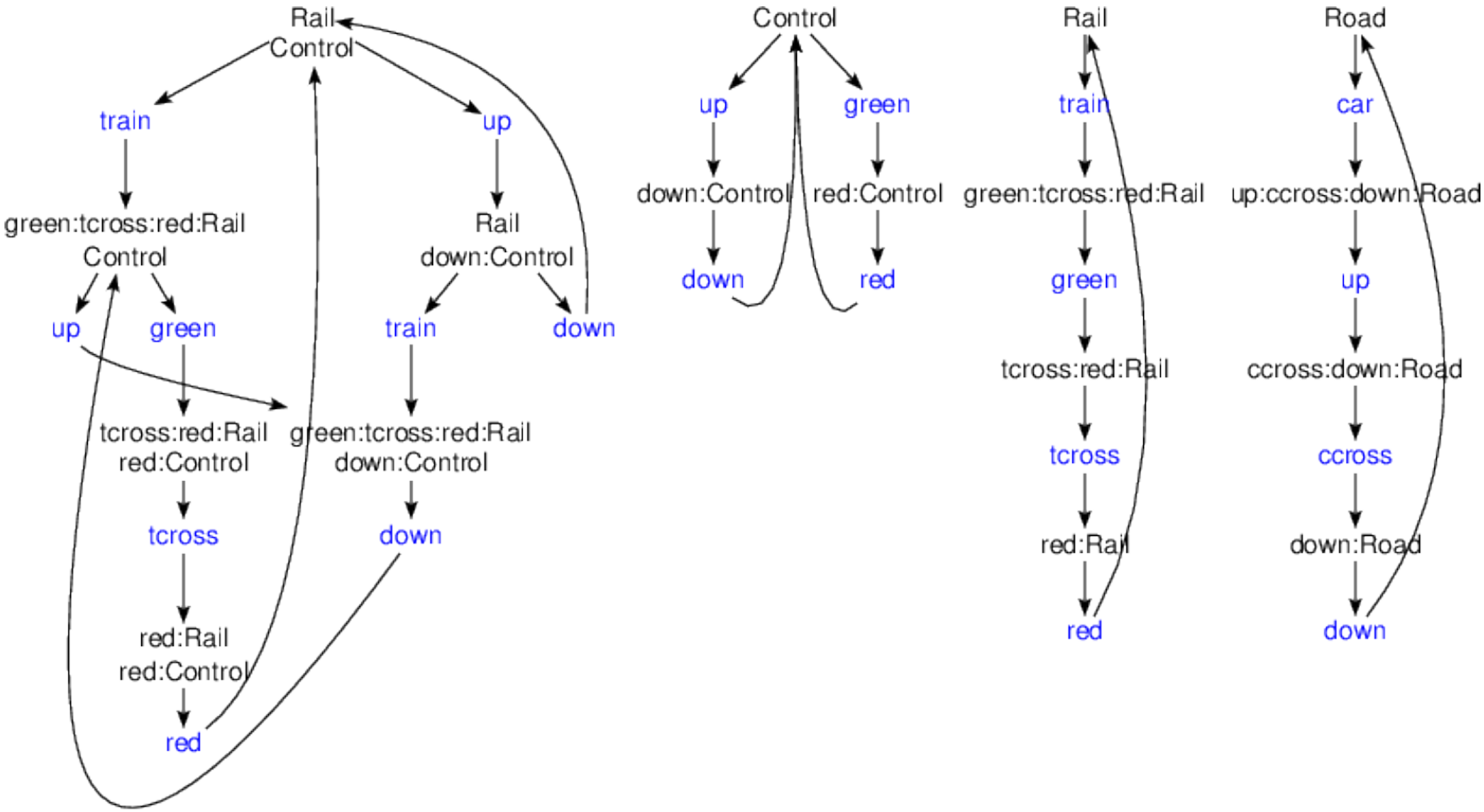
$$E \sim F \text{ implies } \forall\, a \in Act \begin{cases} \forall\, E' \in \delta(E)(a)\ \exists\, F' \in \delta(F)(a) : E' \sim F' \wedge \\ \forall\, F' \in \delta(F)(a)\ \exists\, E' \in \delta(E)(a) : E' \sim F'. \end{cases}$$



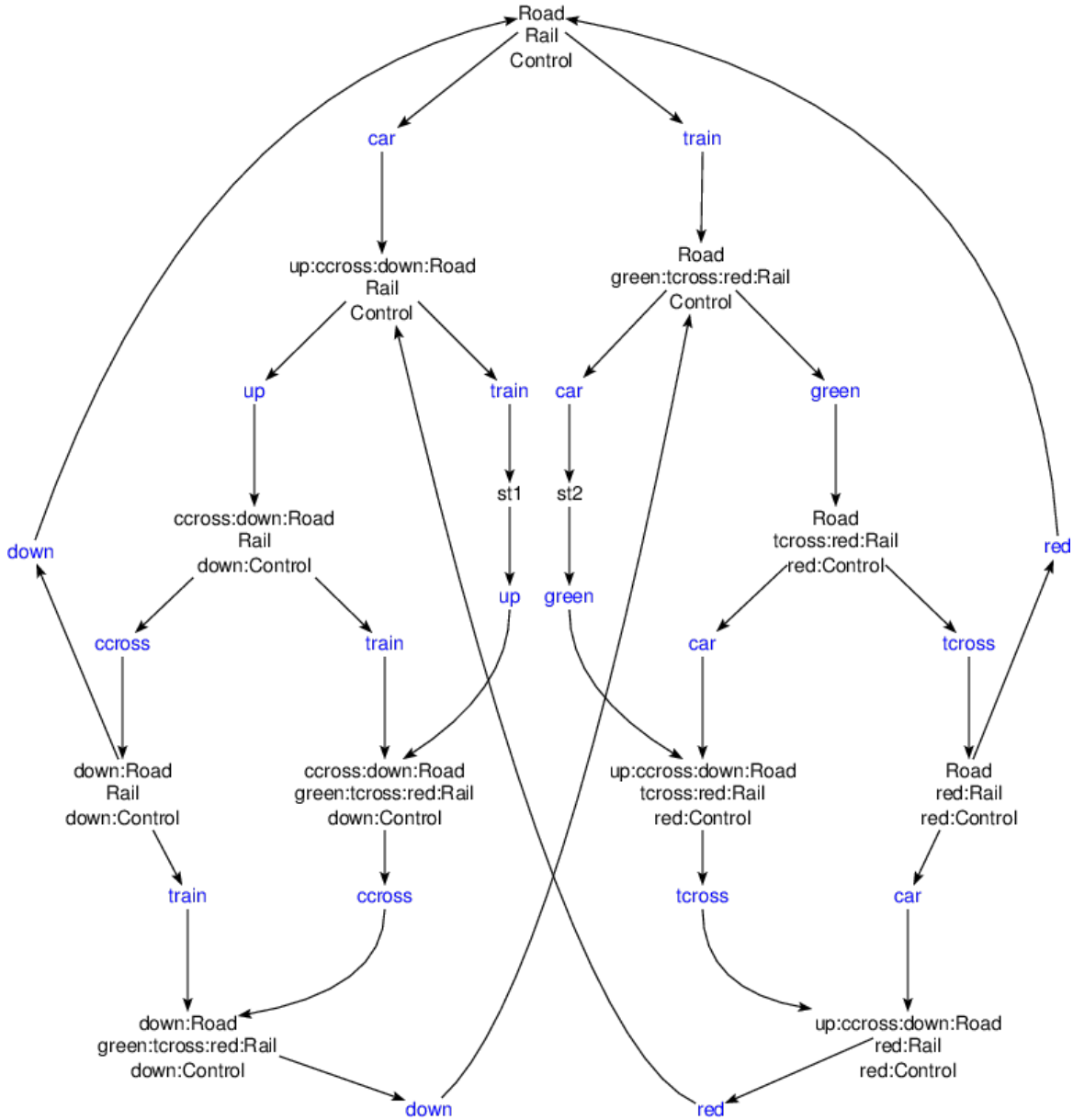
LTS of the Crossing process ([7], figure on page 365)


Road
Rail
Control
car
train
up:ccross:down:Road
Rail
Control
Road
green:tcross:red:Rail
Control
up
train
car
green
ccross:down:Road
Rail
down:Control
up:ccross:down:Road
green:tcross:red:Rail
Control
Road
tcross:red:Rail
red:Control
down
red
ccross
train
up
green
car
tcross
down:Road
Rail
down:Control
ccross:down:Road
green:tcross:red:Rail
down:Control
up:ccross:down:Road
tcross:red:Rail
red:Control
Road
red:Rail
red:Control
train
ccross
tcross
car
down:Road
green:tcross:red:Rail
down:Control
up:ccross:down:Road
red:Rail
red:Control
down
red

Expander2 version of the above LTS (generated by `build last Kripke model`) and `draw` has been applied (see below)



Further states and transitions needed for computing the above LTS


Road
Rail
Control
car
train
up:ccross:down:Road
Rail
Control
Road
green:tcross:red:Rail
Control
up
train
car
green
st1
st2
ccross:down:Road
Rail
down:Control
Road
tcross:red:Rail
red:Control
down
red
up
green
ccross
train
car
tcross
down:Road
Rail
down:Control
ccross:down:Road
green:tcross:red:Rail
down:Control
up:ccross:down:Road
tcross:red:Rail
red:Control
Road
red:Rail
red:Control
train
ccross
tcross
car
down:Road
green:tcross:red:Rail
down:Control
up:ccross:down:Road
red:Rail
red:Control
down
red

Refined version of the above LTS (generated by `build last Kripke model`) and `draw` has been applied (see below).
The middle state has been split into st1 and st2.

Properties

- A car may not cross at the same time as a train. (1)
- If a car arrives, it crosses eventually. (2)
- If a train arrives, it crosses eventually. (3)

(1) its satisfied by both LTS, (2) and (3) only by the refined one.

```
-- CROSSING

constructs:     car train up down green red ccross tcross
                Road Rail Control ||
defuncts:       Crossing draw sync
preds:          P X EF AF EG AG Car Train Green Up
fovars:         act act' E F G E' F'
hovars:         P X

axioms:
```

```
Crossing == Road||Rail||Control &
states == [Road,Rail,Control,Rail||Control,Crossing] &

labels == [car,train,up,down,green,red,ccross,tcross] &

draw == wtree $ fun(x||y||z,shelf(1)$map(text.noBrackets)[x,y,z],
                    x||y,shelf(1)$map(text.noBrackets)[x,y],
                    x,text$noBrackets$x) &

sync$Road    == [up,down] &
sync$Rail    == [green,red] &
sync$Control == [up,down,green,red] &
sync$act:E   == sync$E &
sync$E||F    == sync(E)`join`sync(F) &

(Road,car)      -> up:ccross:down:Road &
(Rail,train)    -> green:tcross:red:Rail &
(Control,up)    -> down:Control &
(Control,green) -> red:Control &
```

```
-- ((up:ccross:down:Road)||Rail||Control,train) -> st1 &
-- (Road||(green:tcross:red:Rail)||Control,car) -> st2 &

-- (st1,up)
     -> (ccross:down:Road)||(green:tcross:red:Rail)||down:Control &
-- (st2,green)
     -> (up:ccross:down:Road)||(tcross:red:Rail)||red:Control &

(act:E,act) -> E &

(act `NOTin` sync(F)
-- & (E,F,act) =/= (Road,(green:tcross:red:Rail)||Control,car)
     ==> (E||F,act) -> branch$map(fun(E',E'||F))$transL(E)$act) &

(act `NOTin` sync(E)
-- & (E,F,act) =/= (up:ccross:down:Road,Rail||Control,train)
     ==> (E||F,act) -> branch$map(fun(F',E||F'))$transL(F)$act) &

(E||F,act) -> branch$map(fun((E',F'),E'||F'))
                    $prodL[transL(E)$act,transL(F)$act]
```

```
-- branching-time fixpoints

(EF$P <==> MU X.(P\/EX$X))&                    -- finally on some path
(AF$P <==> MU X.(P\/(AX(X)/\EX$true))) &       -- finally on each path
(EG$P <==> NU X.(P/\(EX(X)\/AX$false))) &      -- generally on some path
(AG$P <==> NU X.(P/\AX$X))                     -- generally on each path

conjects:

Crossing `sats` AG$(ccross#false)\/tcross#false &       -- (1)
Crossing `sats` AG$car#AF$ccross<>true &                -- (2)
Crossing `sats` AG$train#AF$tcross<>true                -- (3)
```

Mutual exclusion

$idle(i)$	Process i is neither waiting nor in the critical section.
$wait(i)$	Process i asks for entering the critical section.
$crit(i)$	Process i is in the critical section.

safety	There is at most one process in the critical section. $\neg(crit(1) \wedge crit(2))$
liveness	If a process asks for entering the critical section, it will enter it sometime. $wait(i) \;\Rightarrow\; AF(crit(i)),\ i = 1, 2$
non-blocking	Every process may ask for entering the critical section. $idle(i) \;\Rightarrow\; EX(wait(i)),\ i = 1, 2$
no strict sequencing	Every process that has left the critical section may enter it again before another process enters it. $EF(crit(i) \wedge (crit(i) EU (\neg crit(i) \wedge (\neg crit(j) EU crit(i)))))$, $i = 1, 2,\ j = 1, 2,\ i \neq j$

In Fig. 11 below, $idle(i)$, $wait(i)$ and $crit(i)$ are denoted by n_i, t_i and c_i, respectively.

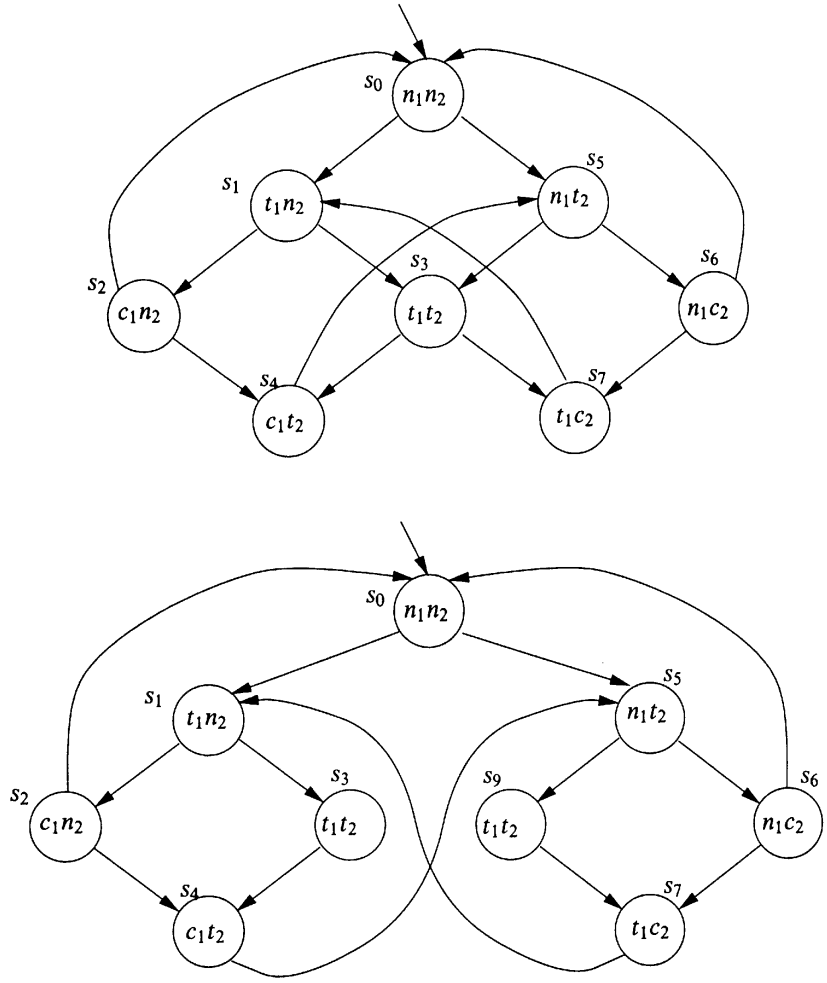


Fig. 11. *Kripke models from [5], Example 3.3.1.*
The first one violates the liveness condition because it does not consider the order in which the processes started waiting.

Generalization to an arbitrary number of processes

The number must be entered into the entry field before calling *specification > build Kripke model*.

```
-- mutex

specs:          modal
constructs:     idle wait crit
preds:          live nonBlock noSeq crit' crits Idle Wait Crit Live NonBlock NoSeq
                Crit' Crits  check
defuncts:       procs start pairs diffpairs pair drawI drawL drawK drawO drawS drawC
fovars:         st i c is ws cs k n

axioms:

states == [start] &
start == (procs,[],[]) &
atoms == map($)$pairs &
pairs == prodL[[idle,wait,crit],procs] &

(is =/= [] ==> (is,ws,cs) -> branch$map(fun(i,(is-[i],i:ws,cs)))$is) &
                                                                  -- i waits
(ws =/= [] ==> (is,ws,[]) -> (is,init(ws),[last(ws)])) &          -- last(ws) enters
(is,ws,[c]) -> (insert(c)(is),ws,[]) &                            -- c leaves
```

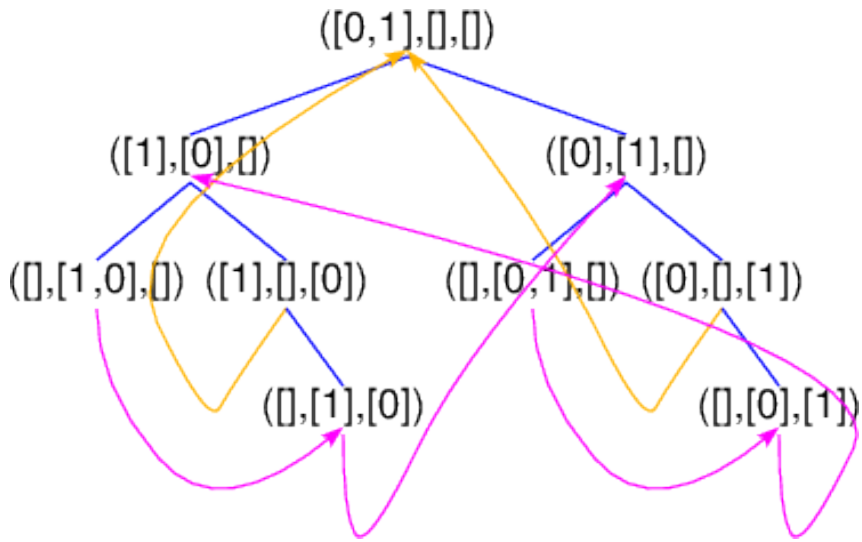
```
-- 2 processes yield 9 states
-- 3 processes yield 31 states
-- 4 processes yield 129 states
-- 5 processes yield 651 states

idle$i -> valid$Idle$i &
wait$i -> valid$Wait$i &
crit$i -> valid$Crit$i &

(Idle(i)(is,ws,cs) <==> i `in` is) &
(Wait(i)(is,ws,cs) <==> i `in` ws) &
(Crit(i)(is,ws,cs) <==> i `in` cs) &

-- modal operator simplification

(live$i <==> (wait$i) `then` AF$crit$i) &

(nonBlock$i <==> (idle$i) `then` EX$wait$i) &

(noSeq$i <==> EF $ (crit$i) /\ ((crit$i) `EU`
                                ((not$crit$i) /\ ((crit'$i) `EU` crit$i)))) &

(crit'$i <==> all(not.crit)$procs-[i]) &

(crits <==> map(crit)$procs) &
```

```
-- modal operator simplification outside eval

(Live$i <==> (Wait$i) `then` AF$crit$i) &

(NonBlock$i <==> (Idle$i) `then` EX$Wait$i) &

(NoSeq$i <==> EF $ (Crit$i) /\ ((Crit$i) `EU`
                                ((not$Crit$i) /\ ((Crit'$i) `EU` Crit$i)))) &

(Crit'$i <==> all(not.Crit)$procs-[i]) &

(Crits <==> map(Crit)$procs) &

-- widget interpreters

drawI == wtree $ fun(st,color(index(st,states),length$states)$circ$11) &

drawL == wtree(2,fun((st,k,n),color(k,n)$circ$11)) &

drawK == wtree $ fun(st,matrix$filter(check$st)$pairs) &

(check(is,ws,cs)(idle,i) <==> i `in` is) &
(check(is,ws,cs)(wait,i) <==> i `in` ws) &
(check(is,ws,cs)(crit,i) <==> i `in` cs) &
```

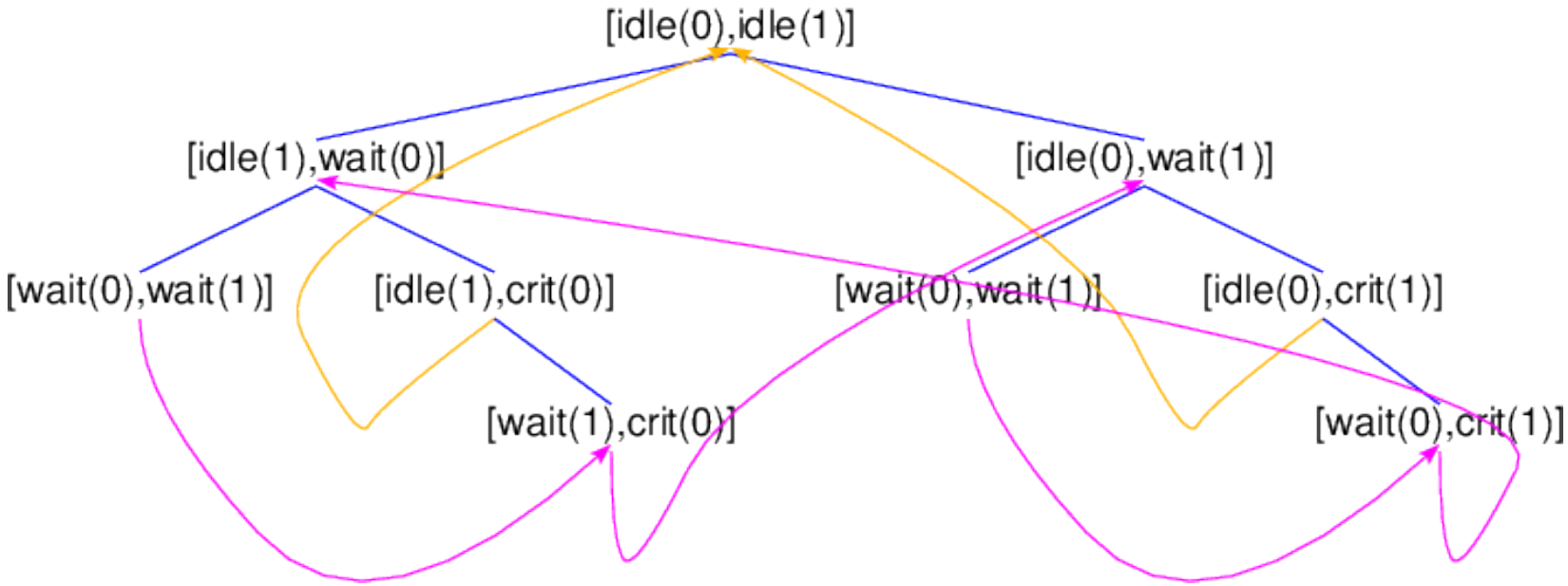
```
drawO == wtree $ fun(st,matrix$map(pair)$st) &

pair(idle$i) == (idle,i) &
pair(wait$i) == (wait,i) &
pair(crit$i) == (crit,i) &

drawS == wtree $ fun(sat$st,frame$text$st) &

drawC == wtree $ fun(sat$st,green$circ$11,st,red$circ$11) &

diffpairs == filter(=/=)$prodL$[procs,procs]


conjects:

(Crit$1)$start &                                          --> False
(EF$Crit$1)$start &                                       --> True
(or$Crits)$start &                                        --> False

(Live$1)$start &                                          --> True
(NonBlock$1)$start &                                      --> True
(NoSeq$1)$start &                                         --> True ?
(EF$or$Crits)$start &                                     --> True
```

```
(EF$and$Crits)$start &                                          --> False ?
(and$map(EF.Crit)$procs)$start &                                --> True
(or$map(rel((x,y),EF$Crit(x)/\Crit$y))$diffpairs)$start &       --> False ?
(and$map(rel((x,y),EF$Crit(x)\/Crit$y))$diffpairs)$start &      --> True

(and$map(Live)$procs)$start &                                   --> True
(and$map(NonBlock)$procs)$start &                               --> True
(and$map(NoSeq)$procs)$start &                                  --> True ?
(and$map($)$prodL[[Live,NonBlock,NoSeq],procs])$start           --> True ?

terms:

eval(or$crits) <+>              --> [([1],[],[0]),([],[1],[0]),([0],[],[1]),
                                --   ([],[0],[1])]
eval(live$1) <+>                                                --> states
eval(nonBlock$1) <+>                                            --> states
eval(noSeq$1) <+>                                               --> states

eval(EF$or$crits) <+>                                           --> states
eval(EF$and$crits) <+>                                          --> []
eval(and$map(EF.crit)$procs) <+>                                --> states
eval(or$map(rel((x,y),EF$crit(x)/\crit$y))$diffpairs) <+>       --> []
eval(and$map(rel((x,y),EF$crit(x)\/crit$y))$diffpairs) <+>      --> states

eval(and$map(live)$procs) <+>                                   --> states
```

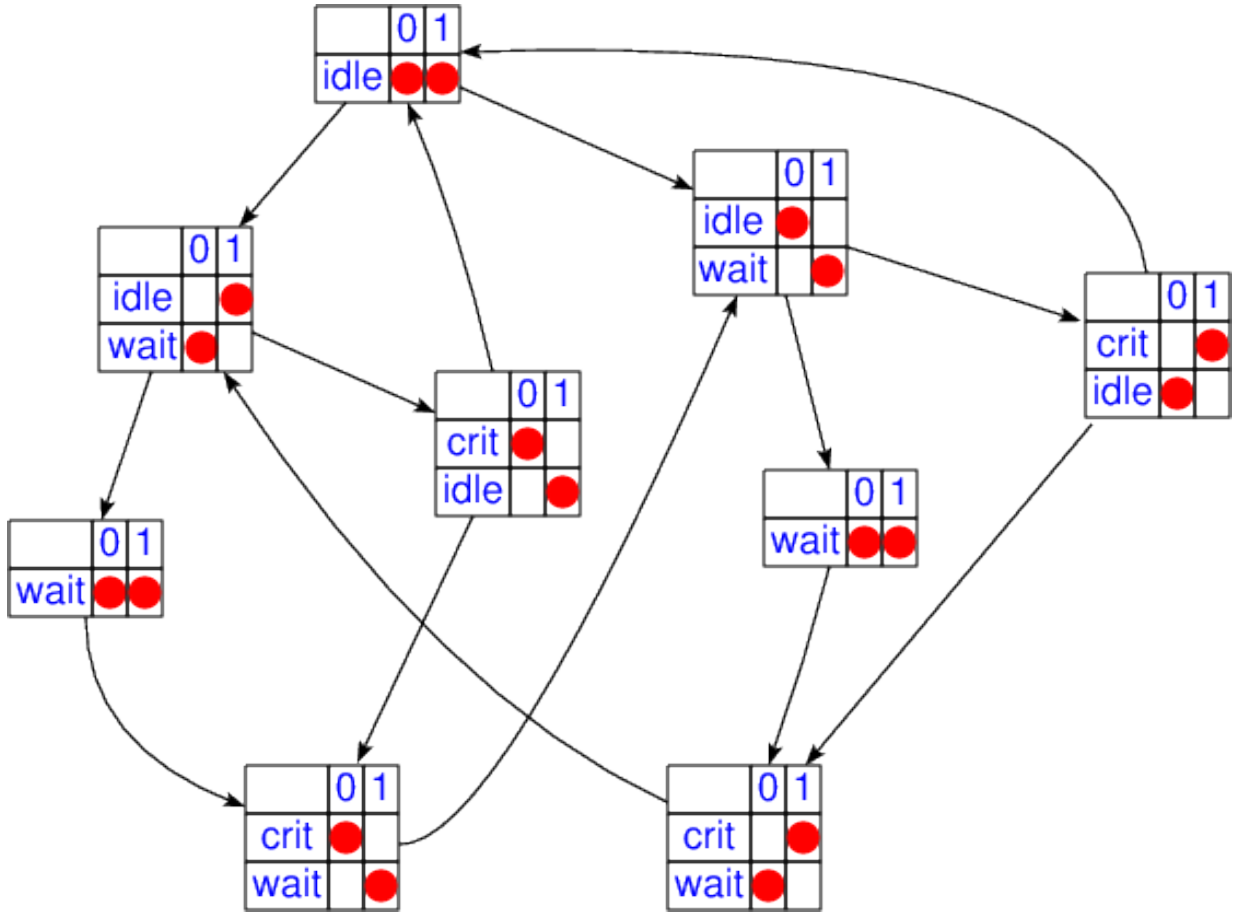
```
eval(and$map(nonBlock)$procs) <+>                                  --> states
eval(and$map(noSeq)$procs) <+>                                     --> states
eval(and$map($)$prodL[[live,nonBlock,noSeq],procs]) <+>            --> states

eval((idle$1)`EU`wait$1) <+>    --> [([0],[1],[]),([],[1,0],[]),([],[0,1],[]),
                                --    ([],[1],[0]),([0,1],[],[]),([1],[0],[]),
                                --    ([1],[],[0])]
eval((idle$1)`EU`((wait$1)`EU`crit$1)) <+>                         --> states
eval(((idle$1)`EU`(wait$1))`EU`crit$1)                             --> states
```



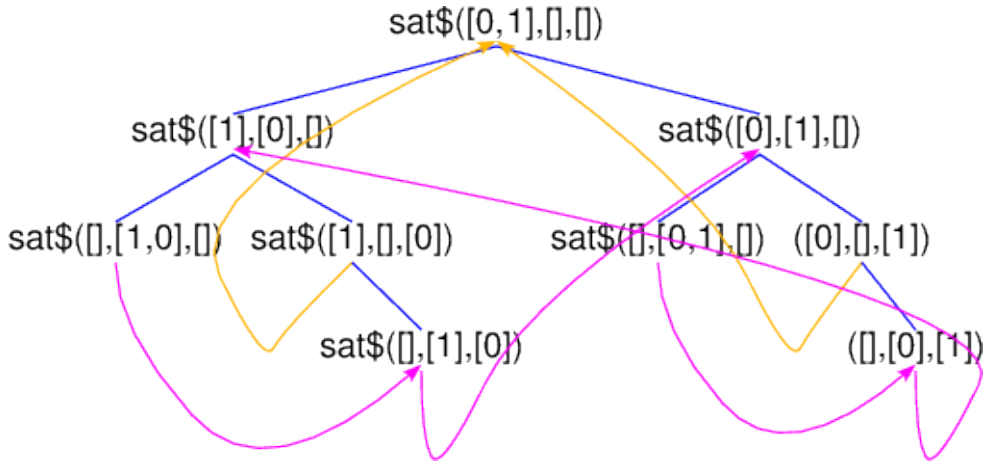
Transition graph of `mutex` *for 2 processes*



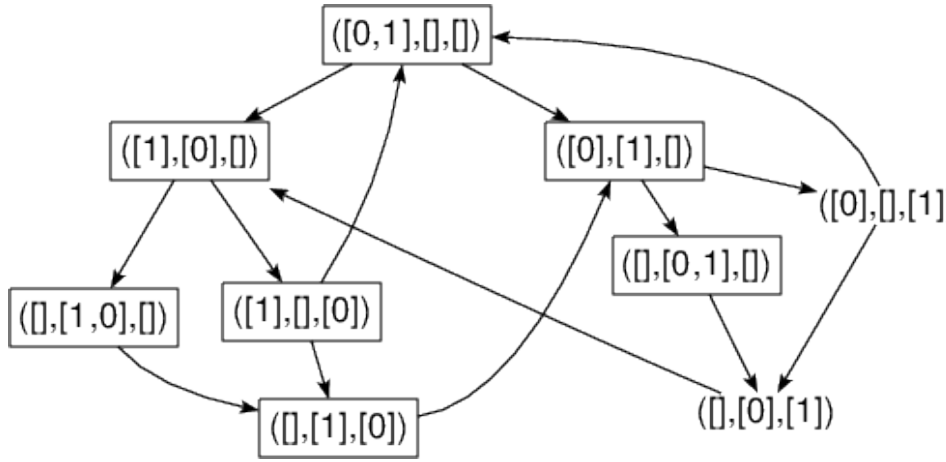
Output graph of `mutex` *for 2 processes*



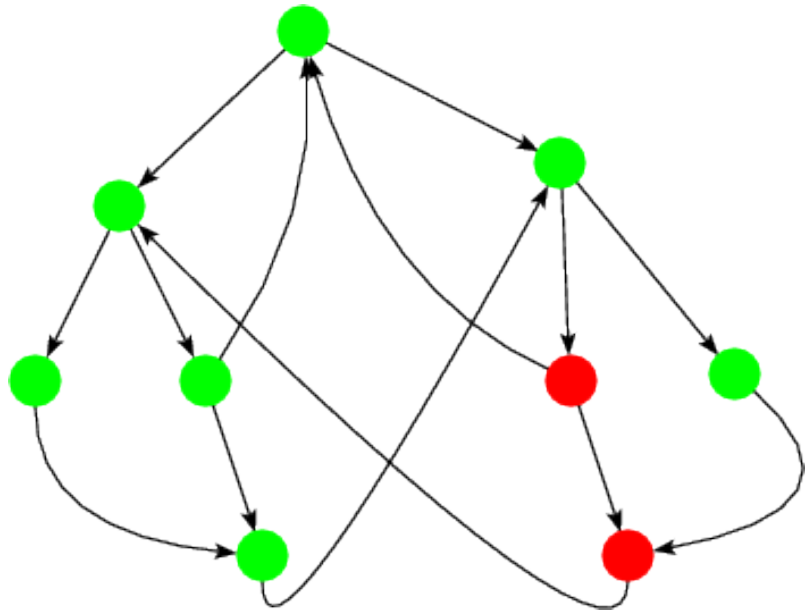
Transition graph of `mutex` *for 2 processes after application of* `drawK`



`evalG(idle(1)‘EU‘wait(1)` *for 2 processes*



Result of `evalG(idle(1)‘EU‘wait(1)` *for 2 processes after application of* `drawS`



Result of `evalG(idle(1)‘EU‘wait(1)` *for 2 processes after application of* `drawC`

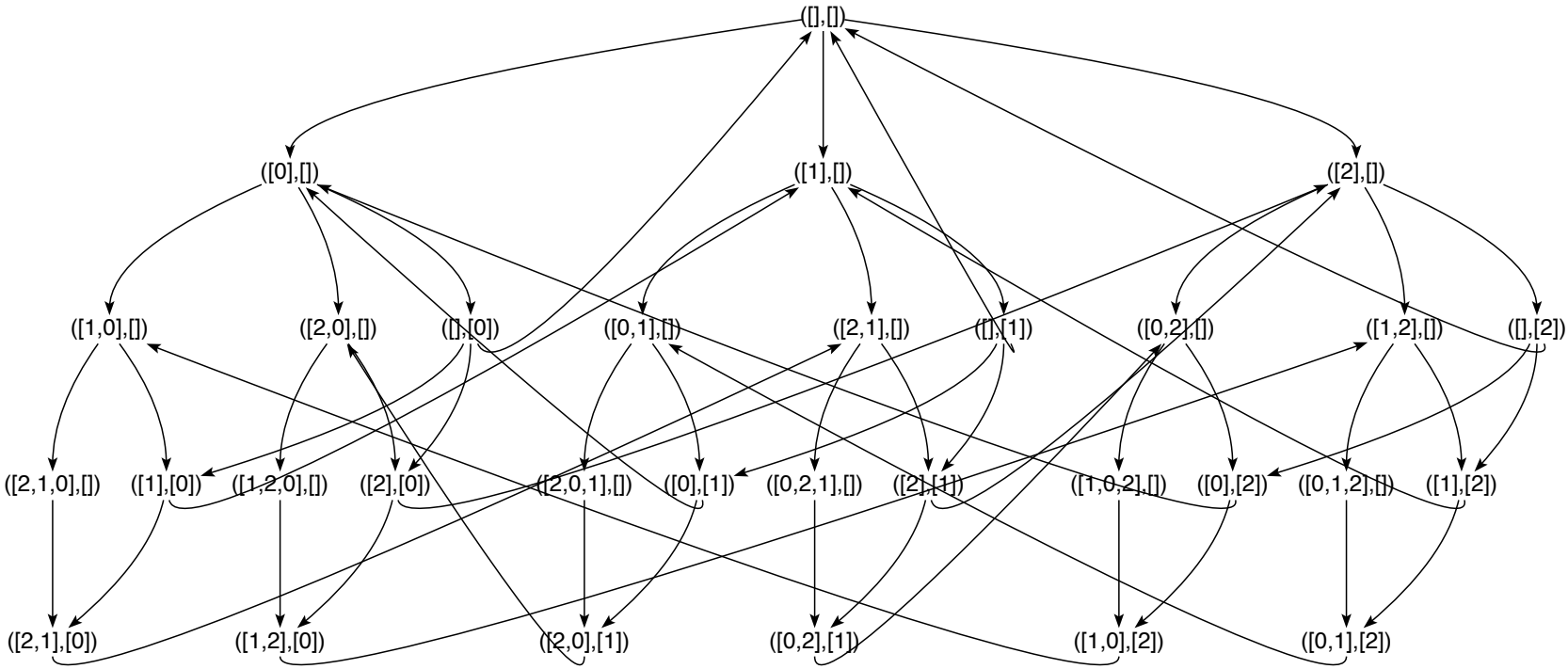


Fig. 12.
Transition graph of `mutex` *for 3 processes*

```
-- MUTEXco

specs:          modal
preds:          Idle Wait Crit enabled safe noSeq
copreds:        others
constructs:     c
defuncts:       request enter leave posi maxwait weight
fovars:         xs ys xs' ys'

axioms:
(st >> st' <==> weight(st) > weight(st')) &
weight(xs,ys) == (length(xs)-posi(xs++ys),maxwait-length(xs),length$ys) &
(st -> st' <=== st' = f$st & enabled(f)$st) &

(enabled(request$x)(xs,ys) <=== Idle(x)(xs,ys) & maxwait > length$xs) &
enabled(enter)(x:xs,[]) &
enabled(leave)(xs,[x]) &

request(x)(xs,ys) == (x:xs,ys) &
enter(xs,ys) == (init$xs,[last$xs]) &
leave(xs,ys) == (xs,[]) &

(Wait(x)(xs,ys) <==> x `in` xs) &
(Crit(x)(xs,ys) <==> x `in` ys) &
(Idle(x)(xs,ys) <==> x `NOTin` xs & x `NOTin` ys) &
```

```
safe(xs,[]) & safe(xs,[x]) &
(others(P)(x)$st ===> (x =/= y ==> P(y)$st)) &
(noSeq$x <==> EF $ Crit(x) /\ (Crit(x) `EU`
                                (not(Crit$x) /\
                                 (others(not.Crit)(x) `EU` Crit$x))))

theorems:
((xs,ys) >> st <=== c `in` xs & (xs,ys) -> st) &        -- transDesc
(x `in` xs' | x `in` ys'
               <=== x `in` xs & (xs,ys) -> (xs',ys')) & -- transIn
(x `in` xs++ys <=== x `in` xs) &
(posi$x:s = suc$posi$s <=== c `in` s) &
(posi$s++s' = posi$s <=== c `in` s) &
(True ===> Any xs ys: st = (xs,ys)) &
(x `in` init$y:s | x = last$y:s <=== x = y | x `in` s)

conjects:
(safe$st ==> AG(safe)$st) &     -- AGsafe: proof by coinduction on AG
(Idle(x)(xs,ys) & length$xs < maxwait ==> EX(Wait$x)(xs,ys)) &
                                -- EXwait: proof by resolution on EX
(c `in` xs | c `in` ys ==> AF(Crit$c)(xs,ys))
                                -- AFcrit: proof by Noetherian induction wrt >>
```

Kann `c ‘in‘ ys` weggelassen werden?

Beweis von `AGsafe`*:*

```
safe(st) ==> AG(safe)$st

Adding

  (AG0(z0)$st <=== safe(st) & z0 = safe)
& (notAG0(z0)$st ===> Not(safe(st)) | z0 =/= safe)

to the axioms and applying coinduction wrt

  (AG(P)$st ===> P(st) & AX(AG(P))$st)

at position [] of the preceding formula leads to

All st:(safe(st) ==> AX(AG0(safe))$st)

The reducts have been simplified.

Narrowing the preceding formula (4 steps) leads to 3 factors.
The current factor is given by

All xs x0:(x0 `NOTin` xs & maxwait > length(xs) ==> AG0(safe)(x0:xs,[]))

The reducts have been simplified.
```

```
Narrowing the preceding factors (4 steps) leads to a single formula,
which is given by

All x xs:(AX(AGO(safe))(xs,[x]))

The reducts have been simplified.

Narrowing the preceding formula (4 steps) leads to 2 factors.
The current factor is given by

All x xs x1:
 (x1 `NOTin` xs & x1 =/= x & maxwait > length(xs) ==> safe(x1:xs,[x]))

The reducts have been simplified.

Narrowing the preceding factors (3 steps) leads to a single formula,
which is given by

True

The reducts have been simplified.

Number of proof steps: 5
```

Beweis von `EXwait`*:*

```
Idle(x)(xs,ys) & length(xs) < maxwait ==> EX(Wait(x))(xs,ys)

Narrowing the preceding formula (1 step) leads to

x `NOTin` xs & x `NOTin` ys & length(xs) < maxwait ==>
Any st':((xs,ys) -> st' & Wait(x)$st')

The reducts have been simplified.

Narrowing the preceding formula (1 step) leads to

x `NOTin` xs & x `NOTin` ys & length(xs) < maxwait ==>
Any f:(enabled(f)(xs,ys) & Wait(x)$f(xs,ys))

The reducts have been simplified.

Narrowing the preceding formula (1 step) leads to

True

The reducts have been simplified.

Number of proof steps: 3
```

Beweis von `AFcrit`*:*

```
c `in` xs | c `in` ys ==> AF(Crit(c))(xs,ys)

Selecting induction variables at positions [1,1,0],[1,1,1]
of the preceding formula leads to

c `in` !xs | c `in` !ys ==> AF(Crit(c))(!xs,!ys)

Narrowing the preceding formula (2 steps) leads to

c `in` !xs | c `in` !ys ==>
Crit(c)(!xs,!ys) | All st':((!xs,!ys) -> st' ==> AF(Crit(c))$st')

Applying the induction hypothesis

AF(Crit(c))(xs,ys) <=== (c `in` xs | c `in` ys) & (!xs,!ys) >> (xs,ys)

at position [1,1,0,1] of the preceding formula leads to

c `in` !xs | c `in` !ys ==>
Crit(c)(!xs,!ys) |
All st':((!xs,!ys) -> st' ==>
         Any xs ys:((c `in` xs | c `in` ys) & (!xs,!ys) >> (xs,ys) &
                    st' = (xs,ys)))
```

```
Applying the theorem

(xs,ys) >> st <=== c `in` xs & (xs,ys) -> st

at position [1,1,0,1,0,1]
of the preceding formula leads to

c `in` !xs | c `in` !ys ==>
Crit(c)(!xs,!ys) |
All st':((!xs,!ys) -> st' ==>
         Any xs ys:((c `in` xs | c `in` ys) &
                    (c `in` !xs & (!xs,!ys) -> (xs,ys)) & st' = (xs,ys)))

Applying the theorem

x `in` xs' | x `in` ys' <=== x `in` xs & (xs,ys) -> (xs',ys')

at position [1,1,0,1,0,0]
of the preceding formula leads to

c `in` !xs | c `in` !ys ==>
Crit(c)(!xs,!ys) |
All st':((!xs,!ys) -> st' ==>
         Any xs ys:(Any xs1 ys1:
```

```
                    (c `in` xs1 & (xs1,ys1) -> (xs,ys)) &
                   (c `in` !xs & (!xs,!ys) -> (xs,ys)) & st' = (xs,ys)))
```

Substituting !xs for xs1 to the preceding formula leads to

```
c `in` !xs | c `in` !ys ==>
Crit(c)(!xs,!ys) |
All st':((!xs,!ys) -> st' ==>
         Any xs ys:(Any xs1 ys1:
                    (c `in` !xs & (!xs,ys1) -> (xs,ys)) &
                   (c `in` !xs & (!xs,!ys) -> (xs,ys)) & st' = (xs,ys)))
```

Substituting !ys for ys1 to the preceding formula leads to

```
c `in` !xs | c `in` !ys ==>
Crit(c)(!xs,!ys) |
All st':((!xs,!ys) -> st' ==>
         Any xs ys:(Any xs1 ys1:
                    (c `in` !xs & (!xs,!ys) -> (xs,ys)) &
                   (c `in` !xs & (!xs,!ys) -> (xs,ys)) & st' = (xs,ys)))
```

Simplifying the preceding formula (10 steps) leads to

```
All st':(c `in` !xs & (!xs,!ys) -> st' ==>
         Any xs ys:(c `in` !xs & (!xs,!ys) -> (xs,ys) & st' = (xs,ys)) |
```

```
         c `in` !ys)
```

Applying the theorem

```
Any xs ys:(st = (xs,ys))
```

at position [0,1,0] of the preceding formula leads to

```
All st':(c `in` !xs & (!xs,!ys) -> st' ==>
         (All st:Any xs2 ys2:
                  (st = (xs2,ys2)) ==>
          Any xs ys:(c `in` !xs & (!xs,!ys) -> (xs,ys) & st' = (xs,ys))) |
         c `in` !ys)
```

Substituting st' for st to the preceding formula leads to

```
All st':(c `in` !xs & (!xs,!ys) -> st' ==>
         (All st:Any xs2 ys2:
                  (st' = (xs2,ys2)) ==>
          Any xs ys:(c `in` !xs & (!xs,!ys) -> (xs,ys) & st' = (xs,ys))) |
         c `in` !ys)
```

Simplifying the preceding formula (15 steps) leads to

```
True
```

```
Number of proof steps: 11
```

Beweis von `transDesc`*:*

```
(xs,ys) >> st <=== c `in` xs & (xs,ys) -> st

The tree has been split.
The current formula is given by

(xs,ys) >> st <=== c `in` xs & (xs,ys) -> st

Simplifying the preceding formula (2 steps) leads to

c `in` xs & (xs,ys) -> st ==>
(length(xs)-posi(xs++ys),maxwait-length(xs),length(ys)) > weight(st)

Narrowing the preceding formula (2 steps) leads to

c `in` xs &
Any f:(st = f((xs,ys)) &
       (Any x:(Idle(x)(xs,ys) & maxwait > length(xs) & f = request(x)) |
        Any x xs0:(f = enter & xs = (x:xs0) & ys = []) |
        Any x:(f = leave & ys = [x]))) ==>
(length(xs)-posi(xs++ys),maxwait-length(xs),length(ys)) > weight(st)
```

```
Simplifying the preceding formula (11 steps) leads to 3 factors.
The current factor is given by

All x:(c `in` xs & st = (request(x)(xs,ys)) & Idle(x)(xs,ys) &
       maxwait > length(xs) ==>
       (length(xs)-posi(xs++ys),maxwait-length(xs),length(ys)) > weight(st))

Simplifying the preceding factors (11 steps) leads to the factor

All x:(st = (x:xs,ys) & c `in` xs & x `NOTin` xs & x `NOTin` ys &
       maxwait > length(xs) ==>
       (length(xs)-posi(xs++ys)) > (suc(length(xs))-posi(x:(xs++ys))) |
       True & (length(xs)-posi(xs++ys)) = (suc(length(xs))-posi(x:(xs++ys))) |
       (maxwait-length(xs)) = (maxwait-suc(length(xs))) &
       length(ys) > length(ys) &
       (length(xs)-posi(xs++ys)) = (suc(length(xs))-posi(x:(xs++ys))))

Simplifying the preceding factors (2 steps) leads to the factor

All x:(st = (x:xs,ys) & c `in` xs & x `NOTin` xs & x `NOTin` ys &
       maxwait > length(xs) ==>
       (length(xs)-posi(xs++ys)) > (suc(length(xs))-posi(x:(xs++ys))) |
       (length(xs)-posi(xs++ys)) = (suc(length(xs))-posi(x:(xs++ys))))

Applying the theorem
```

```
posi(x:s) = suc(posi(s)) <=== c `in` s

at position [0,1,1,1,1] of the preceding factors leads to the factor

All x:(st = (x:xs,ys) & c `in` xs & x `NOTin` xs & x `NOTin` ys &
       maxwait > length(xs) ==>
       (length(xs)-posi(xs++ys)) > (suc(length(xs))-posi(x:(xs++ys))) |
       (length(xs)-posi(xs++ys)) = (suc(length(xs))-suc(posi(xs++ys))) &
       c `in` (xs++ys))

Applying the theorem

x `in` (xs++ys) <=== x `in` xs

at position [0,1,1,1] of the preceding factors leads to the factor

All x:(st = (x:xs,ys) & c `in` xs & x `NOTin` xs & x `NOTin` ys &
       maxwait > length(xs) ==>
       (length(xs)-posi(xs++ys)) > (suc(length(xs))-posi(x:(xs++ys))) |
       (length(xs)-posi(xs++ys)) = (suc(length(xs))-suc(posi(xs++ys))) &
       c `in` xs)

Simplifying the preceding factors (4 steps) leads to 2 factors.
The current factor is given by
```

```
All f:(Any x xs0:(f = enter & xs = (x:xs0) & ys = []) & c `in` xs &
       st = f(xs,ys) ==>
       (length(xs)-posi(xs++ys),maxwait-length(xs),length(ys)) > weight(st))
```

Simplifying the preceding factors (11 steps) leads to the factor

```
All x xs0:(c = x & st = enter(x:xs0,[]) & ys = [] & xs = (x:xs0) |
           c `in` xs0 & st = enter(x:xs0,[]) & ys = [] & xs = (x:xs0) ==>
           (suc(length(xs0))-posi((x:xs0)++[]),maxwait-length(x:xs0),0) >
           weight(st))
```

Simplifying the preceding factors (11 steps) leads to 3 factors.
The current factor is given by

```
All xs0:(st = (init(c:xs0),[last(c:xs0)]) & ys = [] & xs = (c:xs0) ==>
         (suc(length(xs0))-posi(c:xs0)) > (length(xs0)-posi(c:xs0)) |
         (suc(length(xs0))-posi(c:xs0)) = (length(xs0)-posi(c:xs0)) &
         (maxwait-suc(length(xs0)),0) > (maxwait-length(xs0),1))
```

Simplifying the preceding factors (2 steps) leads to 2 factors.
The current factor is given by

```
All x xs0:(c `in` xs0 & st = enter(x:xs0,[]) & ys = [] & xs = (x:xs0) ==>
           (suc(length(xs0))-posi(x:xs0),maxwait-suc(length(xs0)),0) >
```

```
          weight(st))
```

Simplifying the preceding factors (7 steps) leads to a single formula, which is given by

```
All f:(Any x:(f = leave & ys = [x]) & c `in` xs & st = f(xs,ys) ==>
       (length(xs)-posi(xs++ys),maxwait-length(xs),length(ys)) > weight(st))
```

Simplifying the preceding formula (11 steps) leads to

```
All x:(st = (xs,[]) & c `in` xs & ys = [x] ==>
       (length(xs)-posi(xs++[x]),maxwait-length(xs),1) >
       (length(xs)-posi(xs),maxwait-length(xs),0))
```

Simplifying the preceding formula (3 steps) leads to

```
All x:(st = (xs,[]) & c `in` xs & ys = [x] ==>
       (length(xs)-posi(xs++[x])) > (length(xs)-posi(xs)) |
       (length(xs)-posi(xs++[x])) = (length(xs)-posi(xs)))
```

Applying the theorem

```
posi(s++s') = posi(s) <=== c `in` s
```

at position [0,1,1,0,1] of the preceding formula leads to

```
All x:(st = (xs,[]) & c `in` xs & ys = [x] ==>
       (length(xs)-posi(xs++[x])) > (length(xs)-posi(xs)) |
       (length(xs)-posi(xs)) = (length(xs)-posi(xs)) & c `in` xs)

Simplifying the preceding formula (3 steps) leads to

True

Number of proof steps: 17
```

Beweis von `transDesc`*:*

```
x `in` xs' | x `in` ys' <=== x `in` xs & (xs,ys) -> (xs',ys')

Simplifying the preceding formula (1 step) leads to

x `in` xs & (xs,ys) -> (xs',ys') ==> x `in` xs' | x `in` ys'

Narrowing the preceding formula (2 steps) leads to

x `in` xs &
Any f:((xs',ys') = f((xs,ys)) &
       (Any x0:(Idle(x0)(xs,ys) & maxwait > length(xs) & f = request(x0)) |
        Any x0 xs0:
         (f = enter & xs = (x0:xs0) & ys = []) |
```

```
        Any x0:(f = leave & ys = [x0]))) ==>
x `in` xs' | x `in` ys'
```

Simplifying the preceding formula (11 steps) leads to 3 factors.
The current factor is given by

```
All x0:(x `in` xs & (xs',ys') = (request(x0)(xs,ys)) & Idle(x0)(xs,ys) &
        maxwait > length(xs) ==>
        x `in` xs' | x `in` ys')
```

Simplifying the preceding factors (10 steps) leads to 2 factors.
The current factor is given by

```
All f:(Any x0 xs0:
        (f = enter & xs = (x0:xs0) & ys = []) &
       x `in` xs & (xs',ys') = f(xs,ys) ==>
       x `in` xs' | x `in` ys')
```

Simplifying the preceding factors (11 steps) leads to 3 factors.
The current factor is given by

```
All x0 xs0:
 (x = x0 & (xs',ys') = enter(x0:xs0,[]) & ys = [] & xs = (x0:xs0) ==>
  x `in` xs' | x `in` ys')
```

```
Simplifying the preceding factors (7 steps) leads to the factor

All xs0:(ys' = [last(x:xs0)] & xs' = init(x:xs0) & ys = [] & xs = (x:xs0) ==>
         x `in` xs' | x = last(x:xs0))

Applying the theorem

x `in` init(y:s) | x = last(y:s) <=== x = y | x `in` s

at position [0,1] of the preceding factors leads to the factor

All xs0:(ys' = [last(x:xs0)] & xs' = init(x:xs0) & ys = [] & xs = (x:xs0) ==>
         (x = x | x `in` xs0) & xs' = init(x:xs0))

Simplifying the preceding factors (3 steps) leads to 2 factors.
The current factor is given by

All x0 xs0:
 (x `in` xs0 & (xs',ys') = enter(x0:xs0,[]) & ys = [] & xs = (x0:xs0) ==>
  x `in` xs' | x `in` ys')

Simplifying the preceding factors (5 steps) leads to the factor

All x0 xs0:
 (ys' = [last(x0:xs0)] & xs' = init(x0:xs0) & x `in` xs0 & ys = [] &
```

```
  xs = (x0:xs0) ==>
  x `in` xs' | x = last(x0:xs0))
```

Replacing the subtrees at position [0,1,0,1]
of the preceding factors leads to the factor

```
All x0 xs0:
 (ys' = [last(x0:xs0)] & xs' = init(x0:xs0) & x `in` xs0 & ys = [] &
  xs = (x0:xs0) ==>
  x `in` init(x0:xs0) | x = last(x0:xs0))
```

Applying the theorem

```
x `in` init(y:s) | x = last(y:s) <=== x = y | x `in` s
```

at position [0,1] of the preceding factors leads to the factor

```
All x0 xs0:
 (ys' = [last(x0:xs0)] & xs' = init(x0:xs0) & x `in` xs0 & ys = [] &
  xs = (x0:xs0) ==>
  x = x0 | x `in` xs0)
```

Simplifying the preceding factors (2 steps) leads to a single formula,
which is given by

```
All f:(Any x0:(f = leave & ys = [x0]) & x `in` xs & (xs',ys') = f(xs,ys) ==>
       x `in` xs' | x `in` ys')

Simplifying the preceding formula (11 steps) leads to

All x0:(ys' = [] & xs' = xs & x `in` xs & ys = [x0] ==> x `in` xs' | False)

Simplifying the preceding formula (1 step) leads to

All x0:(ys' = [] & xs' = xs & x `in` xs & ys = [x0] ==> x `in` xs')

Replacing the subtrees at position [0,1,1] of the preceding formula leads to

All x0:(ys' = [] & xs' = xs & x `in` xs & ys = [x0] ==> x `in` xs)

Simplifying the preceding formula (2 steps) leads to

True

Number of proof steps: 16
```

N-queens problem

```
-- queens

specs:          modal
constructs:     final
defuncts:       procs start add shift draw drawAll pic
preds:          Final safe queens loop rec select select'
fovars:         val

axioms:

  states == [start]
& start == (procs,[])
& atoms == [final]

& (s=/=[] ==> (s,val) -> branch$map(shift$s)$filter(safe$1)$map(add$val)$s)
& add(val)(x) == x:val
& (safe(n)[x] <==> True)
& (safe(n)(x:y:s) <==> x =/= y+n & y =/= x+n & safe(n+1)(x:s))
& shift(s)(x:val) == (s-[x],x:val)

& final -> valid$Final
& (Final <==> null.get0)
```

```
-- 4 queens yield 2 solutions and 15 further states
-- 5 queens yield 10 solutions and 44 further states
-- 6 queens yield 4 solutions and 149 further states
-- 7 queens yield 40 solutions and 512 further states
-- 8 queens yield 92 solutions and 1965 further states
-- 9 queens yield 352 solutions and 8042 further states
-- 10 queens yield 724 solutions and 34815 further states

& pic(s,val) == mat$zip(drop(length$s)$procs)$val

& draw == wtree$fun(st,ite(Final$st,pic$st,text$st))

& drawAll == wtree$pic

& drawT == wtree$fun(st,ite(Final$st,text$final,text$o))

-- logic programs for computing safe board valuations

& (queens(n,val) <=== loop([1..n],[],([],val)))
& (loop(s,val,st) <=== select(s,x) & safe(1)(x:val) & loop(s-[x],x:val,st))

& (queens(n,val) <=== rec([1..n],val))
& rec([],[])
& (rec(x:s,y:val) <=== select(x:s,y) & rec(x:s-[y],val) & safe(1)(y:val))
```

```
& select(x:s,x)
& (select(x:s,y) <=== select(s,y))

conjects:

-- 4 queens
  EF(Final)$start       --> True        42 parallel simplification steps
& AF(Final)$start       --> False       219 depthfirst simplification steps
& EG(Final)$start       --> False       11 parallel simplification steps
& AG(Final)$start       --> False       10 parallel simplification steps

& queens(4,val) -- derive/simplify/refute --> val=[3,1,4,2]|val=[2,4,1,3]

& queens(5,val) {- derive/simplify/refute -->
                        val=[4,2,5,3,1]|val=[3,5,2,4,1]|val=[5,3,1,4,2]|
                        val=[4,1,3,5,2]|val=[5,2,4,1,3]|val=[1,4,2,5,3]|
                        val=[2,5,3,1,4]|val=[1,3,5,2,4]|
                        val=[3,1,4,2,5]|val=[2,4,1,3,5] -}

& select([1..4],x)

& (x=[(0,2),(1,0),(2,3),(3,1)]&[(0,1),(1,3),(2,0),(3,2)]=z)

terms:
```

```
    eval$final

    --> [([],[2,0,3,1]),([],[1,3,0,2])]                          if procs = [0..3]

    --> [([],[3,1,4,2,0]),([],[2,4,1,3,0]),([],[4,2,0,3,1]),([],[3,0,2,4,1]),
    --   ([],[4,1,3,0,2]),([],[0,3,1,4,2]),([],[1,4,2,0,3]),([],[0,2,4,1,3]),
    --   ([],[2,0,3,1,4]),([],[1,3,0,2,4])]                      if procs = [0..4]
    --> [([],[5,3,1,6,4,2]),([],[4,1,5,2,6,3]),([],[3,6,2,5,1,4]),
    --   ([],[2,4,6,1,3,5])]                                     if procs = [1..6]

<+> eval$EF$final

    --> [([],[2,0,3,1]),([],[1,3,0,2]),([0,1,2,3],[]),([0,2,3],[1]),([0,1,3],[2]),
         ([0,2],[3,1]),([1,3],[0,2]),([2],[0,3,1]),([1],[3,0,2])]
                                                                 if procs = [0..3]
    --> [([],[5,3,1,6,4,2]),([],[4,1,5,2,6,3]),([],[3,6,2,5,1,4]),
    --   ([],[2,4,6,1,3,5]),
    --   ([1,2,3,4,5,6],[]),([1,3,4,5,6],[2]),([1,2,4,5,6],[3]),([1,2,3,5,6],[4]),
    --   ([1,2,3,4,6],[5]),([1,3,5,6],[4,2]),([1,2,4,5],[6,3]),([2,3,5,6],[1,4]),
    --   ([1,2,4,6],[3,5]),([1,3,5],[6,4,2]),([1,4,5],[2,6,3]),([2,3,6],[5,1,4]),
    --   ([2,4,6],[1,3,5]),([3,5],[1,6,4,2]),([1,4],[5,2,6,3]),([3,6],[2,5,1,4]),
    --   ([2,4],[6,1,3,5]),([5],[3,1,6,4,2]),([4],[1,5,2,6,3]),([3],[6,2,5,1,4]),
    --   ([2],[4,6,1,3,5])]                                      if procs = [1..6]
```

```
<+> eval$AF$final

    --> [([],[2,0,3,1]),([],[1,3,0,2]),([0,2,3],[1]),([0,1,3],[2]),([0,2],[3,1]),
         ([1,3],[0,2]),([2],[0,3,1]),([1],[3,0,2])]                if procs = [0..3]

<+> eval$EG$final

    --> [([],[2,0,3,1]),([],[1,3,0,2])]                            if procs = [0..3]

<+> eval$AG$final

    --> [([],[2,0,3,1]),([],[1,3,0,2])]                            if procs = [0..3]
```

enter a number of queens into the entry field; press *specification* > *build Kripke model*
press *graph* > *show graph of transitions* > *here*; enter *draw* into the entry field
press *tree* > *tree*; press *paint*

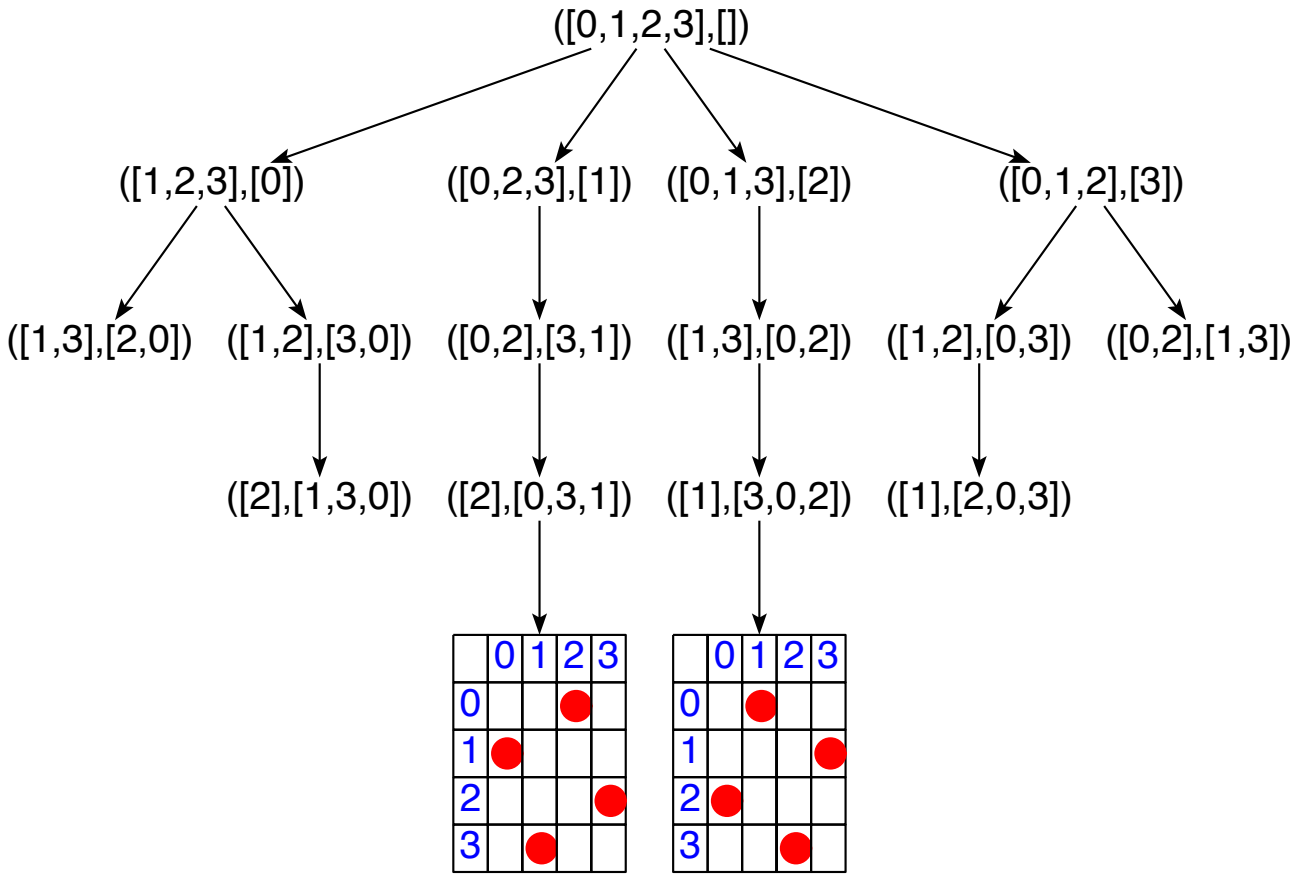
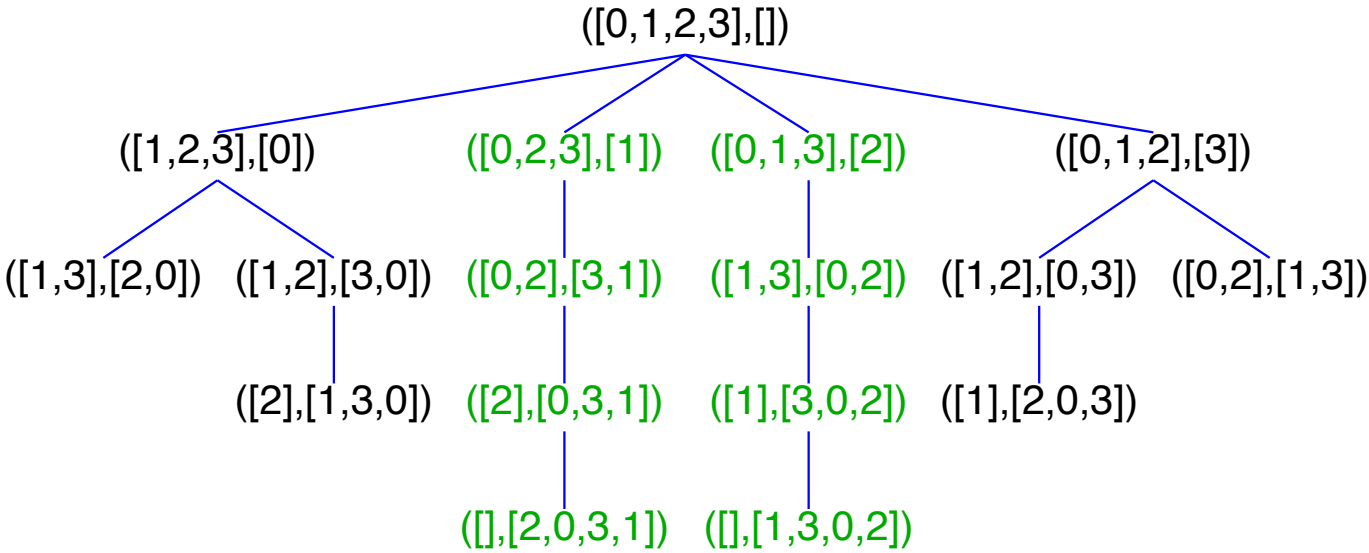


Fig. 13. *Transition graph for 4 queens*

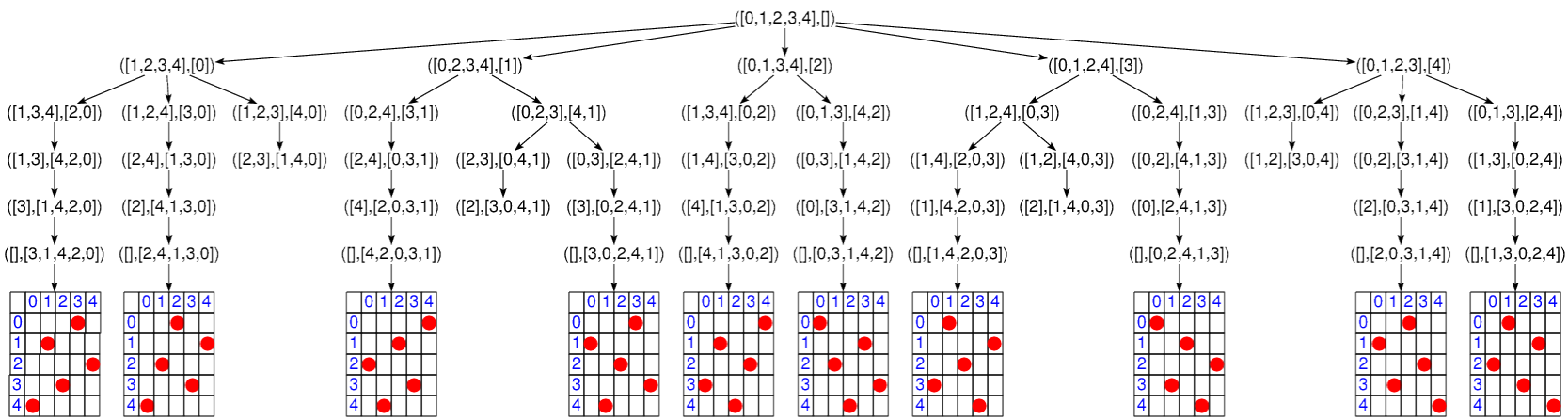
enter *evalGAFfinal* into the text field
press *parse up*; press *simplify*



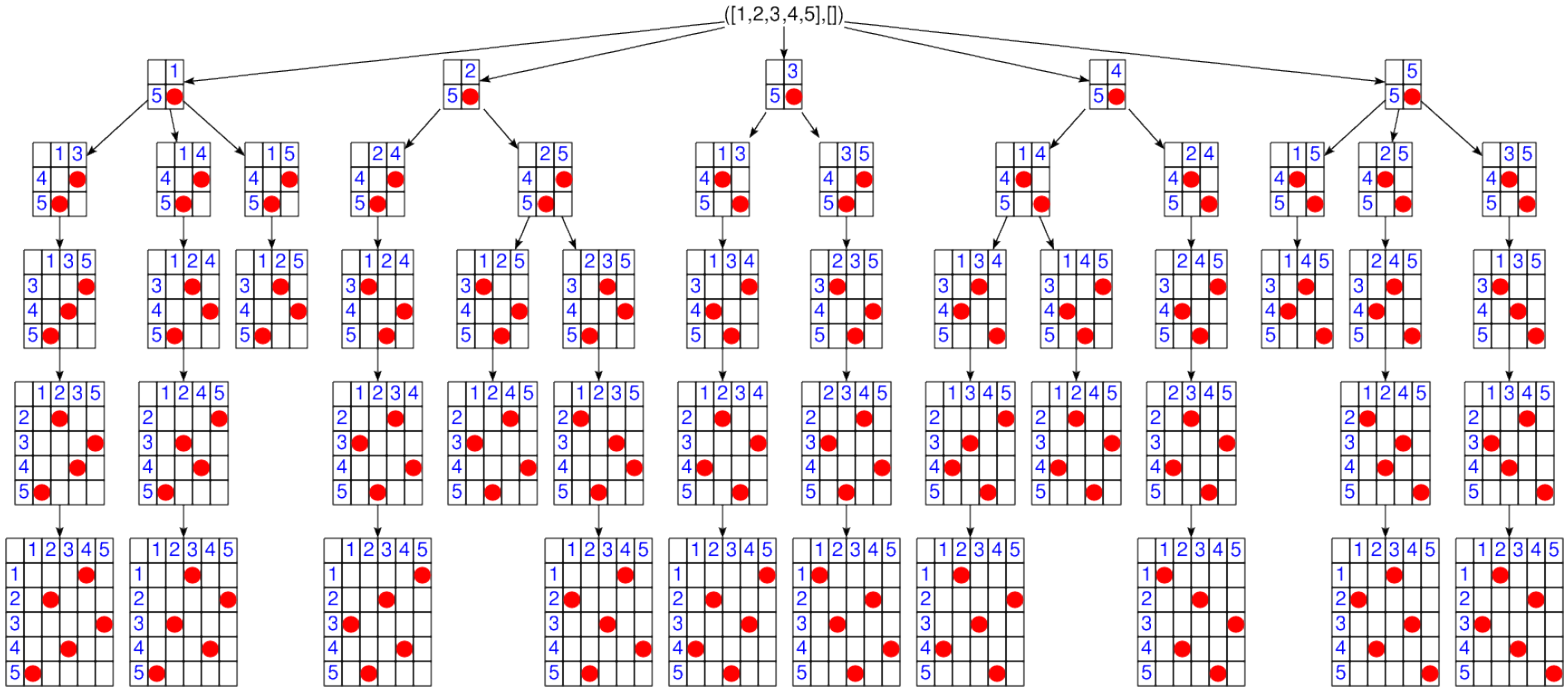
Expander2 computes Kripke models usually from a small number of axioms (here it is just a single one). This simplifies the specification, but may lead to time-consuming computations, especially if the numbers of processes (here: queens) is increased. In such cases, the transition system should be computed with the help of a faster language and then passed to Expander2, only for the purpose of reasoning about or displaying it.

For instance, the pictures below are based on the specifications *queensIO5* and *queensIO6*, respectively (see the *Examples* directory in your Expander2 package), which were generated by running a corresponding Haskell program.

Transition graphs for 5 queens

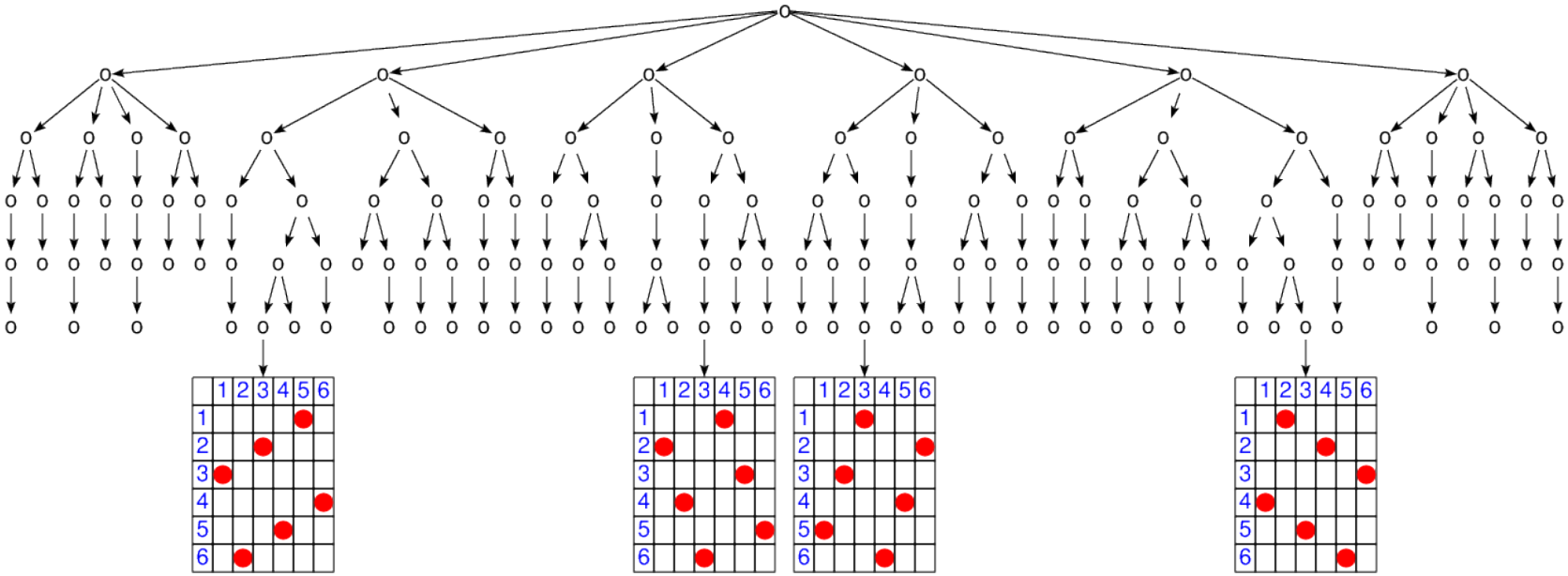


enter *drawAll* instead of *draw* into the entry field

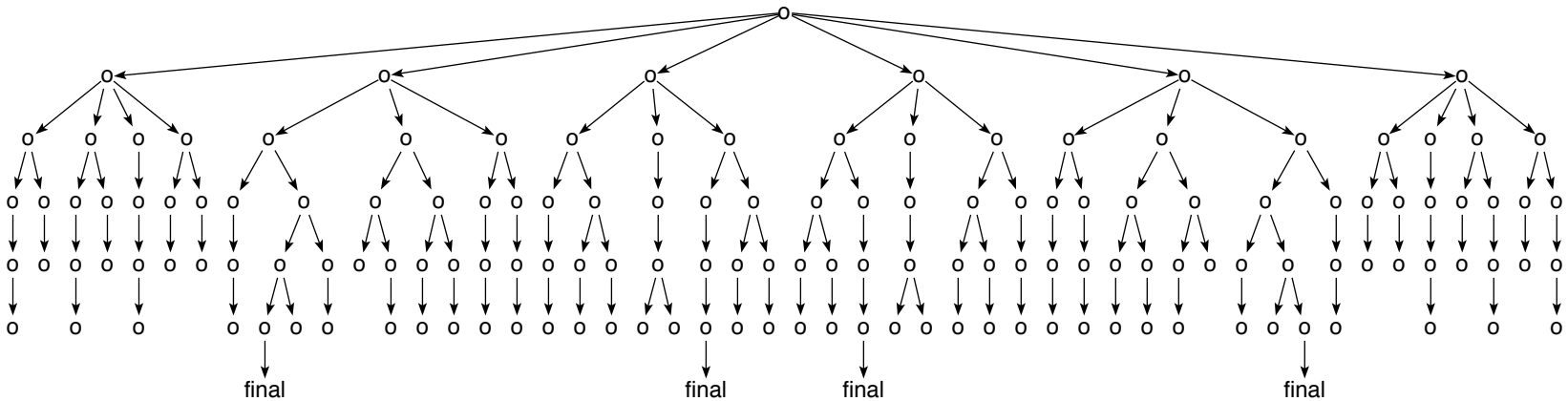


Transition graphs for 6 queens

enter *drawO* instead of *draw* into the entry field



enter *drawT* instead of *draw* into the entry field



Robots

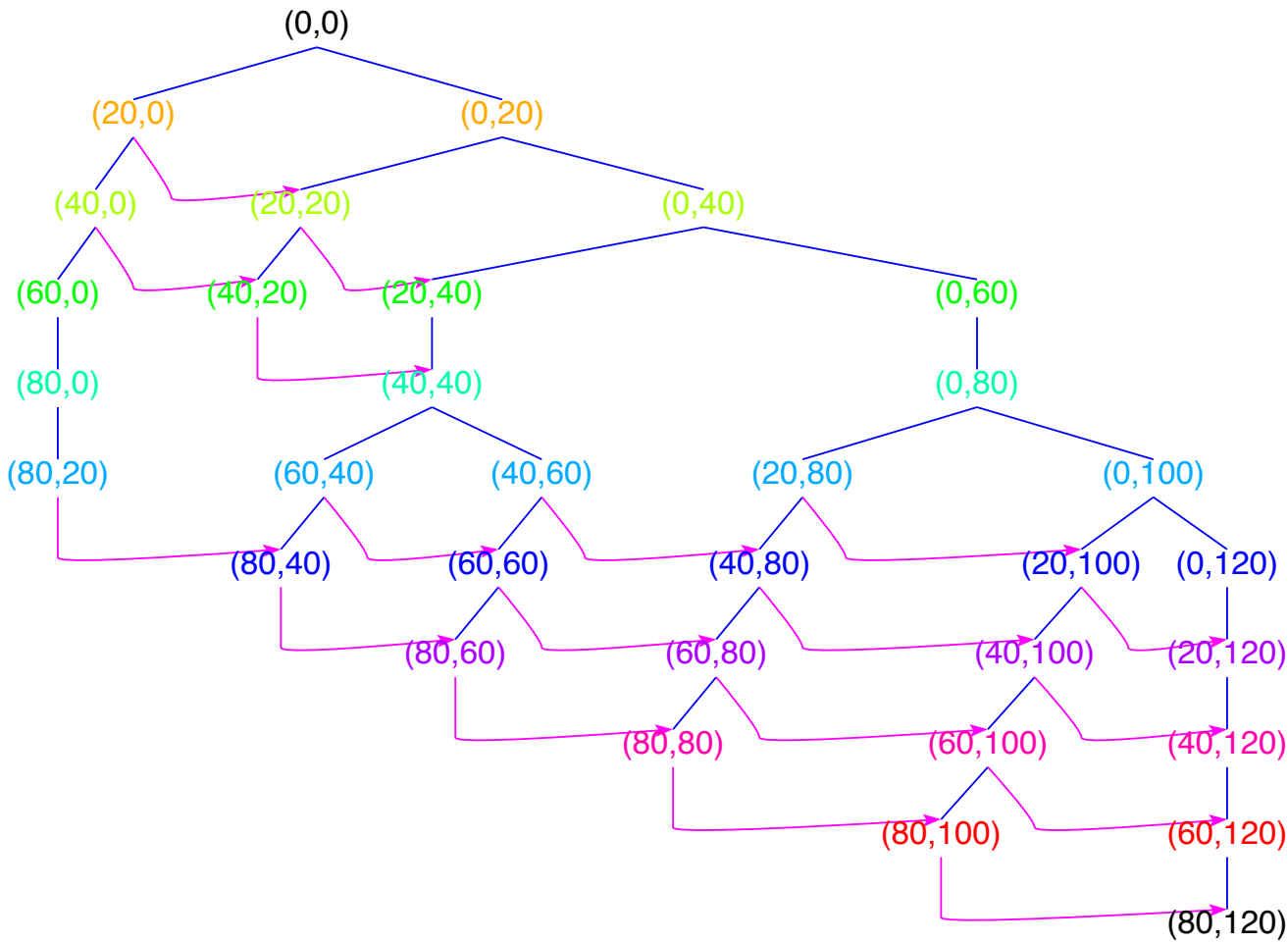
```
-- robot

specs:          modal
constructs:     final
defuncts:       start blocks blocksCode blocksPic pic
fovars:         pa

axioms:

states == [start] &
start == (0,0) &
atoms == [final] &

blocks == [(20,60),(60,20)] &

(x < 80  & (x+20,y) `NOTin` blocks ==> (x,y) -> (x+20,y)) &
(y < 120 & (x,y+20) `NOTin` blocks ==> (x,y) -> (x,y+20)) &

final -> (80,120) &

blocksCode == turt$map(fun(x,place(x)$red$circ$20))$blocks &
```

```
pic1$pa == turt[path(pa),blocksCode] &

pic2$pa == turt[path(pa),load$blocksPic]

conjects:

EF(=(80,120))$start --> True (98 depth- or breadthfirst simplification steps;
                    --        83 simplification steps in parallel with
                    --        selection > permute subtrees after 30 steps)

& set(eval$EF$final) = set$states       --> True
& (80,120)`in`succs(0,0)                --> True

terms:

path[(0,0),(20,0),(40,0),(40,20),(40,40),(40,60),(40,80),(40,100),(40,120)] <+>
                                        --> path[(0,0),(40,0),(40,120)]

map(pic1)$traces(0,0)(40,120) <+>       -- 12 traces from (0,0) to (40,120)
map(pic2)$traces(0,0)(40,120) <+>

map(pic2)$traces(0,0)(80,120) <+>       -- 106 traces from (0,0) to (80,120)

save(blocksCode,blocksPic) <+> load$blocksPic
```

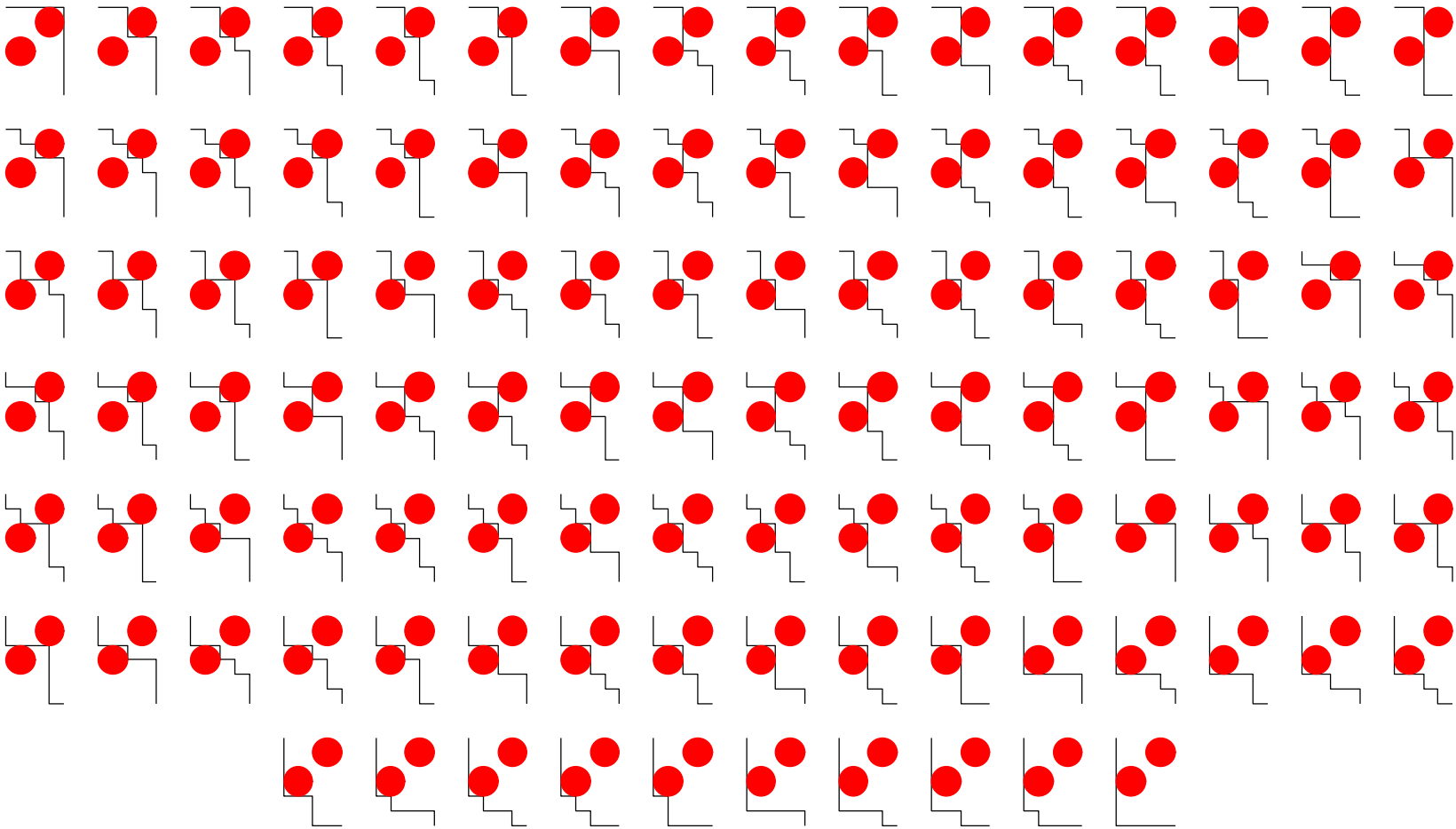
press *specification* > *build Kripke model*

press *graph* > *show graph of transitions* > *with state equivalence*

States in the same equivalence class are colored equally. Every black state is equivalent only to itself.



enter *map(pic1)$traces(0,0)(80,120)* into the text field; press *parse up*; press *simplify*
press *tree* > *widgets*; press *paint*; enter *m116* into the mode field
enter *3* into the space field; press *arrange*



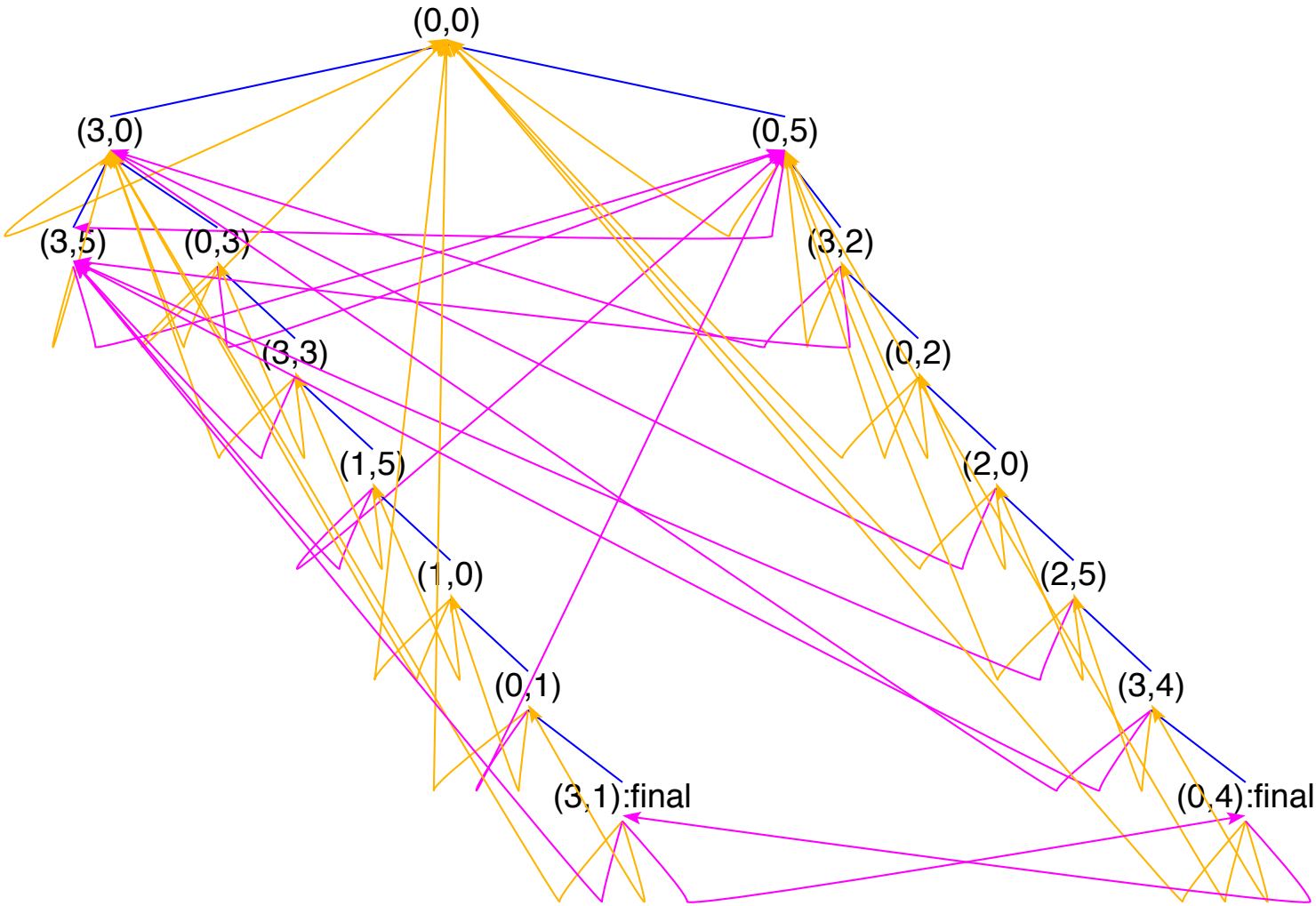
Filling problem

```
-- bottle

specs:          modal
constructs:     final
defuncts:       draw pic
preds:          Final

axioms:

  states == [(0,0)] & atoms == [final]

& (x < 3 ==> (x,y) -> (3,y))                      -- fill x
& (y < 5 ==> (x,y) -> (x,5))                      -- fill y
& (x > 0 ==> (x,y) -> (0,y))                      -- clear x
& (y > 0 ==> (x,y) -> (x,0))                      -- clear y
& (x > 0 & x+y = z & z <= 5 ==> (x,y) -> (0,z))  -- add x to y
& (y > 0 & x+y = z & z <= 3 ==> (x,y) -> (z,0))  -- add y to x
& (x < 3 & x+y = z & z > 3 ==> (x,y) -> (3,z-3)) -- fill x from y
& (y < 5 & x+y = z & z > 5 ==> (x,y) -> (z-5,5)) -- fill y from x

& final -> valid$Final
& (Final(x,y) <==> x+y=4)
```

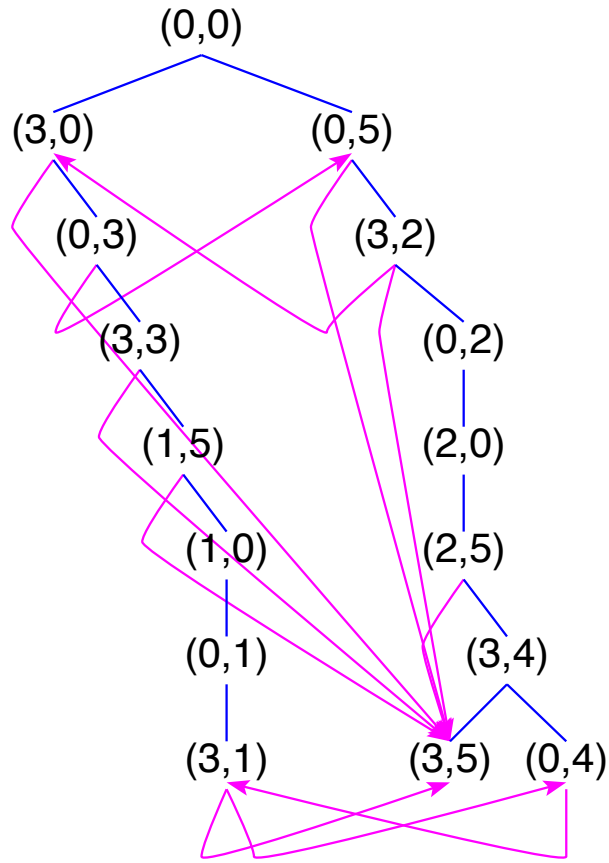
```
& draw == wtree$fun(st,ite(Final$st,pic$st,frame(3)$text$st))
& pic(x,y) == center$turt[pile(3,x),J$20,pile(5,y)]

conjects: EF(Final)(0,0)   -->  True  (69 parallel simplification steps)

terms:

map(pic)[(0,0),(3,0),(0,3),(3,3),(1,5),(1,0),(0,1),(3,1)] <+>   -- trace1
map(pic)[(0,0),(0,5),(3,2),(0,2),(2,0),(2,5),(3,4),(0,4)] <+>   -- trace2

traces(0,0)(3,1) <+>
 -- [[(0,0),(3,0),(0,3),(0,5),(3,2),(0,2),(2,0),(2,5),(3,4),(0,4),(3,1)],
 --  [(0,0),(3,0),(0,3),(3,3),(1,5),(1,0),(0,1),(3,1)],
 --  [(0,0),(0,5),(3,2),(3,0),(0,3),(3,3),(1,5),(1,0),(0,1),(3,1)],
 --  [(0,0),(0,5),(3,2),(0,2),(2,0),(2,5),(3,4),(0,4),(3,1)]]

traces(0,0)(0,4) <+>
 -- [[(0,0),(3,0),(0,3),(0,5),(3,2),(0,2),(2,0),(2,5),(3,4),(0,4)],
 --  [(0,0),(3,0),(0,3),(3,3),(1,5),(1,0),(0,1),(3,1),(0,4)],
 --  [(0,0),(0,5),(3,2),(3,0),(0,3),(3,3),(1,5),(1,0),(0,1),(3,1),(0,4)],
 --  [(0,0),(0,5),(3,2),(0,2),(2,0),(2,5),(3,4),(0,4)]]

eval$final <+>      --> [(3,1),(0,4)]
evalG$EF$final
```

press *specification* > *build Kripke model*

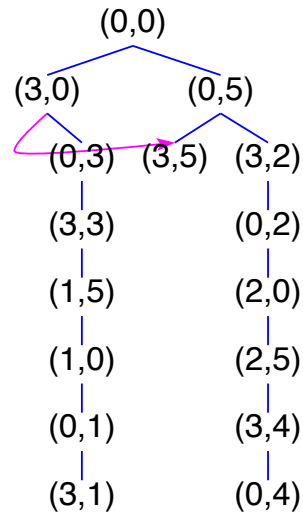
press *graph* > *show graph of Kripke model* > *here*

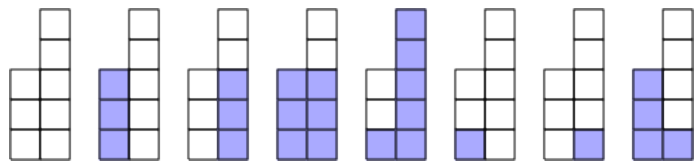


press *specification* > *build Kripke model* .. *cycle-free*
press *graph* > *show graph of transitions* > *here*

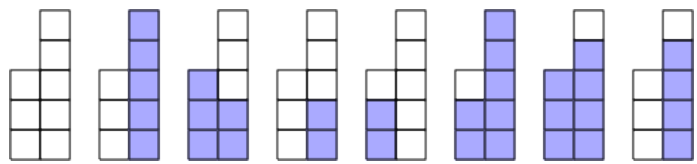


press *specification* > *build Kripke model .. pointer-free*
press *graph* > *show graph of transitions* > *here*



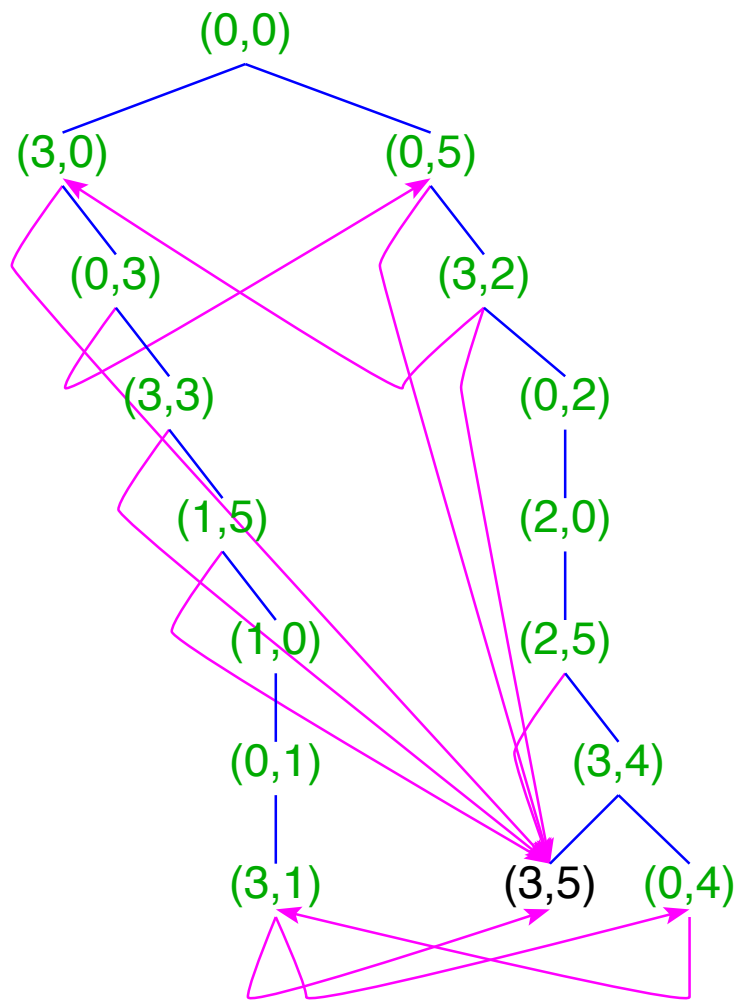


`trace1`

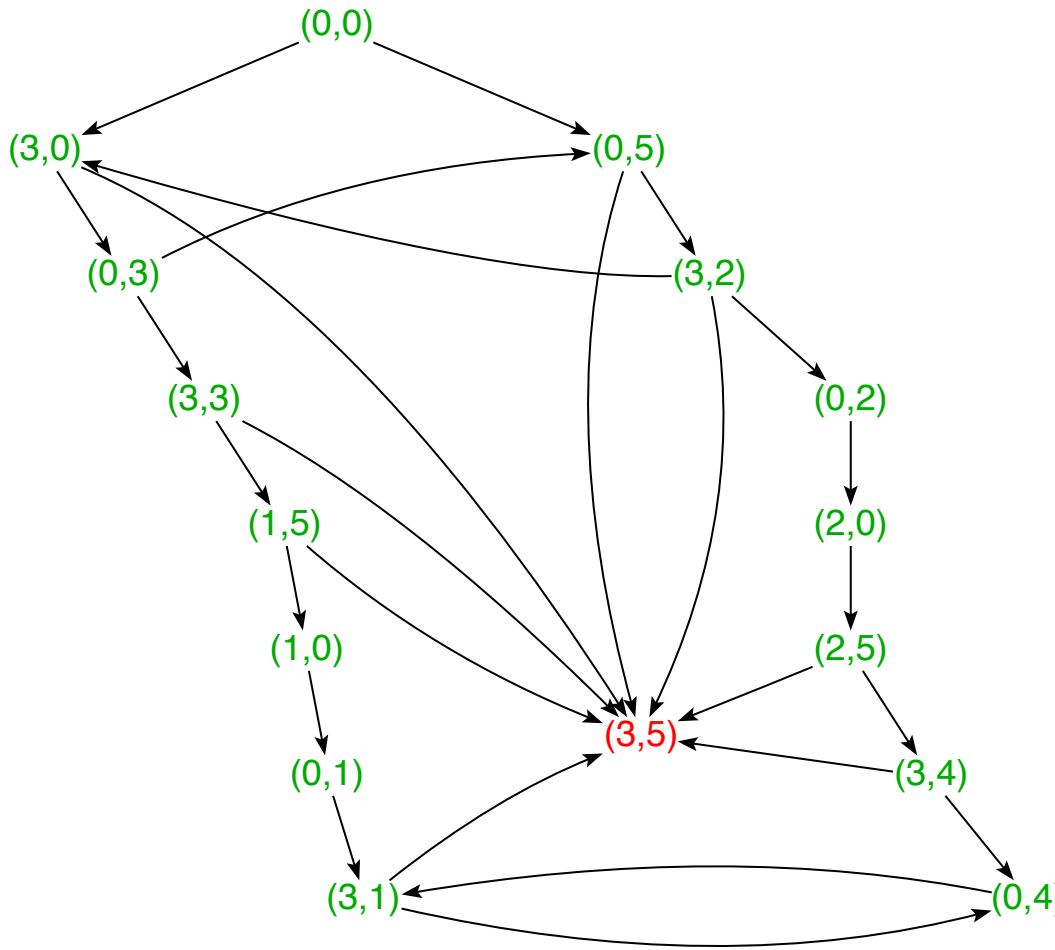


`trace2`

enter *evalGEFfinal* into the text field
press *parse up*; press *simplify*



enter *evalGEFfinal* into the text field; press *parse up*; press *simplify*
enter *draw* into the entry field; press *tree* > *tree*; press *paint*
adapt red support points and press *combis* to remove them



Elevator

Towers of Hanoi

```
-- HANOI

preds:          Initial Final EF
constructs:     A B C
defuncts:       procs lg start draw piles pic
fovars:         st xs ys zs

axioms:
lg == length(procs) &
start == A(procs)^B[]^C[] & states == [start] &

A(x:xs)^B[] -> A(xs)^B[x] &
A(x:xs)^C[] -> A(xs)^C[x] &
A[]^B(x:xs)^C(y:ys) -> A[x]^B(xs)^C(y:ys)<+>A[y]^B(x:xs)^C(ys) &

(x < y ==> A(x:xs)^B(y:ys) -> A(xs)^B(x:y:ys))  &
(y < x ==> A(x:xs)^B(y:ys) -> A(y:x:xs)^B(ys))  &
(x < y ==> A(x:xs)^C(y:ys) -> A(xs)^C(x:y:ys))  &
(y < x ==> A(x:xs)^C(y:ys) -> A(y:x:xs)^C(ys))  &
(x < y ==> B(x:xs)^C(y:ys) -> B(xs)^C(x:y:ys))  &
(y < x ==> B(x:xs)^C(y:ys) -> B(y:x:xs)^C(ys))  &
```

```
(Initial$st <==> st = start) &
(Final$A(xs)^B(ys)^C(zs) <==> ys = procs | zs = procs) &

draw == wtree $ fun(st,ite(Initial$st|Final$st)(piles$st,st)) &
piles$A(xs)^B(ys)^C(zs) == turt[tow$xs,J$16,tow$ys,J$16,tow$zs] &
tow$xs == scale(8)$pileR$lg:xs

conjects: start `sats` EF$Final                --> True
```

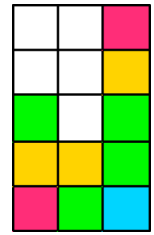


Fig. 14. draw$piles$A[0,1,2]^B[1,0]^C[2,1,0,0,4], falls lg = 5

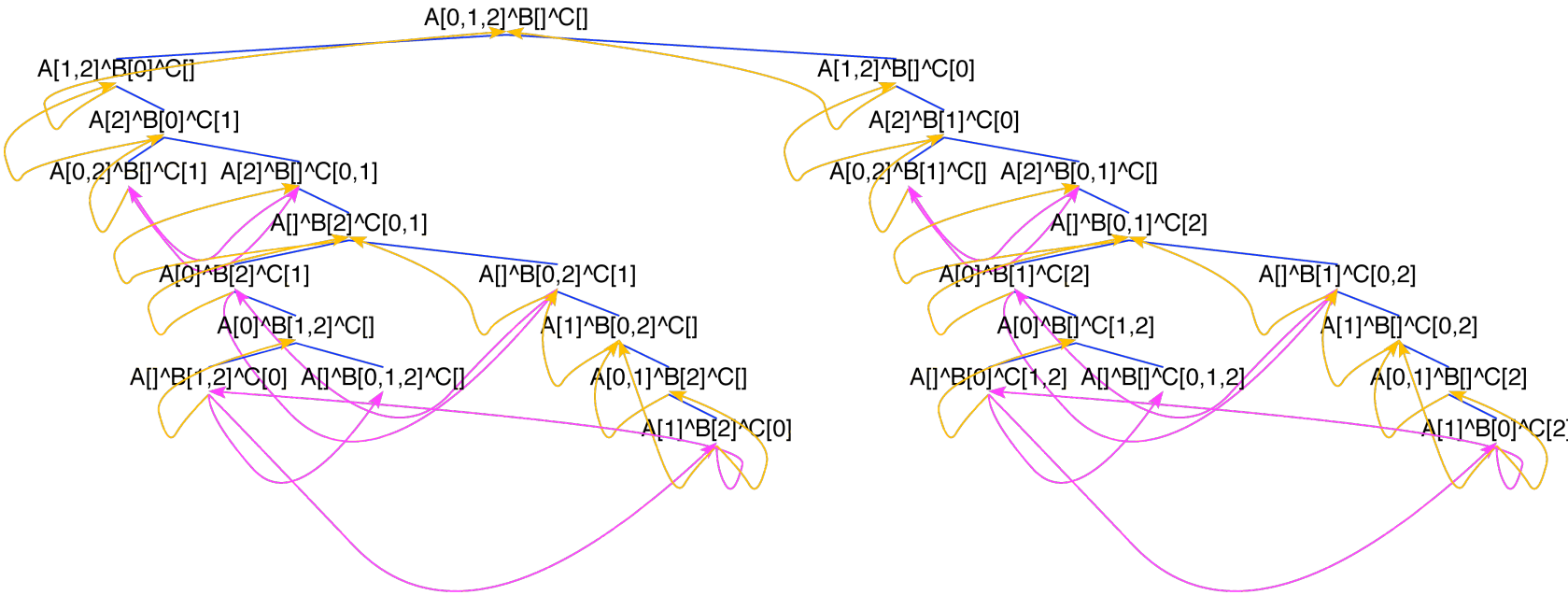


Fig. 15. *Transitionsgraph von* HANOI *für 3 Prozesse*

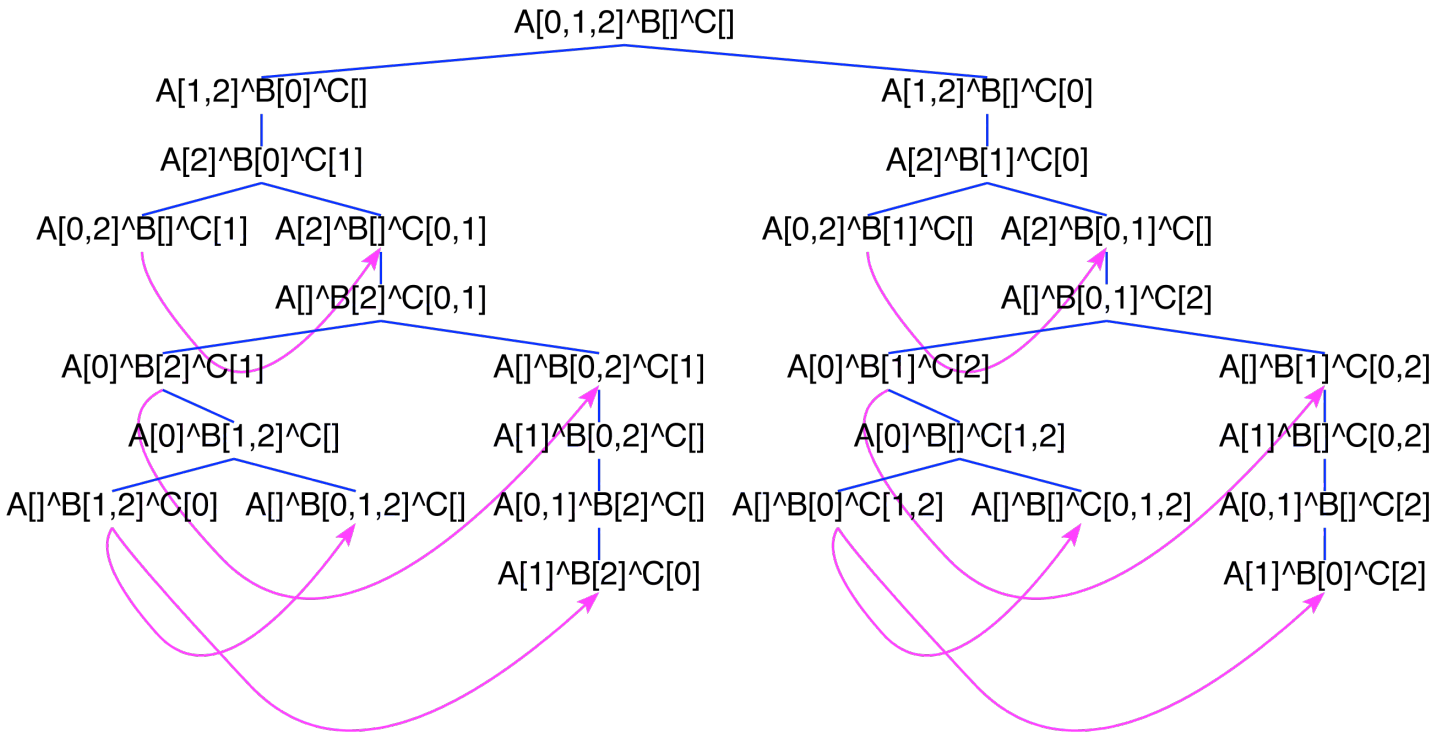


Fig. 16. … *bei Aufbau mit* `build cycle-free model`

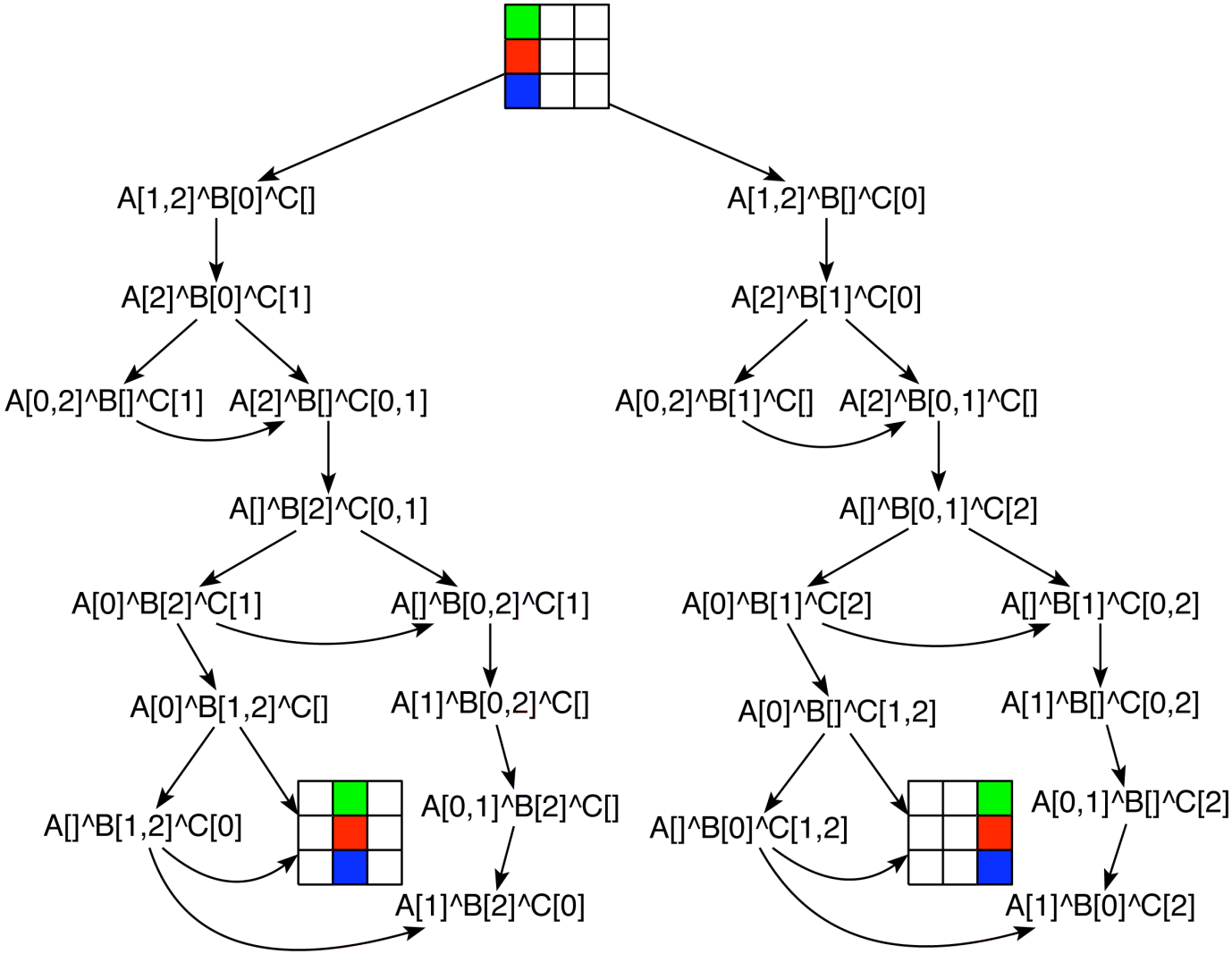


Fig. 17. … *bei Aufbau mit* `build cycle-free model` *und Aufruf des Painters mit* `draw`

```
-- HANOIacts

preds:          Initial Final EF
constructs:     A B C AB BA AC CA BC CB
defuncts:       procs lg start draw piles pic
fovars:         xs ys zs

axioms:
lg == length(procs) &
start == A(procs)^B[]^C[] & states == [start] &
labels == [AB,BA,AC,CA,BC,CB] &

(A(x:xs)^B[],AB) -> A(xs)^B[x]  &
(A(x:xs)^C[],AC) -> A(xs)^C[x]  &
(A[]^B(x:xs),BA) -> A[x]^B(xs)  &
(A[]^C(x:xs),CA) -> A[x]^C(xs)  &
(B[]^C(x:xs),CB) -> B[x]^C(xs)  &
(B(x:xs)^C[],BC) -> B(xs)^C[x]  &

(x < y ==> (A(x:xs)^B(y:ys),AB) -> A(xs)^B(x:y:ys))     &
(y < x ==> (A(x:xs)^B(y:ys),BA) -> A(y:x:xs)^B(ys))     &
(x < y ==> (A(x:xs)^C(y:ys),AC) -> A(xs)^C(x:y:ys))     &
(y < x ==> (A(x:xs)^C(y:ys),CA) -> A(y:x:xs)^C(ys))     &
(x < y ==> (B(x:xs)^C(y:ys),BC) -> B(xs)^C(x:y:ys))     &
(y < x ==> (B(x:xs)^C(y:ys),CB) -> B(y:x:xs)^C(ys))     &
```

```
(Initial$st <==> st = start) &
(Final$A(xs)^B(ys)^C(zs) <==> ys = procs | zs = procs) &

draw == wtree $ fun(x,ite(x `in` labels)
                        (frameS$text$x,ite(Initial$x|Final$x)
                                           (piles$x,x))) &
piles$A(xs)^B(ys)^C(zs) == center$turt[tow$xs,J$16,tow$ys,J$16,tow$zs] &
tow$xs == scale(8)$pileR$lg:xs

conjects: start `sats` EF$Final                 --> True
```

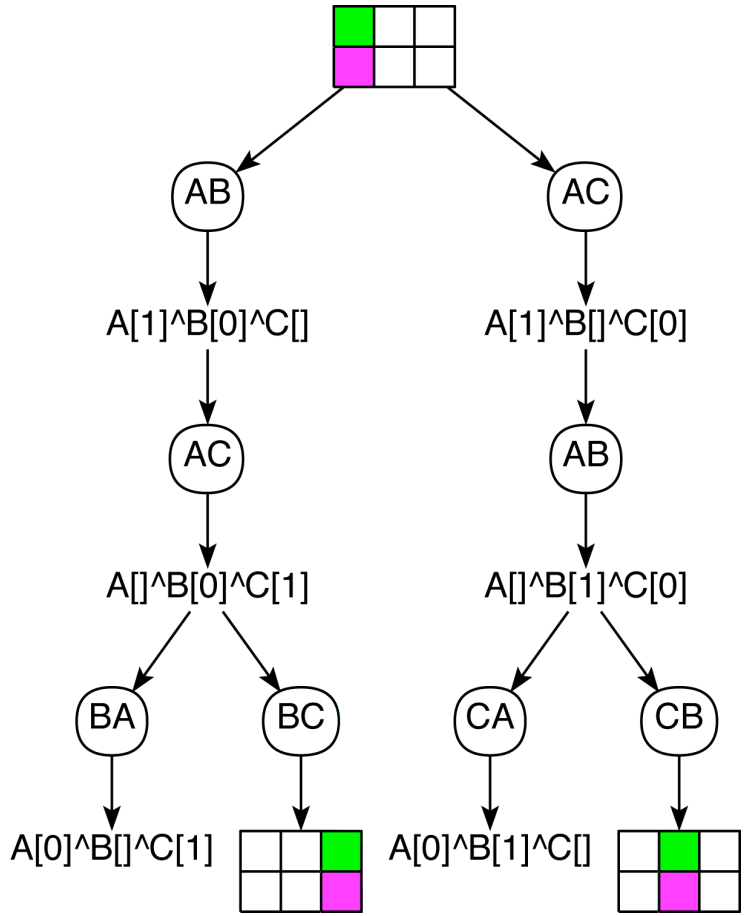


Fig. 18. *Transitionsgraph von* `HANOIacts` *für 2 Prozesse bei Aufbau mit* `build pointer-free model` *und Aufruf des Painters mit* `draw`

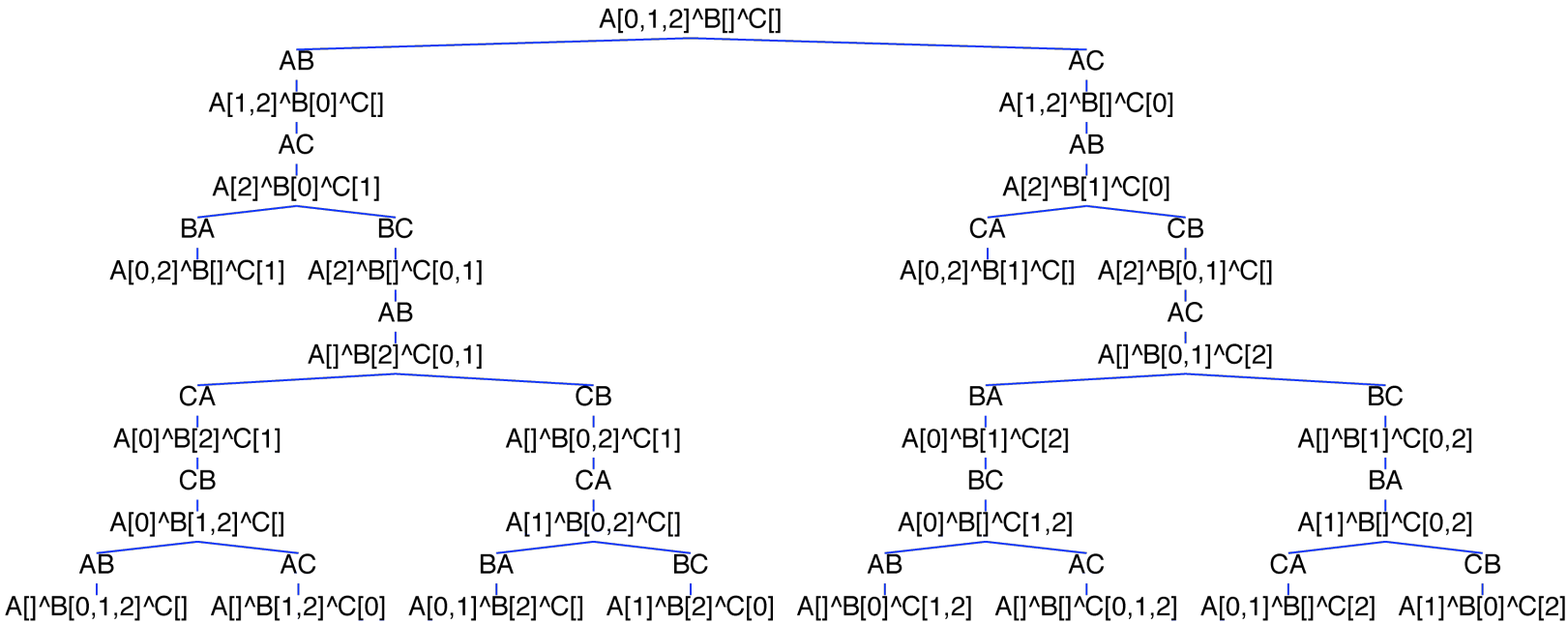


Fig. 19. *Transitionsgraph von* `HANOIacts` *für 3 Prozesse bei Aufbau mit* `build pointer-free model`

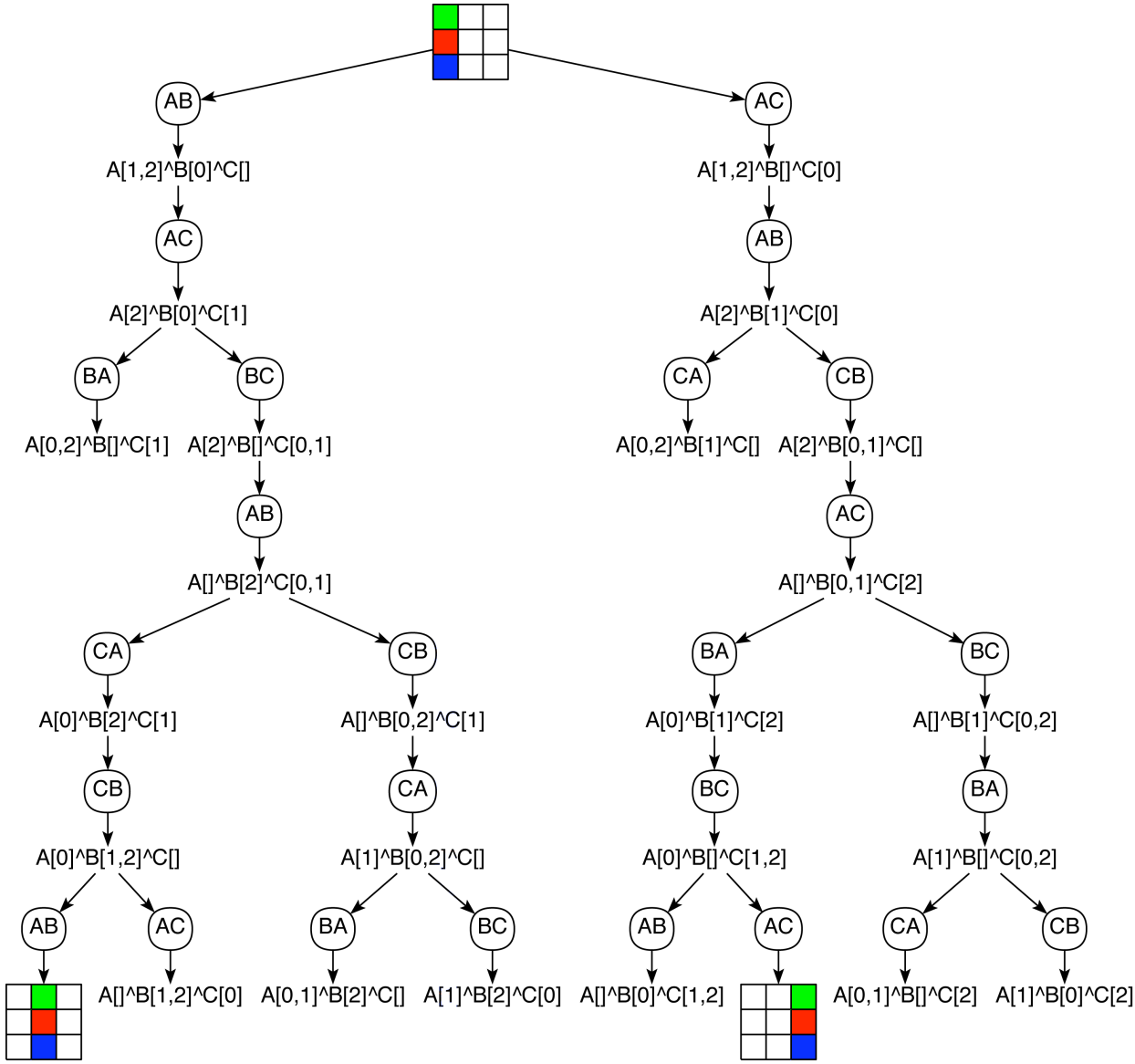

AB
AC
A[1,2]^B[0]^C[]
A[1,2]^B[]^C[0]
AC
AB
A[2]^B[0]^C[1]
A[2]^B[1]^C[0]
BA
BC
CA
CB
A[0,2]^B[]^C[1]
A[2]^B[]^C[0,1]
A[0,2]^B[1]^C[]
A[2]^B[0,1]^C[]
AB
AC
A[]^B[2]^C[0,1]
A[]^B[0,1]^C[2]
CA
CB
BA
BC
A[0]^B[2]^C[1]
A[]^B[0,2]^C[1]
A[0]^B[1]^C[2]
A[]^B[1]^C[0,2]
CB
CA
BC
BA
A[0]^B[1,2]^C[]
A[1]^B[0,2]^C[]
A[0]^B[]^C[1,2]
A[1]^B[]^C[0,2]
AB
AC
BA
BC
AB
AC
CA
CB
A[]^B[1,2]^C[0]
A[0,1]^B[2]^C[]
A[1]^B[2]^C[0]
A[]^B[0]^C[1,2]
A[0,1]^B[]^C[2]
A[1]^B[0]^C[2]

Fig. 20. *... bei Aufbau mit* `build pointer-free model` *und Aufruf des Painters mit* `draw`

Natural numbers

```
-- NAT

preds:      X Nat even odd
defuncts:   div fib loop fibL loop1 loop2 sum
fovars:     q r n
hovars:     f X

axioms:

  sum(0) = 0
& sum(suc(x)) = sum(x)+x+1
& (div(x,y) = (0,x) <=== x < y)
& (div(x,y) = (q+1,r) <=== 0 < y & y <= x & div(x-y,y) = (q,r))
& fib(0) == 0
& fib(1) == 1
& fib(suc(suc(n))) == fib(n)+fib(n+1)

& Nat(0)
& (Nat(suc(x)) <=== Nat(x))

& even(0)
& (even(suc(x)) <=== odd(x))
```

```
& (odd(suc(x)) <=== even(x))

-- & div(x,y) = loop(y,0,x)
& (r < y ==> loop(y,q,r) == (q,r))
& (r >= y ==> loop(y,q,r) == loop(y,q+1,r-y))
& (INV(x,y,q,r) <=== x = (y*q)+r)

& fibL(n) = loop1(n,0,1)
& loop1(0,x,y) = x
& loop1(suc(n),x,y) = loop1(n,y,x+y)

& loop2(f)(0)(x) == x
& loop2(f)(suc(n))(x) == f$loop2(f)(n)(x)

& suc(x) >> x

-- & (Nat(0) <==> True)
-- & (Nat(suc(x)) <==> Nat(x))

-- & (Nat <==> MU X.rel(x,x=0|Any y:x=suc(y)&X(y)))

-- & (INV(n,x,y,z) <=== n >= x & y = fib(n-x) & z = fib(n-x+1))

& (x >> y <=== x > y)
```

```
conjects:

  (sum(x) = y ==> x*(x+1) = 2*y)                                  -- sum1
& (div(x,y) = (q,r) ==> x = (y*q)+r & r < y)                      -- div
& (x = (y*q)+r ==> loop(y,q,r) = div(x,y))                        -- divloop
& fibL(x) = fib(x)                                                -- fib
& (Nat(x) ==> x+y = y+x)                                          -- comm
& (Nat(x) ==> x+(y+z) = (x+y)+z)                                  -- assoc
& (Nat(x) ==> x < 2**x)                                           -- exp
& (Nat(x) ==> even(x) | odd(x))                                   -- evenodd
& (Nat(x) ==> suc(x)*x = x**2+x)                                  -- pot
& (Nat(n) ==> loop2(f)(n)$f$x = f$loop2(f)(n)(x))                 -- natloop
& (even(x) ==> x = 0 | Any y z:(x = y+z & odd(y) & odd(z)))      -- evenany

& div(5,4) = x
& div(5,x) = (1,1)
& Any x y:(x < y & div(5,3)=(x,y))

terms:
fun((suc(x),y),x+x+y)(6,10) <+>
fun((suc(x),y),fun(z,x+y+z)(5))(suc(z),10) <+>
filter(rel(x,x<5))[1,2,3,4,5,6] <+>
filter(rel(x,Int(x)))[1,2,3.6,4,5,6]
```

Proof of `evenodd` *by fixpoint induction:*

```
0. Derivation of

Nat(x) ==> even(x) | odd(x)

All simplifications are admitted.
Equation removal is safe.

1. Adding

  (Nat0(x) ===> even(x) | odd(x))

to the axioms and applying FIXPOINT INDUCTION wrt

  Nat(0)
& (Nat(suc(x)) <=== Nat(x))

at positions
[]
of the preceding formulas leads to

All x:(even(0) | odd(0)) & All x:(even(suc(x)) | odd(suc(x)) <=== Nat0(x))

2. NARROWING the preceding formulas (one step) leads to

All x:(True | odd(0)) & All x:(even(suc(x)) | odd(suc(x)) <=== Nat0(x))
```

```
The axioms have been MATCHED against their redices.
THIS GOAL COINCIDES WITH GOAL NO. 2

3. NARROWING the preceding formulas (one step) leads to

All x:(True | odd(0)) & All x:(odd(x) | odd(suc(x)) <=== Nat0(x))

The axioms have been MATCHED against their redices.

4. NARROWING the preceding formulas (one step) leads to

All x:(True | odd(0)) & All x:(odd(x) | even(x) <=== Nat0(x))

The axioms have been MATCHED against their redices.

5. NARROWING the preceding formulas (one step) leads to

All x:(True | odd(0)) & All x:(odd(x) | even(x) <=== even(x) | odd(x))

The axioms have been MATCHED against their redices.

6. SIMPLIFYING the preceding formulas (one step) leads to

True
```

Proof of `evenodd` *by Noetherian induction:*

```
0. Derivation of

Nat(x) ==> even(x) | odd(x)

All simplifications are admitted.
Equation removal is safe.

1. SELECTING INDUCTION VARIABLES at positions
[0,0]
of the preceding formulas leads to

Nat(!x) ==> even(!x) | odd(!x)

2. Applying the axioms

  even(0)
& (even(suc(x)) <=== odd(x))

at positions
[1,0]
of the preceding formulas leads to

Nat(!x) ==> (!x = 0 | Any x:(odd(x) & !x = suc(x))) | odd(!x)
```

```
3. Applying the theorem

  (odd(suc(x)) <=== even(x))

at positions
[1,1]
of the preceding formulas leads to

Nat(!x) ==>
(!x = 0 | Any x:(odd(x) & !x = suc(x))) | Any x0:(even(x0) & !x = suc(x0))

4. SIMPLIFYING the preceding formulas (one step) leads to

Nat(!x) ==>
!x = 0 | Any x:(odd(x) & !x = suc(x)) | Any x0:(even(x0) & !x = suc(x0))

5. Applying the axioms

  Nat(0)
& (Nat(suc(x)) <=== Nat(x))

at positions
[0]
of the preceding formulas leads to
```

```
!x = 0 | Any x1:(Nat(x1) & !x = suc(x1)) ==>
!x = 0 | Any x:(odd(x) & !x = suc(x)) | Any x0:(even(x0) & !x = suc(x0))

6. Applying the induction hypothesis

Nat(x) ===> (!x >> x ==> even(x) | odd(x))

at positions
[0,1,0,0]
of the preceding formulas leads to

!x = 0 | Any x1:((!x >> x1 ==> even(x1) | odd(x1)) & !x = suc(x1)) ==>
!x = 0 | Any x:(odd(x) & !x = suc(x)) | Any x0:(even(x0) & !x = suc(x0))

7. NARROWING at positions
[0,1,0,0,0]
of the preceding formulas (one step) leads to

!x = 0 |
Any x1:((!x = suc(x1) | !x > x1 ==> even(x1) | odd(x1)) & !x = suc(x1)) ==>
!x = 0 | Any x:(odd(x) & !x = suc(x)) | Any x0:(even(x0) & !x = suc(x0))

8. SIMPLIFYING the preceding formulas (19 steps) leads to
True
```

Lists

```
-- LIST

specs:      nat
preds:      P any zipAny sorted part NOTsorted >>
copreds:    all zipAll ~
defuncts:   F bag map foldl flatten ext scan zip zipWith
            evens odds mergesort split merge isort insert splitAt
fovars:     x y z xs ys s s' s1 s2 z1 z2 p
hovars:     F P

axioms:

  x:s >> s
& (s >> s' <=== s >> s1 & s1 >> s')

& bag(x:s) = x^bag(s)
& bag(s++s') = bag(s)^bag(s')

& map(F)[] = []
& map(F)(x:s) = F(x):map(F)(s)

& foldl(F)(x)[] = x
```

```
& foldl(F)(x)(y:s) = foldl(F)(F(x,y))(s)
--& sum(s) = foldl(+)(0)(s)
--& product(s) = foldl(*)(1)(s)

& flatten[] == []
& flatten(s:p) == s++flatten(p)

& ext(F)(s) = flatten(map(F)(s))

& scan(F)(x)[] = [x]
& scan(F)(x)(y:s) = x:scan(F)(F(x,y))(s)

& zip[][] = []
& zip(x:s)(y:s') = (x,y):zip(s)(s')
& zipWith(F)[][] = []
& zipWith(F)(x:s)(y:s') = F(x,y):zipWith(F)(s)(s')

& (any(P)(x:s) <=== P(x) | any(P)(s))
& (all(P)(x:s) ===> P(x) & all(P)(s))
& (zipAny(P)(x:s)(y:s') <=== P(x,y) | zipAny(P)(s)(s'))
& (zipAll(P)(x:s)(y:s') ===> P(x,y) & zipAll(P)(s)(s'))

& part([x],[[x]])
& (part(x:y:s,[x]:p) <=== part(y:s,p))
& (part(x:y:s,(x:s'):p) <=== part(y:s,s':p))
```

```
& evens[] = []
& evens(x:s) = x:odds(s)
& odds[] = []
& odds(x:s) = evens(s)

& mergesort[] = []
& mergesort[x] = [x]
& (mergesort(x:y:s) = merge(mergesort(x:s1),mergesort(y:s2))
   <=== split(s) = (s1,s2))

& split[] = ([],[])
& split[x] = ([x],[])
& (split(x:(y:s)) = (x:s1,y:s2) <=== split(s) = (s1,s2))

& (merge(x:s,y:s') = x:merge(s,y:s') <=== x <= y)
& (merge(x:s,y:s') = y:merge(x:s,s') <=== x > y)
& merge([],s) = s
& merge(s,[]) = s

& isort[] = []
& isort[x] = [x]
& isort(x:s) = insert(x,isort(s))

& insert(x,[]) = [x]
```

```
& (insert(x,y:s) = x:y:s <=== x <= y)
& (insert(x,y:s) = y:insert(x,s) <=== x > y)

& sorted([])
& sorted([x])
& (sorted(x:y:s) <=== x <= y & sorted(y:s))

& (s ~ s' ===> bag(s) = bag(s'))

& splitAt(0)(s)         = ([],s)
& splitAt(suc(n))[]     = ([],[])
& (splitAt(suc(n))(x:s) = (x:s1,s2) <=== splitAt(n)(s) = (s1,s2))

& x:s>>s
& (s >> s' <=== s >> s1 & s1 >> s')

theorems:

  NOTsorted(s) <=== Not(sorted(s))
& (sorted(s) & sorted(s') ===> sorted(merge(s,s')))
& (sorted(s) ===> sorted(insert(x,s)))
& (split(s) = (s1,s2) ===> s ~ s1++s2)
& (s ~ merge(s1,s2) <=== s ~ s1++s2)
& (s ~ insert(x,s') <=== s ~ x:s')
& (sorted(x:s) ===> sorted(s))
```

```
& (sorted(x:s) & sorted(y:s') & x <= y & sorted(s1) & s1~(s++y:s')
               ===> sorted(x:s1))
& (x > y ===> y <= x)
& y:x:s++s' ~ x:s++y:s'
& s'++x:s ~ x:s++s'

conjects:

(part(s,p) ==> flatten(p) = s)                    &  -- partflatten
(mergesort(s) = s' ==> sorted(s'))                &
(mergesort(s) = s' ==> s ~ s')                    &
(isort(s) = s' ==> sorted(s'))                    &
(isort(s) = s' ==> s ~ s')                        &
(merge(s1,s2) = s & sorted(s1) & sorted(s2)
   ==> sorted(s) & s ~ s1++s2)                    &
(map(F)(s) = s' ==> lg(s) = lg(s'))               &
zip(evens(s),odds(s)) = s                         &
-- prem subsumes conc:
All x s z:
 (sorted(x:s) & All s': (NOTsorted(s') | x:s = s')
   ==> NOTsorted(z++[x]) | x:s = z++[x])          &
splitAt(3)[5..12] = s

terms:
```

```
flatten([[1,2,3],[5,6,7,8],[1,2,3]])    <+>
merge([1,3,5],[2,4,6,8])
```

Proof of `partflatten` *by fixpoint induction:*

```
0. Derivation of

(part(s,p)==>(flatten(p)=s))

All simplifications are admitted.
Equation removal is safe.

1. Adding

  (part0(s,p) ===> flatten(p) = s)

to the axioms and applying FIXPOINT INDUCTION wrt

  part([x],[[x]])
& (part(x:(y:s),[x]:p) <=== part(y:s,p))
& (part(x:(y:s),(x:s'):p) <=== part(y:s,s':p))

at positions
[]
of the preceding formulas leads to
```

```
(All x p y s s':((flatten([[x]])=[x]))&All x p y s s':(((flatten(([x]:p))=(x:(y:s)))<==

2. SIMPLIFYING the preceding formulas (12 steps) leads to

(All p y s:((part0((y:s),p)==>(flatten(p)=(y:s))))&All p y s s':((part0((y:s),(s':p))==

3. Applying the theorem

  (part0(s,p) ===> flatten(p) = s)

at positions
[1,0,0]
[0,0,0]
of the preceding formulas leads to

(All p y s:(((flatten(p)=(y:s))==>(flatten(p)=(y:s))))&All p y s s':(((flatten((s':p))=

The axioms have been MATCHED against their redices.

4. SIMPLIFYING the preceding formulas (3 steps) leads to

True
```

Proof of `partflatten` *by Noetherian induction:*

```
0. Derivation of
```

```
part(s,p) ==> flatten(p) = s

All simplifications are admitted.
Equation removal is safe.

1. SELECTING INDUCTION VARIABLES at positions
[0,0]
of the preceding formulas leads to

All p:(part(!s,p) ==> flatten(p) = !s)

2. NARROWING the preceding formulas (one step) leads to

All p:(Any x:(!s = [x] & p = [[x]]) |
       Any x y s p0:
        (part(y:s,p0) & !s = (x:(y:s)) & p = ([x]:p0)) |
       Any x y s s' p0:
        (part(y:s,s':p0) & !s = (x:(y:s)) & p = ((x:s'):p0)) ==>
       flatten(p) = !s)

3. SIMPLIFYING the preceding formulas (28 steps) leads to

All x y s p0:
 (!s = (x:(y:s)) & part(y:s,p0) ==> flatten(p0) = (y:s)) &
```

```
All x y s s' p0:
 (!s = (x:(y:s)) & part(y:s,s':p0) ==> (s'++flatten(p0)) = (y:s))

4. Applying the induction hypothesis

part(s,p) ===> (!s >> s ==> flatten(p) = s)

at positions
[1,0,0,1]
[0,0,0,1]
of the preceding formulas leads to

All x y s p0:
 (!s = (x:(y:s)) & (!s >> (y:s) ==> flatten(p0) = (y:s)) ==>
  flatten(p0) = (y:s)) &
All x y s s' p0:
 (!s = (x:(y:s)) & (!s >> (y:s) ==> flatten(s':p0) = (y:s)) ==>
  (s'++flatten(p0)) = (y:s))

5. Applying the theorem

  (x:s) >> s

at positions
[1,0,0,1,0]
```

```
[0,0,0,1,0]
of the preceding formulas leads to

All x y s p0:
 (!s = (x:(y:s)) & (Any x0:(!s = (x0:(y:s))) ==> flatten(p0) = (y:s)) ==>
  flatten(p0) = (y:s)) &
All x y s s' p0:
 (!s = (x:(y:s)) & (Any x1:(!s = (x1:(y:s))) ==> flatten(s':p0) = (y:s)) ==>
  (s'++flatten(p0)) = (y:s))

6. SIMPLIFYING the preceding formulas (15 steps) leads to

True
```

Video for both proofs: http://fldit-www.cs.tu-dortmund.de/~peter/Expander2/PAFL.mp4

Streams

```
-- stream

defuncts:   zeros ones blink blink' evens odds map zipWith zip bzip
            fact facts iter1 iter2 fib fib0 fib1 addTail nats loop
            inv switch morse morse1 morse2 morsef
preds:      Nat X P true false not \/ /\ `then` `leadsto` `ble`
            Fmu Gnu Fpath GF FG `U` `Umu` `Wnu` `Rnu` eqstream
copreds:    NatStream Gpath `le` `W` `Rpath` ~
fovars:     n z z' s s'
hovars:     X

axioms:

  head(zeros) == 0
& tail(zeros) == zeros

& head(ones) == 1
& tail(ones) == ones

& head(blink) == 0
& tail(blink) == 1:blink
```

```
-- & (blink = 1:blink <==> False)       -- used in fairblink2
                                        -- and notfairblink2
& blink' == mu s.(0:1:s)

& head(evens(s)) == head(s)
& tail(evens(s)) == evens(tail(tail(s)))
& odds == evens.tail

-- & tail(evens(s)) == odds(tail(s))
-- & head(odds(s)) == head(tail(s))
-- & tail(odds(s)) == odds(tail(tail(s)))

& head(map(F)(s)) == F(head(s))
& tail(map(F)(s)) == map(F)(tail(s))

& head(zipWith(F)(s)(s')) == F(head(s),head(s'))
& tail(zipWith(F)(s)(s')) == zipWith(F)(tail(s),tail(s'))

& head(zip(s,s')) == head(s)
& tail(zip(s,s')) == zip(s',tail(s))

& head(bzip(s,s',0)) == head(s)
& head(bzip(s,s',1)) == head(s')
& tail(bzip(s,s',0)) == bzip(tail(s),s',1)
& tail(bzip(s,s',1)) == bzip(s,tail(s'),0)
```

```
& fact(0) == 1
& fact(suc(n)) == n*fact(n)

& head(facts(n)) == fact(n)
& tail(facts(n)) == facts(suc(n))

& head(iter1(f,x)) == x
& tail(iter1(f,x)) == iter1(f,f(x))

& head(iter2(f,x)) == x
& tail(iter2(f,x)) == map(f)(iter2(f,x))

& nats(n) == iter1(suc,n)
& nats'(n) == iter2(suc,n)

& loop(f)(0)(x) == x
& loop(f)(suc(n))(x) == f(loop(f)(n)(x))

& head$inv$s == switch$head$s
& tail$inv$s == inv$tail$s

& switch(0) == 1
& switch(1) == 0
& switch$switch$x == x                          -- used by invinv
```

```
& head$morse == 0
& head$tail$morse == 1
& tail$tail$morse == zip(tail$morse,inv$tail$morse)

& head$morsef$s == head$s
& head$tail$morsef$s == switch$head$s
& tail$tail$morsef$s == morsef$tail$s

& head$morse1$s == head$s
& tail$morse1$s == morse2$s
& head$morse2$s == switch$head$s
& tail$morse2$s == morse1$tail$s

& fib(0) == 1
& fib(1) == 1
& fib(suc(suc(n))) == fib(n)+fib(n+1)

& head(fib0(n)) == fib(n)
& tail(fib0(n)) == fib0(n+1)

& head(fib1(n)) == fib(n)
& head(tail(fib1(n))) == fib(n+1)
& tail(tail(fib1(n))) == addTail(zip(fib1(n),tail(fib1(n))))
```

```
& head(addTail(s)) == head(s)+head(tail(s))
& tail(addTail(s)) == addTail(tail(tail(s)))

& head$s+s' == head(s)+head(s')
& tail$s+s' == tail(s)+tail(s')

-- state equivalence

& (s ~ s' ===> head$s = head$s' & tail$s ~ tail$s')

& (eqstream <==> NU X.rel((s,s'),head$s=head$s' & X(tail$s,tail$s')))

& (Fpath(P)$s <=== P$s | Fpath(P)$tail$s)                -- finally
& (Gpath(P)$s ===> P$s & Gpath(P)$tail$s)                -- generally
& ((P`U`Q)$s <=== Q$s | P$s & (P`U`Q)$tail$s)            -- until
& ((P`W`Q)$s ===> Q$s | P$s & (P`W`Q)$tail$s)            -- weak until
& ((P`Rpath`Q)$s ===> Q$s & (P$s | (P`Rpath`Q)$tail$s)) -- release

& (GF <==> Gpath.Fpath)
& (FG <==> Fpath.Gpath)

& (Fmu$P <==> MU X.(P\/X.tail))                          -- finally
& (Gnu$P <==> NU X.(P/\X.tail))                          -- generally
& (P`Umu`Q <==> MU X.(Q\/(P/\X.tail)))                   -- until
& (P`Wnu`Q <==> NU X.(Q\/(P/\X.tail)))                   -- weak until
```

```
& (P`Rnu`Q <==> NU X.(Q/\(P\/X.tail)))                    -- release
& (P`leadsto`Q <==> Gmu$P`then`Fmu$Q)

& ((P/\Q)$s <==> P$s & Q$s)
& ((P\/Q)$s <==> P$s | Q$s)

& (Nat$x <==> Int(x) & x >= 0)

& (NatStream(x:s) ===> Nat(x) & NatStream$s)

& (s `le` s' ===> head$s `ble` head(s'))
& (s `le` s' ===> (head$s = head$s' ==> tail$s `le` tail$s'))

theorems:

  iter1(f,loop(f)(n)(x)) = map(loop(f)(n))(iter2(f,x))  -- used in iter1iter1
& map(loop(f)(0))(s) = s                                 -- used in iter1iter1
& map(loop(f)(suc(n)))(s) = map(loop(f)(n))(map(f)(s))  -- used in iterloop
& loop(f)(n)(f(x)) = f(loop(f)(n)(x))                    -- used in maploop
                                                         -- see NAT (natloop)
& tail(fib1(n)) = fib1(n+1)                              -- used in fib01
& fib((n+1)+1) = (fib(n)+fib(n+1))                       -- used in fibtail
& evens(zip(s,s')) = s                                   -- used in evodmorse
& s ~ s                                                  -- used in evodmorse
& zip(s,inv(s)) = morsef(s)                              -- used in morsef
```

```
& (x `ble` z <=== x `ble` y & y `ble` z)
& (x `ble` y & y `ble` z & x = z ===> y = z & x = y)
& (x+z `ble` y+z' <=== x `ble` y & z `ble` z')

& (Fpath(Q)$s <=== (true`U`Q)$s)
& (Gpath(P)$s <=== (P`W`false)$s)
& ((P`U`Q)$s <=== (P`W`Q)$s & Fpath(Q)$s)
& ((P`W`Q)$s <=== (P`U`Q)$s | Gpath(P)$s)
& (not(Fpath$P)$s <=== (Gpath$not$P)$s)
& (not(Gpath$P)$s <=== (Fpath$not$P)$s)
& (not(P`Rpath`Q)$s <=== (not(P)`U`not(Q))$s)

conjects:

  blink ~ zip(zeros,ones)                                  -- blinkzip
& evens$zip(s,s') ~ s                                      -- evenszip
& odds$zip(s,s') ~ s'                                      -- oddszip
& zip(evens$s,odds$s) ~ s                                  -- zipEvensOdds
& evens$x:s ~ x:odds$s                                     -- evensodds
& bzip(s,s',0) ~ zip(s,s')                                 -- bzip
& bzip(s,s',1) ~ zip(s',s)
& inv$inv$s ~ s                                            -- invinv
& iter1(f,n) ~ iter2(f,n)                                  -- iter1iter2
& iter1(f,loop(f)(n)(x)) ~ map(loop(f)(n))(iter2(f,x))  -- iterloop
```

```
& map(loop(f)(0))(s) ~ s                                  -- maploop0
& map(loop(f)(suc(n)))(s) ~ map(loop(f)(n))(map(f)(s))  -- maploop
& map(f)(iter1(f,x)) ~ iter1(f,f(x))                      -- mapiter1
& map(fact)(nats(n)) ~ facts(n)                           -- mapfact
& fib0(n) ~ fib1(n)                                       -- fib01
& fib1(n+1) ~ tail(fib1(n))                               -- fibtail
& tail(fib1(n+1)) ~ addTail(zip(fib1(n),tail(fib1(n))))  -- tailfib
& evens$morse ~ morse & odds$tail$morse ~ tail$morse     -- evodmorse
& morsef$inv$s ~ inv$morsef$s                             -- morseinv
& morse1$inv$s ~ inv$morse1$s                             -- morseinv2
& morsef$morse ~ morse                                    -- morsef
& zip(s,inv$s) ~ morsef$s                                 -- zipmorse
& morse ~ 0:zip(inv$morse,tail$morse)

& Fmu((=0).head)$1:2:3:blink            --> True

& GF((=0).head)$blink                   --> True        -- fairblink
& Not(GF((=0).head)$blink)              --> False       -- notfairblink
& GF((=2).head)$blink                   --> False       -- fairblink2
& Not(GF((=2).head)$blink)              --> True        -- notfairblink2

& GF((=0).head)$mu s.(0:1:s)            --> True        -- fairblinkmu
& NatStream$mu s.(1:2:3:s)              --> True        -- natstream

& eqstream(blink,0:1:blink)
```

```
& (s = x & s' = zip(evens$x,odds$x) |
   s = tail$x & s' = zip(odds$x,evens$x) ==> eqstream(s,s'))

& (x `le` y & y `le` z ==> x `le` z)                    -- kozen1
& (x `le` y & z `le` z' ==> x+z `le` y+z')              -- kozen2

-- & GF((=!x).head)$blink       --> !x=0 | !x=1         -- fairblinkx
-- & NatStream$1:2:3:!s         --> !s = 1:2:3:!s |     -- natstreamSol
                                --  !s = 2:3:!s | !s = 3:!s
```

Proof of `blinkzip` *by coinduction:*

```
0. Derivation of

blink ~ zip(zero,one)

All simplifications are admitted.
Equation removal is safe.

1. Adding

  (~0(z0,z1) <=== z0 = blink & z1 = zip(zero,one))

to the axioms and applying COINDUCTION wrt

  (s ~ s' ===> head(s) = head(s') & tail(s) ~ tail(s'))
```

```
at positions
[]
of the preceding formulas leads to

All s s':(s = blink & s' = zip(zero,one) ===>
          head(s) = head(s') & ~0(tail(s),tail(s')))

2. SIMPLIFYING the preceding formulas (12 steps) leads to

~0(1:blink,zip(one,zero))

3. Adding

  (~0(z2,z3) <=== z2 = (1:blink) & z3 = zip(one,zero))

to the axioms and applying COINDUCTION wrt

  (s ~ s' ===> head(s) = head(s') & tail(s) ~ tail(s'))

at positions
[]
of the preceding formulas leads to

All s s':(s = (1:blink) & s' = zip(one,zero) ===>
```

```
          head(s) = head(s') & ~0(tail(s),tail(s')))

4. SIMPLIFYING the preceding formulas (12 steps) leads to

~0(blink,zip(zero,one))

5. NARROWING the preceding formulas (one step) leads to

blink = blink & zip(zero,one) = zip(zero,one) |
blink = (1:blink) & zip(zero,one) = zip(one,zero)

The axioms have been MATCHED against their redices.

6. SIMPLIFYING the preceding formulas (one step) leads to

True
```

Proof of `bzip` *by coinduction:*

```
0. Derivation of

bzip(s,s',0) ~ zip(s,s')

All simplifications are admitted.
Equation removal is safe.
```

```
1. Adding

  (~0(z0,z1) <=== z0 = bzip(s,s',0) & z1 = zip(s,s'))

to the axioms and applying COINDUCTION wrt

  (s ~ s' ===> head(s) = head(s') & tail(s) ~ tail(s'))

at positions
[]
of the preceding formulas leads to

All s s':(Any s0 s'0:
           (s = bzip(s0,s'0,0) & s' = zip(s0,s'0)) ===>
          head(s) = head(s') & ~0(tail(s),tail(s')))
THIS GOAL COINCIDES WITH GOAL NO. 1

2. SIMPLIFYING the preceding formulas (12 steps) leads to

All s0 s'0:
 ~0(bzip(tail(s0),s'0,1),zip(s'0,tail(s0)))

3. Adding

  (~0(z2,z3) <=== z2 = bzip(tail(s0),s'0,1) & z3 = zip(s'0,tail(s0)))
```

```
to the axioms and applying COINDUCTION wrt

  (s ~ s' ===> head(s) = head(s') & tail(s) ~ tail(s'))

at positions
[0]
of the preceding formulas leads to

All s0 s'0:
 All s s':(Any s0 s'0:
            (s = bzip(tail(s0),s'0,1) & s' = zip(s'0,tail(s0))) ===>
           head(s) = head(s') & ~0(tail(s),tail(s')))

4. SIMPLIFYING the preceding formulas (12 steps) leads to

All s0 s'0:
 ~0(bzip(tail(s0),tail(s'0),0),zip(tail(s0),tail(s'0)))

5. Applying the theorem

  (~0(z0,z1) <=== z0 = bzip(s,s',0) & z1 = zip(s,s'))

at positions
[0]
```

```
of the preceding formulas leads to

All s0 s'0:
 Any s s':(bzip(tail(s0),tail(s'0),0) = bzip(s,s',0) &
           zip(tail(s0),tail(s'0)) = zip(s,s'))

6. SUBSTITUTING tail(s'0) FOR s' to the preceding formulas leads to

All s0 s'0:
 Any s s':(bzip(tail(s0),tail(s'0),0) = bzip(tail(s0),tail(s'0),0) &
           zip(tail(s0),tail(s'0)) = zip(tail(s0),tail(s'0)))

7. SIMPLIFYING the preceding formulas (one step) leads to

True
```

Proof of `fairblink` *by coinduction:*

```
0. Derivation of

(G(F(((=0).head)))$blink)

All simplifications are admitted.
Equation removal is safe.

1. Adding
```

```
  (G0(z0)$z1 <=== z0 = F((=0).head) & z1 = blink)

to the axioms and applying COINDUCTION wrt

  (G(P)$s ===> P(s) & G(P)$tail(s))

at positions
[]
of the preceding formulas leads to

All P s:((((P=F(((=0).head)))&(s=blink))===>(P(s)&(G0(P)$tail(s)))))

2. SIMPLIFYING the preceding formulas (6 steps) leads to

((F(((=0).head))$blink)&(G0(F(((=0).head)))$(1:blink)))

3. Applying the theorem

  (F(P)$s <=== P(s) | F(P)$tail(s))

at positions
[0]
of the preceding formulas leads to
```

```
(((((=0).head)$blink)|(F(((=0).head))$tail(blink)))&(G0(F(((=0).head)))$(1:blink)))

The axioms have been MATCHED against their redices.

4. SIMPLIFYING the preceding formulas (6 steps) leads to

(G0(F(((=0).head)))$(1:blink))

5. Adding

  (G0(z2)$z3 <=== z2 = F((=0).head) & z3 = (1:blink))

to the axioms and applying COINDUCTION wrt

  (G(P)$s ===> P(s) & G(P)$tail(s))

at positions
[]
of the preceding formulas leads to

All P s:((((P=F(((=0).head)))&(s=(1:blink)))===>(P(s)&(G0(P)$tail(s)))))

6. SIMPLIFYING the preceding formulas (6 steps) leads to

((F(((=0).head))$(1:blink))&(G0(F(((=0).head)))$blink))
```

```
7. Applying the axioms

  (G0(z0)$z1 <=== z0 = F((=0).head) & z1 = blink)
& (G0(z2)$z3 <=== z2 = F((=0).head) & z3 = (1:blink))

at positions
[1]
of the preceding formulas leads to

((F(((=0).head))$(1:blink))&(((F(((=0).head))=F(((=0).head)))&(blink=blink))|
((F(((=0).head))=F(((=0).head)))&(blink=(1:blink)))))

The axioms have been MATCHED against their redices.

8. SIMPLIFYING the preceding formulas (one step) leads to

(F(((=0).head))$(1:blink))
THIS GOAL COINCIDES WITH GOAL NO. 8

9. NARROWING the preceding formulas (one step) leads to

((((=0).head)$(1:blink))|(F(((=0).head))$tail((1:blink))))

The axioms have been MATCHED against their redices.
```

```
10. SIMPLIFYING the preceding formulas (5 steps) leads to

(F(((=0).head))$blink)

11. NARROWING the preceding formulas (one step) leads to

((((=0).head)$blink)|(F(((=0).head))$tail(blink)))

The axioms have been MATCHED against their redices.

12. SIMPLIFYING the preceding formulas (4 steps) leads to

True
```

Proof of `fairblinkS` *by simplification:*

```
0. Derivation of

s = blink | s = (1:blink) ==> GS(FS((=0).head))$s

All simplifications are admitted.
Equation removal is safe.

1. SIMPLIFYING the preceding formulas (one step) leads to
```

```
s = blink | s = (1:blink) ==> NU X.(FS((=0).head)/\(X.tail))$s

2. SIMPLIFYING the preceding formulas (one step) leads to

All s:(s = blink | s = (1:blink) ==>
       (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s)

3. SIMPLIFYING the preceding formulas (one step) leads to

All s:((s = blink ==>
        (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s) &
       (s = (1:blink) ==>
        (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s))

4. SIMPLIFYING the preceding formulas (one step) leads to

All s:(s = blink ==> (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s) &
All s:(s = (1:blink) ==>
       (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s)

5. SIMPLIFYING the preceding formulas (one step) leads to

All s:((FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$blink) &
All s:(s = (1:blink) ==>
       (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s)
```

```
6. SIMPLIFYING the preceding formulas (one step) leads to

(FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$blink &
All s:(s = (1:blink) ==>
       (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s)

7. SIMPLIFYING the preceding formulas (one step) leads to

(FS((=0).head)$blink & (rel(s,(s = blink | s = (1:blink))).tail)$blink) &
All s:(s = (1:blink) ==>
       (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s)

8. SIMPLIFYING the preceding formulas (11 steps) leads to

(1:blink) = blink & MU X.(((=0).head)\/(X.tail))$(1:blink) &
(rel(s,(s = blink | s = (1:blink))).tail)$(1:blink) &
MU X.(((=0).head)\/(X.tail))$blink |
tail(blink) = (1:blink) &
All s:(s = (1:blink) ==>
       (FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$s) &
MU X.(((=0).head)\/(X.tail))$blink

9. SIMPLIFYING the preceding formulas (11 steps) leads to
```

```
(FS((=0).head)/\(rel(s,(s = blink | s = (1:blink))).tail))$(1:blink) &
MU X.(((=0).head)\/(X.tail))$blink |
MU X.(((=0).head)\/(X.tail))$blink & (1:blink) = blink &
MU X.(((=0).head)\/(X.tail))$(1:blink)

10. SIMPLIFYING the preceding formulas (11 steps) leads to

MU X.(((=0).head)\/(X.tail))$blink & MU X.(((=0).head)\/(X.tail))$(1:blink)

11. SIMPLIFYING the preceding formulas (11 steps) leads to

(=0)$head(1:blink) | (MU X.(((=0).head)\/(X.tail)).tail)$(1:blink)

12. SIMPLIFYING the preceding formulas (11 steps) leads to

True
```

Binary trees

```
-- infbintree

defuncts:   left root right mirror
fovars:     t t'

axioms:

  left(mirror(t)) == right(t)
& root(mirror(t)) == root(t)
& right(mirror(t)) == left(t)

& (t ~ t' ===> left(t) ~ left(t') & root(t) = root(t') & right(t) ~ right(t'))

conjects:   mirror(mirror(t)) ~ t
```

Beweis:

```
Adding

  (x ~0 y <=== x = y)
& (x ~0 y <=== y ~0 x)
& (x ~0 z <=== x ~0 y & y ~0 z)
```

```
& (z0 ~0 z1 <=== z0 = mirror(mirror(t)) & z1 = t)
& (x ~/~0 y ===> x =/= y)
& (x ~/~0 y ===> Not(y ~0 x))
& (x ~/~0 z ===> Not(x ~0 y) | Not(y ~0 z))
& (z0 ~/~0 z1 ===> z0 =/= mirror(mirror(t)) | z1 =/= t)

to the axioms and applying coinduction w.r.t.

  (t ~ t' ===> left(t) ~ left(t') & root(t) = root(t') & right(t) ~ right(t'))

at position [] of the preceding formula leads to

All t0:(left(t0) ~0 left(t0)) & All t0:(right(t0) ~0 right(t0))

The reducts have been simplified.

Narrowing the preceding formula (1 step) leads to

All t0:(right(t0) ~0 right(t0))

The axioms were MATCHED against their redices.
The reducts have been simplified.

Narrowing the preceding formula (1 step) leads to
```

```
True

The axioms were MATCHED against their redices.
The reducts have been simplified.

Number of proof steps: 3

-- cobintree

constructs: undef
defuncts:   lrr mt subtree mirror
preds:      finite exists P
copreds:    infinite forall ~pr
fovars:     t u t' u'
hovars:     P

axioms:

  (lrr(mirror(t)) = undef <=== lrr(t) = undef)
& (lrr(mirror(t)) = (mirror(u'),x,mirror(u)) <=== lrr(t) = (u,x,u'))

& (t ~ t' ===> lrr(t) ~pr lrr(t'))
& ((t,x,u) ~pr (t',y,u') ===> t ~ t' & x = y & u ~ u')
& (undef ~pr (t,x,u) ===> False)
& ((t,x,u) ~pr undef ===> False)
```

```
& lrr(mt) = undef

& subtree(t,[]) = t
& (subtree(t,x:s) = undef <=== lrr(t) = undef)
& (subtree(t,0:s) = subtree(u,s) <=== lrr(t) = (u,x,u'))
& (subtree(t,1:s) = subtree(u',s) <=== lrr(t) = (u,x,u'))

& (finite(t) <=== lrr(t) = undef | lrr(t) = (u,x,u') & finite(u) & finite(u'))
& (infinite(t) ===> Any u x u':(lrr(T) = (u,x,u') & (infinite(u) | infinite(u'))))

& (exists(P)(t) <=== lrr(t) = (u,x,u') & (P(x) | exists(P)(u) | exists(P)(u')))
& (forall(P)(t) ===> (lrr(t) = (u,x,u') ==> forall(P)(u) & P(x) & forall(P)(u')))

conjects:   mirror(mirror(t)) ~ t
```

Beweis:

```
Adding

  (x ~0 y <=== x = y)
& (x ~0 y <=== y ~0 x)
& (x ~0 z <=== x ~0 y & y ~0 z)
& (z0 ~0 z1 <=== z0 = mirror(mirror(t)) & z1 = t)
& (x ~/~0 y ===> x =/= y)
& (x ~/~0 y ===> Not(y ~0 x))
```

```
& (x ~/~0 z ===> Not(x ~0 y) | Not(y ~0 z))
& (z0 ~/~0 z1 ===> z0 =/= mirror(mirror(t)) | z1 =/= t)
```

to the axioms and applying coinduction w.r.t.

```
  (t ~ t' ===> lrr(t) ~pr lrr(t'))
```

at position [] of the preceding formula leads to

```
All t t':(Any t0:(t = mirror(mirror(t0)) & t' = t0) ===> lrr(t) ~pr lrr(t'))
```

Simplifying the preceding formula (7 steps) leads to

```
All t0:(lrr(mirror(mirror(t0))) ~pr lrr(t0))
```

Adding

```
  (x ~pr0 y <=== x = y)
& (x ~pr0 y <=== y ~pr0 x)
& (x ~pr0 z <=== x ~pr0 y & y ~pr0 z)
& (z2 ~pr0 z3 <=== z2 = lrr(mirror(mirror(t0))) & z3 = lrr(t0))
& (x ~/~pr0 y ===> x =/= y)
& (x ~/~pr0 y ===> Not(y ~pr0 x))
& (x ~/~pr0 z ===> Not(x ~pr0 y) | Not(y ~pr0 z))
& (z2 ~/~pr0 z3 ===> z2 =/= lrr(mirror(mirror(t0))) | z3 =/= lrr(t0))
```

to the axioms and applying coinduction w.r.t.

```
  ((t,x,u) ~pr (t',y,u') ===> t ~ t' & x = y & u ~ u')
& (undef ~pr (t,x,u) ===> False)
& ((t,x,u) ~pr undef ===> False)
```

at position [0] of the preceding formula leads to

```
All t0:(All t x u t' y u':(Any t0:((t,x,u) = lrr(mirror(mirror(t0))) &
                                   (t',y,u') = lrr(t0)) ===>
                            t ~0 t' & x = y & u ~0 u') &
        All t x u t' y u':(Any t0:(undef = lrr(mirror(mirror(t0))) &
                                   (t,x,u) = lrr(t0)) ===>
                            False) &
        All t x u t' y u':(Any t0:((t,x,u) = lrr(mirror(mirror(t0))) &
                                   undef = lrr(t0)) ===>
                            False))
```

Simplifying the preceding formula (1 step) leads to

```
All t x u t' y u':(Any t0:((t,x,u) = lrr(mirror(mirror(t0))) &
                            (t',y,u') = lrr(t0)) ==>
                    t ~0 t' & x = y & u ~0 u') &
All t x u t0:(Not(undef = lrr(mirror(mirror(t0)))) | Not((t,x,u) = lrr(t0))) &
```

```
All t x u t0:(Not((t,x,u) = lrr(mirror(mirror(t0)))) | Not(undef = lrr(t0)))

The tree has been split.
The current formula is given by

All t x u t' y u':(Any t0:((t,x,u) = lrr(mirror(mirror(t0))) &
                           (t',y,u') = lrr(t0)) ==>
                   t ~0 t' & x = y & u ~0 u')

Applying the axioms

  (lrr(mirror(t)) = undef <=== lrr(t) = undef)
& (lrr(mirror(t)) = (mirror(u'),x,mirror(u)) <=== lrr(t) = (u,x,u'))

at position [0,0,0,0,1] of the preceding factors leads to the factor

All t x u t' y u':(Any t0:(((t,x,u) = undef & lrr(mirror(t0)) = undef |
                            Any u'0 x0 u0:((t,x,u) = (mirror(u'0),x0,mirror(u0)) &
                                           lrr(mirror(t0)) = (u0,x0,u'0))) &
                           (t',y,u') = lrr(t0)) ==>
                   t ~0 t' & x = y & u ~0 u')

The axioms were MATCHED against their redices.

Simplifying at position [0,0,0,0]
```

of the preceding factors (5 steps) leads to the factor

```
All t x u t' y u':(Any t0:(Any u'0 x0 u0:(u = mirror(u0) &
                                          lrr(mirror(t0)) = (u0,x,u'0) &
                                          t = mirror(u'0)) &
                          (t',y,u') = lrr(t0)) ==>
                  t ~0 t' & x = y & u ~0 u')
```

Applying the axioms

```
  (lrr(mirror(t)) = undef <=== lrr(t) = undef)
& (lrr(mirror(t)) = (mirror(u'),x,mirror(u)) <=== lrr(t) = (u,x,u'))
```

at position [0,0,0,0,0,1,0] of the preceding factors leads to the factor

```
All t x u t' y u':(Any t0:(Any u'0 x0 u0:(u = mirror(u0) &
                                          (undef = (u0,x,u'0) & lrr(t0) = undef |
                                           Any u'1 x1 u1:((mirror(u'1),x1,
                                                           mirror(u1)) = (u0,x,
                                                                          u'0) &
                                                          lrr(t0) = (u1,x1,u'1))) &
                                          t = mirror(u'0)) &
                          (t',y,u') = lrr(t0)) ==>
                  t ~0 t' & x = y & u ~0 u')
```

```
The axioms were MATCHED against their redices.

Simplifying at position [0,0,0,0,0,1]
of the preceding factors (5 steps) leads to the factor

All t x u t' y u':(Any t0:(Any u'0 x0 u0:(u = mirror(u0) &
                                          Any u'1 x1 u1:(mirror(u1) = u'0 &
                                                         lrr(t0) = (u1,x,u'1) &
                                                         mirror(u'1) = u0) &
                                          t = mirror(u'0)) &
                              (t',y,u') = lrr(t0)) ==>
                    t ~0 t' & x = y & u ~0 u')

Simplifying the preceding factors (29 steps) leads to the factor

All x t0 u'1 u1:(lrr(t0) = (u1,x,u'1) ==> mirror(mirror(u'1)) ~0 u'1)

Applying the axioms

  (x ~0 y <=== x = y)
& (x ~0 y <=== y ~0 x)
& (x ~0 z <=== x ~0 y & y ~0 z)
& (z0 ~0 z1 <=== z0 = mirror(mirror(t)) & z1 = t)

at position [0,1] of the preceding factors leads to the factor
```

```
All x t0 u'1 u1:(lrr(t0) = (u1,x,u'1) ==>
                 mirror(mirror(u'1)) = u'1 | u'1 ~0 mirror(mirror(u'1)) |
                 Any y:(mirror(mirror(u'1)) ~0 y & y ~0 u'1) |
                 Any t:(mirror(mirror(u'1)) = mirror(mirror(t)) & u'1 = t))

The axioms were MATCHED against their redices.

Substituting u'1 for t to the preceding factors leads to the factor

All x t0 u'1 u1:(lrr(t0) = (u1,x,u'1) ==>
                 mirror(mirror(u'1)) = u'1 | u'1 ~0 mirror(mirror(u'1)) |
                 Any y:(mirror(mirror(u'1)) ~0 y & y ~0 u'1) |
                 Any t u'1:(mirror(mirror(u'1)) = mirror(mirror(u'1)) &
                            u'1 = u'1))

Simplifying the preceding factors (1 step) leads to 2 factors.
The current factor is given by

All t x u t0:(Not(undef = lrr(mirror(mirror(t0)))) | Not((t,x,u) = lrr(t0)))

Applying the axioms

  (lrr(mirror(t)) = undef <=== lrr(t) = undef)
& (lrr(mirror(t)) = (mirror(u'),x,mirror(u)) <=== lrr(t) = (u,x,u'))
```

```
at position [0,0,0,1] of the preceding factors leads to the factor

All t x u t0:(Not(undef = undef & lrr(mirror(t0)) = undef |
                  Any u' x3 u2:(undef = (mirror(u'),x3,mirror(u2)) &
                                lrr(mirror(t0)) = (u2,x3,u'))) |
              Not((t,x,u) = lrr(t0)))

The axioms were MATCHED against their redices.

Simplifying the preceding factors (4 steps) leads to the factor

All t x u t0:(lrr(mirror(t0)) =/= undef | (t,x,u) =/= lrr(t0))

Applying the axioms

  (lrr(mirror(t)) = undef <=== lrr(t) = undef)
& (lrr(mirror(t)) = (mirror(u'),x,mirror(u)) <=== lrr(t) = (u,x,u'))

at position [0,0,0] of the preceding factors leads to the factor

All t x u t0:((undef =/= undef & lrr(t0) = undef |
               Any u' x4 u3:((mirror(u'),x4,mirror(u3)) =/= undef &
                             lrr(t0) = (u3,x4,u'))) |
              (t,x,u) =/= lrr(t0))
```

```
The axioms were MATCHED against their redices.

Simplifying the preceding factors (11 steps) leads to a single formula,
which is given by

All t x u t0:(Not((t,x,u) = lrr(mirror(mirror(t0)))) | Not(undef = lrr(t0)))

Applying the axioms

  (lrr(mirror(t)) = undef <=== lrr(t) = undef)
& (lrr(mirror(t)) = (mirror(u'),x,mirror(u)) <=== lrr(t) = (u,x,u'))

at position [0,0,0,1] of the preceding formula leads to

All t x u t0:(Not((t,x,u) = undef & lrr(mirror(t0)) = undef |
                  Any u' x5 u4:((t,x,u) = (mirror(u'),x5,mirror(u4)) &
                                lrr(mirror(t0)) = (u4,x5,u'))) |
               Not(undef = lrr(t0)))

The axioms were MATCHED against their redices.

Simplifying the preceding formula (31 steps) leads to

All t0 u4 x5 u':(undef =/= lrr(t0) | lrr(mirror(t0)) =/= (u4,x5,u'))
```

```
Applying the axioms

  (lrr(mirror(t)) = undef <=== lrr(t) = undef)
& (lrr(mirror(t)) = (mirror(u'),x,mirror(u)) <=== lrr(t) = (u,x,u'))

at position [0,1,0] of the preceding formula leads to

All t0 u4 x5 u':(undef =/= lrr(t0) |
                 (undef =/= (u4,x5,u') & lrr(t0) = undef |
                  Any u'2 x u:((mirror(u'2),x,mirror(u)) =/= (u4,x5,u') &
                               lrr(t0) = (u,x,u'2))))

The axioms were MATCHED against their redices.

Simplifying the preceding formula (5 steps) leads to

True

Number of proof steps: 21
```

Lazy evaluation

```
-- lazy

specs:      bool
constructs: leaf #
defuncts:   is_even mergesortL splitL mergeL mergesort split merge
            mergesortF splitF mergeF replaceL replace foldRepL foldRep
            palL pal palF reveqL reveq reveqI reveqF sortTL sortT sortTI
            leavesRepL leavesRep leavesRepI foo goo nats nats' fibs fibs'
            RequestsL ClientL Requests Client CSI
            || tree1 tree2 tree3
preds:      even odd
fovars:     t u r a b requests acc t1 t2 u1 u2 s1 s2 ls ls1 ls2
hovars:     f g client

axioms:

(even$0 <==> True) &
(even$suc$x <==> odd$x) &
(odd$x <==> Not(even$x)) &

is_even == fun(x||even$x,true,x||odd$x,false) &
```

```
-- MERGESORT

(mergesortL(x:y:s) = mergeL(mergesortL$x:s1)$mergesortL$y:s2
                     <=== splitL$s = (s1,s2)) &
mergesortL[x] = [x] &
mergesortL[]  = [] &

(splitL$x:y:s = (x:s1,y:s2) <=== splitL$s = (s1,s2)) &
splitL[x] = ([x],[]) &
splitL[]  = ([],[]) &

(mergeL(x:s,y:s') = x:mergeL(s,y:s') <=== x <= y) &
(mergeL(x:s,y:s') = y:mergeL(x:s,s') <=== x > y) &
mergeL([],s) = s &
mergeL(s,[]) = s &

(split$s = (s1,s2)
            ==> mergesort$x:y:s == merge(mergesort$x:s1,mergesort$y:s2)) &
mergesort(s) == s &

(split$s = (s1,s2) ==> split$x:y:s == (x:s1,y:s2)) &
split$s == (s,[]) &

(x <= y ==> merge(x:s,y:s') == x:merge(s,y:s')) &
merge(x:s,y:s') == y:merge(x:s,s') &
```

```
merge([],s)     == s &
merge(s,[])     == s &

mergesortF(x:y:s) == fun((s1,s2),mergeF(mergesortF$x:s1)$mergesortF$y:s2)
                        $splitF$s &
mergesortF[x] == [x] &
mergesortF[]  == [] &

splitF(x:y:s) == fun((s1,s2),(x:s1,y:s2))$splitF$s &
(length(s) <= 1 ==> splitF(s) == (s,[])) &

mergeF(x:s)$y:s' == ite(x<=y,x:mergeF(s)$y:s',y:mergeF(x:s)$s') &
mergeF[]$s       == s &
mergeF(s)[]      == s &

-- PALINDROMES

(palL$s = b <=== reveqL(s)$r = (r,b)) &

reveqL[]$s        = ([],true) &
(reveqL(x:s)$y:s' = (r++[x],eq(x,y)`and`b) <=== reveqL(s)$s' = (r,b)) &

pal$s == get1$mu r b.reveq(s)(r) &             -- reveq(s)$s' = (reverse$s,s=s')

reveq[]$s  == ([],true) &
```

```
(reveq(s)$tail$s' = (r,b) ==> reveq(x:s)$s' == (r++[x],eq(x,head$s')`and`b)) &

palI$s == get1$mu r b.reveqI(s)(r)[] &
                                    -- revEqI(s)(s')$acc = (reverse(s)++acc,s=s')
reveqI[](s)$acc == (acc,true) &
(reveqI(s)(tail$s')$x:acc = (r,b)
                  ==> reveqI(x:s)(s')$acc == (r,eq(x,head$s')`and`b)) &

(reveqF$s = (r,f) ==> palF$s == f$r) &

reveqF[] == ([],fun(s,true)) &
(reveqF$s = (r,f) ==> reveqF$x:s == (r++[x],fun(y:s',eq(x,y)`and`f$s'))) &

-- BINARY TREES

(replaceL(f)$t = u <=== foldRepL(f)(t)$x = (x,u)) &

foldRepL(f)(leaf$x)$y  = (x,leaf$y) &
(foldRepL(f)(t1#t2)$x  = (f(y,z),u1#u2) <=== foldRepL(f)(t1)$x = (y,u1) &
                                             foldRepL(f)(t2)$x = (z,u2)) &

replace(f)$t == get1$mu x u.foldRep(f)(t)(x) &

foldRep(f)(leaf$x)$y == (x,leaf(y)) &
(foldRep(f)(t1)$x = (y,u1) & foldRep(f)(t2)$x = (z,u2)
```

```
                    ==> foldRep(f)(t1#t2)$x == (f(y,z),u1#u2)) &

tree1 == leaf(3)#(leaf(2)#leaf(6)) &

tree2 == (leaf(9)#leaf(3))#leaf(2) &

tree3 == (leaf(9)#leaf(3))#(leaf(2)#leaf(6)) &

-- TREE SORTING

(sortTL$t = u <=== leavesRepL(t)(sort$ls) = (ls,u,s)) &

(leavesRepL(leaf$x)$ls = ([x],leaf$y,s) <=== ls = y:s) &
(leavesRepL(t1#t2)$ls  = (ls1++ls2,u1#u2,s2)
                         <=== leavesRepL(t1)$ls = (ls1,u1,s1) &
                              leavesRepL(t2)$s1 = (ls2,u2,s2)) &

sortT$t == get1$mu ls u s.leavesRep(t)(sort$ls) &

leavesRep(leaf$x)$ls == ([x],leaf$head$ls,tail$ls) &
(leavesRepL(t1)$ls = (ls1,u1,s1) & leavesRepL(t2)$s1 = (ls2,u2,s2)
                ==> leavesRep(t1#t2)$ls == (ls1++ls2,u1#u2,s2)) &

sortTI$t == get1$mu ls u s.leavesRepI(t)(sort$ls)[] &
```

```
leavesRepI(leaf$x)(ls)$acc == (x:acc,leaf$head$ls,tail$ls) &
(leavesRepI(t1)(ls)$acc = (ls1,u1,s1) & leavesRepI(t2)(s1)$ls1 = (ls2,u2,s2)
                ==> leavesRepI(t1#t2)(ls)$acc == (ls2,u1#u2,s2)) &

-- INFINITE OBJECTS

nats == 0:map(+1)$nats &

nats' == mu s.(0:map(+1)$s) &

natsf == mu f.fun(n,n:f$n+1) &             -- nats = natsf$0

fibs  == 1:1:zipWith(+)(fibs)$tail$fibs &  -- simplify breadthfirst/parallel

fibs' == mu s.(1:1:zipWith(+)(s)$tail$s) & -- simplify breadthfirst/parallel

-- CLIENT-SERVER INTERACTION

RequestsL(f)$g = ClientL(g)(0)$map(f)$RequestsL(f)$g &

(ClientL(g)(a)$s = a:ClientL(g)(g$b)$s' <=== s = b:s') &

Requests(f)(g)$a == Client(g)(a)$map(f)$Requests(f)(g)$a &

Client(g)(a)$s == a:Client(g)(g$head$s)$tail$s &
```

```
CSI(f)(g)$a == get0$mu requests client.
                       (client(a)$map(f)$requests,
                        fun(a,fun(s,a:client(g$head$s)$tail$s))) &

-- LAZY MULTIPLICATION

foo(x) == ite(x=0,0,foo(x-1)*foo(x+1)) &
goo(x) == ite(x=0,0,goo(x+1)*goo(x-1))

conjects:

mergesortL[3,2,1,4] = s                          & -- match and narrow
replaceL(min)$tree1 = t                          & -- match and narrow
replaceL(+)$tree1 = t                            & -- match and narrow

palL[2,3,2] = b                                  & -- unify and narrow, simpl
palL[2,3,1,2] = b                                & -- unify and narrow, simpl

sortTL$tree1 = t                                   -- match and narrow, simpl

take(3)$RequestsL(+1)(*2) = s                      -- match and narrow, simpl
                                                   -- 9 steps          (10.2.20)
terms:
```

```
mergesort[3,2,1,4]                                  <+>
mergesortF[3,2,1,4]                                 <+>

replace(min)$tree1                                  <+>
replace(+)$tree1                                    <+>

pal[2,3,2]                                          <+>
               --> bool(3 = head[]) `and` bool(2 = head(tail(tail[])))  (4.2.20)

palI[2,3,2]                                         <+> -- 9 steps          (10.2.20)
palI[2,3,1,2]                                       <+> -- 5 steps          (10.2.20)
palF[2,3,2]                                         <+> -- 10 steps         (10.2.20)
palF[2,3,1,2]                                       <+> -- 9 steps          (10.2.20)

sortT$tree1                                         <+> -- wrong result      (4.2.20)

sortTI$tree1/2                                      <+> -- 11 steps          (4.2.20)
sortTI$tree3                                        <+> -- 12 steps          (5.2.20)

take(3)$nats                                        <+> -- 21 steps         (1.11.17)

take(3)$nats'                                       <+> -- 22 steps         (1.11.17)

take(3)$Requests(+1)(*2)$0                          <+> --> [0,2,6]         (10.2.20)
                                                        -- 30 steps
```

```
take(4)$Requests(+1)(*2)$0                    <+> --> [0,2,6,14]      (10.2.20)
                                                  -- 47 steps
take(5)$Requests(+1)(*2)$0                    <+> --> [0,2,6,14,30]   (10.2.20)
                                                  -- 66 steps
take(6)$Requests(+1)(*2)$0                    <+> --> [0,2,6,14,30,62](10.2.20)
                                                  -- 89 steps
take(3)$CSI(+1)(*2)$0                         <+> --> [0,2,6]          (9.2.20)
                                                  -- 42 steps
foo(3)                                        <+> --> 0                (4.2.20)
                           -- 15 depthfirst, 19 parallel, 27 breadthfirst steps
goo(3)                                            --> 0                (4.2.20)
            -- depthfirst is non-terminating, 19 parallel, 48 breadthfirst steps
```

Further examples

[15], Examples 1 to 5

phil, echo, election, alternating bit protocol, knight, puzzle, lift, message delivery, peterson

Beispiele aus [14]

[14], §4.2: Logische Programme

```
append([],L,L)
append(X:L,L1,X:L2) <== append(L,L1,L2)

sorted([])
sorted(X:[])
sorted(X:Y:L) <== X <= Y /\ sorted(Y:L)

quicksort([],[])
quicksort(Z:L,L3) <== filter(Z,L,Low,High) /\ quicksort(Low,L1) /\
                      quicksort(High,L2) /\ append(L1,Z:L2,L3)

filter(Z,[],[],[]).
filter(Z,X:L,X:Low,High) <== X <= Z /\ filter(Z,L,Low,High)
filter(Z,X:L,Low,X:High) <== X > Z /\ filter(Z,L,Low,High)

perm([],[])
perm(X:L,P) <== perm(L,Q) /\ insert(X,Q,P)

insert(X,Q,X:Q)
insert(X,Y:Q,Y:P) <== insert(X,Q,P)
```

```
        part([],[])
        part(X:L,[X]:Q) <== part(L,Q)
        part(X:L,P) <== part(L,Q) /\ glue(X,Q,P)

        glue(X,[],[X]:[])
        glue(X,L:Q,(X:L):Q)
        glue(X,L:Q,L:P) <== glue(X,Q,P)

Any q:(perm([1,2,3],p) & q++[2] = p)

Narrowing the preceding formula leads to

Any q:(Any q0:(perm([2,3],q0) & insert(1,q0,p)) & q++[2] = p)

Simplifying the preceding formula (2 steps) leads to

Any q0:(perm([2,3],q0) & insert(1,q0,p)) & Any q:(q++[2] = p)

Narrowing the preceding formula leads to

Any q0:(Any q1:(perm([3],q1) & insert(2,q1,q0)) & insert(1,q0,p)) &
Any q:(q++[2] = p)

Narrowing the preceding formula leads to
```

```
Any q0:(Any q1:(Any q2:(perm([],q2) & insert(3,q2,q1)) & insert(2,q1,q0)) &
        insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Narrowing the preceding formula leads to

```
Any q0:(Any q1:(Any q2:(q2 = [] & insert(3,q2,q1)) & insert(2,q1,q0)) &
        insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Simplifying the preceding formula (2 steps) leads to

```
Any q0:(Any q1:(insert(3,[],q1) & insert(2,q1,q0)) & insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Narrowing the preceding formula leads to

```
Any q0:(Any q1:(q1 = 3:[] & insert(2,q1,q0)) & insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Simplifying the preceding formula (3 steps) leads to

```
Any q0:(insert(2,[3],q0) & insert(1,q0,p)) & Any q:(q++[2] = p)
```

```
Narrowing the preceding formula leads to

Any q0:((q0 = 2:[3] | Any p5:(insert(2,[],p5) & q0 = 3:p5)) & insert(1,q0,p)) &
Any q:(q++[2] = p)

Simplifying the preceding formula (6 steps) leads to

insert(1,[2,3],p) & Any q:(q++[2] = p) |
Any q0:(Any p5:(insert(2,[],p5) & q0 = 3:p5) & insert(1,q0,p)) &
Any q:(q++[2] = p)

Narrowing the preceding formula leads to

(p = 1:[2,3] | Any p6:(insert(1,[3],p6) & p = 2:p6)) & Any q:(q++[2] = p) |
Any q0:(Any p5:(insert(2,[],p5) & q0 = 3:p5) & insert(1,q0,p)) &
Any q:(q++[2] = p)

Simplifying the preceding formula (9 steps) leads to

Any p6:(insert(1,[3],p6) & p = 2:p6) & Any q:(q++[2] = p) |
Any q0:(Any p5:(insert(2,[],p5) & q0 = 3:p5) & insert(1,q0,p)) &
Any q:(q++[2] = p)

Narrowing the preceding formula leads to
```

```
Any p6:((p6 = 1:[3] | Any p7:(insert(1,[],p7) & p6 = 3:p7)) & p = 2:p6) &
Any q:(q++[2] = p) |
Any q0:(Any p5:(insert(2,[],p5) & q0 = 3:p5) & insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Simplifying the preceding formula (14 steps) leads to

```
Any p6:(Any p7:(insert(1,[],p7) & p6 = 3:p7) & p = 2:p6) & Any q:(q++[2] = p) |
Any q0:(Any p5:(insert(2,[],p5) & q0 = 3:p5) & insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Narrowing the preceding formula leads to

```
Any p6:(Any p7:(p7 = 1:[] & p6 = 3:p7) & p = 2:p6) & Any q:(q++[2] = p) |
Any q0:(Any p5:(insert(2,[],p5) & q0 = 3:p5) & insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Simplifying the preceding formula (17 steps) leads to

```
Any q0:(Any p5:(insert(2,[],p5) & q0 = 3:p5) & insert(1,q0,p)) &
Any q:(q++[2] = p)
```

Narrowing the preceding formula leads to

```
Any q0:(Any p5:(p5 = 2:[] & q0 = 3:p5) & insert(1,q0,p)) & Any q:(q++[2] = p)
```

```
Simplifying the preceding formula (9 steps) leads to

insert(1,[3,2],p) & Any q:(q++[2] = p)

Narrowing the preceding formula leads to

(p = 1:[3,2] | Any p10:(insert(1,[2],p10) & p = 3:p10)) & Any q:(q++[2] = p)

Simplifying the preceding formula (7 steps) leads to

p = [1,3,2] | Any p10:(insert(1,[2],p10) & p = 3:p10) & Any q:(q++[2] = p)

Narrowing the preceding formula leads to

p = [1,3,2] |
Any p10:((p10 = 1:[2] | Any p11:(insert(1,[],p11) & p10 = 2:p11)) & p = 3:p10) &
Any q:(q++[2] = p)

Simplifying the preceding formula (13 steps) leads to

p = [1,3,2] | p = [3,1,2] |
Any p10:(Any p11:(insert(1,[],p11) & p10 = 2:p11) & p = 3:p10) &
Any q:(q++[2] = p)
```

```
Narrowing the preceding formula leads to

p = [1,3,2] | p = [3,1,2] |
Any p10:(Any p11:(p11 = 1:[] & p10 = 2:p11) & p = 3:p10) & Any q:(q++[2] = p)

Simplifying the preceding formula (17 steps) leads to

p = [1,3,2] | p = [3,1,2]

Number of proof steps: 24
```

[14], §5.1: Partielle Unifikation

```
f(f(x,y),z) = 0

Narrowing the preceding formula leads to

f(f(x,y),0) = 0 & z = 0 | Any y0:(2 = 0 & z = suc(y0))

Narrowing the preceding formula leads to

(f(0,0) = 0 & x = 0 & y = 0 | Any x1:(f(1,0) = 0 & x = suc(x1) & y = 0) |
 Any y1:(f(2,0) = 0 & y = suc(y1))) &
z = 0 |
Any y0:(2 = 0 & z = suc(y0))
```

```
Narrowing the preceding formula leads to

(0 = 0 & x = 0 & y = 0 | Any x1:(f(1,0) = 0 & x = suc(x1) & y = 0) |
 Any y1:(f(2,0) = 0 & y = suc(y1))) &
z = 0 |
Any y0:(2 = 0 & z = suc(y0))

Narrowing the preceding formula leads to

(0 = 0 & x = 0 & y = 0 | Any x1:(1 = 0 & x = suc(x1) & y = 0) |
 Any y1:(f(2,0) = 0 & y = suc(y1))) &
z = 0 |
Any y0:(2 = 0 & z = suc(y0))

Narrowing the preceding formula leads to

(0 = 0 & x = 0 & y = 0 | Any x1:(1 = 0 & x = suc(x1) & y = 0) |
 Any y1:(1 = 0 & y = suc(y1))) &
z = 0 |
Any y0:(2 = 0 & z = suc(y0))

Simplifying the preceding formula (21 steps) leads to

x = 0 & y = 0 & z = 0
```

```
Number of proof steps: 6

Solutions:

x = 0 & y = 0 & z = 0
```

[14], Beispiel 5.1.3

NAT

`sorts`	nat
`constructs`	$0 :\rightarrow nat$
	$suc : nat \rightarrow nat$
`defuncts`	$1, 2, 3, \ldots :\rightarrow nat$
	$+, -, *, min, max : nat \times nat \rightarrow nat$
`preds`	$\leq, \geq, <, >: nat \times nat$
`vars`	$m, n : nat$
`axioms`	$1 \equiv suc(0)$
	$2 \equiv suc(1)$
	$3 \equiv suc(2)$
	$\ldots$
	$n + 0 \equiv n$
	$m + suc(n) \equiv suc(m + n)$
	$n - 0 \equiv n$
	$0 - n \equiv 0$
	$suc(m) - suc(n) \equiv m - n$

$$
\begin{aligned}
& n * 0 \equiv 0 \\
& m * suc(n) \equiv (m * n) + m \\
& min(m, n) \equiv m \;\Leftarrow\; m \leq n \\
& min(m, n) \equiv n \;\Leftarrow\; n \leq m \\
& max(m, n) \equiv n \;\Leftarrow\; m \leq n \\
& max(m, n) \equiv m \;\Leftarrow\; n \leq m \\
& 0 \leq n \\
& suc(m) \leq suc(n) \Leftarrow\; m \leq n \\
& m \geq n \;\Leftarrow\; n \leq m \\
& 0 < suc(n) \\
& suc(m) < suc(n) \Leftarrow\; m < n \\
& m > n \;\Leftarrow\; n < m
\end{aligned}
$$

STACK = NAT `and`

`sorts`	$stack$
`constructs`	$() :\rightarrow 1 + nat$
	$just : nat \rightarrow 1 + nat$
	$empty :\rightarrow stack$
	$push : nat \times stack \rightarrow stack$
`defuncts`	$top : stack \rightarrow 1 + nat$
	$pop : stack \rightarrow stack$
`vars`	$x : nat \quad s : stack$
`Horn axioms`	$top(empty) \equiv ()$
	$pop(empty) \equiv empty$
	$top(push(x, s)) \equiv just(x)$

$$pop(push(x,s)) \equiv s$$

DIVREP = NAT `then`

`sorts`	$tree$
`constructs`	$() :\rightarrow 1 + (nat \times nat)$
	$(\_,\_) : nat \times nat \rightarrow 1 + (nat \times nat)$
	$leaf : nat \rightarrow tree$
	$\_\#\_ : tree \times tree \rightarrow tree$
`defuncts`	$div : nat \times nat \rightarrow 1 + (nat \times nat)$
	$rep\&min : tree \times nat \rightarrow tree \times nat$
	$repByMin : tree \rightarrow tree$
`vars`	$m, n, n', q, r : nat \quad T, T', U, U' : tree$
`Horn axioms`	$div(m,n) \equiv (0,m) \;\Leftarrow\; m < n$
	$div(m,n) \equiv (suc(q), r) \;\Leftarrow\; 0 < n \wedge n \leq m \wedge div(m-n,n) \equiv (q,r)$
	$div(m,0) \equiv ()$
	$rep\&min(leaf(n),m) \equiv (leaf(m),n)$
	$rep\&min(T\#T',m) \equiv (U\#U', min(n,n'))$
	$\Leftarrow\; rep\&min(T,m) \equiv (U,n) \wedge rep\&min(T',m) \equiv (U',n')$
	$repByMin(T) \equiv T' \;\Leftarrow\; rep\&min(T,m) \equiv (T',m)$

$div(m,n)$ berechnet den ganzzahligen Quotienten von m und n und gleichzeitig den verbleibenden Rest.

$rep\&min(T,m)$ liefert den minimalen Eintrag von T sowie einen neuen Baum, der aus T entsteht, wenn man alle Einträge durch m ersetzt. $repByMin(T)$ liefert den Baum, der aus T entsteht, wenn man alle Einträge durch den minimalen Eintrag von T ersetzt.

```
repByMin(leaf(3)#(leaf(2)#leaf(6))) = T

Narrowing the preceding formula leads to

Any T' m:(T' = T & repAndMin(leaf(3)#(leaf(2)#leaf(6)),m) = (T',m))

Narrowing the preceding formula leads to

Any T' m:(T' = T &
          Any U U' n n':((U#U',min(n,n')) = (T',m) &
                         repAndMin(leaf(3),m) = (U,n) &
                         repAndMin(leaf(2)#leaf(6),m) = (U',n')))

Narrowing the preceding formula leads to

Any T' m:(T' = T &
          Any U U' n n':((U#U',min(n,n')) = (T',m) & (leaf(m),3) = (U,n) &
                         repAndMin(leaf(2)#leaf(6),m) = (U',n')))

Narrowing the preceding formula leads to

Any T' m:(T' = T &
          Any U U' n n':((U#U',min(n,n')) = (T',m) & (leaf(m),3) = (U,n) &
                         Any U1 U'1 n1 n'1:((U1#U'1,min(n1,n'1)) = (U',n') &
                                            repAndMin(leaf(2),m) = (U1,n1) &
```

```
                                        repAndMin(leaf(6),m) = (U'1,n'1))))
```

Narrowing the preceding formula leads to

```
Any T' m:(T' = T &
          Any U U' n n':((U#U',min(n,n')) = (T',m) & (leaf(m),3) = (U,n) &
                         Any U1 U'1 n1 n'1:((U1#U'1,min(n1,n'1)) = (U',n') &
                                            (leaf(m),2) = (U1,n1) &
                                            repAndMin(leaf(6),m) = (U'1,n'1))))
```

Narrowing the preceding formula leads to

```
Any T' m:(T' = T &
          Any U U' n n':((U#U',min(n,n')) = (T',m) & (leaf(m),3) = (U,n) &
                         Any U1 U'1 n1 n'1:((U1#U'1,min(n1,n'1)) = (U',n') &
                                            (leaf(m),2) = (U1,n1) &
                                            (leaf(m),6) = (U'1,n'1))))
```

Simplifying the preceding formula (38 steps) leads to

```
leaf(2)#(leaf(2)#leaf(2)) = T
```

[14], Beispiel 6.4.3: MAP implementiert STACK

Keller werden implementiert als Paare, bestehend aus einem Feld (= Funktion mit endlichem Definitionsbereich) und einem Zeiger auf das zuletzt eingetragene Element. Die Kelleroperationen werden entsprechend verfeinert. Wir beginnen mit einer parametrisierten Version der Spezifikation STACK aus [14], Beispiel 5.1.4, die das Komplement der *stack*-Gleichheit enthält (siehe [14], Beispiel 6.3.2):

$SP = \mathrm{STACK}[\mathrm{NEQ}(entry)]$ `where` STACK =

`sorts`	$stack(entry)$
`constructs`	$() :\rightarrow 1 + entry$
	$just : entry \rightarrow 1 + entry$
	$empty :\rightarrow stack(entry)$
	$push : entry \times stack(entry) \rightarrow stack(entry)$
`defuncts`	$top : stack(entry) \rightarrow 1 + entry$
	$pop : stack(entry) \rightarrow stack(entry)$
`preds`	$\not\equiv : stack(entry) \times stack(entry)$
`vars`	$x : entry \quad s : stack(entry)$
`Horn axioms`	$top(empty) \equiv ()$
	$pop(empty) \equiv empty$
	$top(push(x, s)) \equiv just(x)$
	$pop(push(x, s)) \equiv s$
	Axiome für $\not\equiv$ gemäß [14], Satz 6.2.7

$\mathrm{MAP}[\mathrm{NEQ}(index),\mathrm{NEQ}(entry)]$ `where` MAP =

`sorts`	$map(index, entry)$
`constructs`	$index : index \rightarrow index + entry$

	$entry : entry \to index + entry$	
	$new :\to map(index, entry)$	
	$upd : index \times entry \times map(index, entry) \to map(index, entry)$	*"update"*
`defuncts`	$get : map(index, entry) \times index \to index + entry$	
`preds`	$\not\equiv : map(index, entry) \times map(index, entry)$	
	$\not\equiv : (index + entry) \times (index + entry)$	
`vars`	$i, j : index \quad x : entry \quad f : map(index, entry)$	
`axioms`	$get(new, i) \equiv index(i)$	
	$get(upd(i, x, f), i) \equiv entry(x)$	
	$get(upd(i, x, f), j) \equiv get(f, j) \quad \Leftarrow \quad i \not\equiv j$	
	Axiome für $\not\equiv$ gemäß [14], Satz 6.2.7	

Die Herbrandmodelle der Aktualisierungen von MAP[NEQ(*index*),NEQ(*entry*)] liefern leider keine *eindeutigen* Repräsentationen von Feldern. Das ist für die hier behandelte Verwendung von MAP als Verfeinerung aber auch irrelevant. Zu einer adäquateren Modellierung von Feldern vgl. [14], §6.4.

Der Repräsentationsmorphismus *rep* bildet alle SP-Symbole bis auf die folgenden auf sich selbst ab:

$$\begin{aligned}
rep(1) &= nat, \\
rep(stack(entry)) &= map(nat, entry) \times nat, \\
rep(\equiv : stack(entry) \times stack(entry)) &= \sim : (map(nat, entry) \times nat) \times (map(nat, entry) \times nat), \\
rep(\not\equiv : stack(entry \times stack(entry)) &= \not\sim : (map(nat, entry) \times nat) \times (map(nat, entry) \times nat).
\end{aligned}$$

Es folgt die Verfeinerung als Extension einer Aktualisierung des ersten Parameters von MAP[NEQ(*index*),NEQ(*entry*)] durch NAT (siehe [14], Def. 6.3.4):

$SP' = \mathrm{MAP}_{nat/index}[\mathrm{NAT}][\mathrm{NEQ}(entry)]$ `and`

`defuncts` $() :\to nat + entry$

$just : entry \to nat + entry$

$empty :\to map(nat, entry) \times nat$

$push : entry \times (map(nat, entry) \times nat) \to map(nat, entry) \times nat$

$top : map(nat, entry) \times nat \to nat + entry$

$pop : map(nat, entry) \times nat \to map(nat, entry) \times nat$

`preds` $\not\sim : (map(nat, entry) \times nat) \times (map(nat, entry) \times nat)$

`copreds` $\sim : (map(nat, entry) \times nat) \times (map(nat, entry) \times nat)$

`vars` $i, j : nat \quad x : entry \quad f : map(nat, entry) \quad s, s' : map(nat, entry) \times nat$

`axioms` $() \equiv index(0)$

$just(x) \equiv entry(x)$

$empty \equiv (new, 0)$

$push(x, s) \equiv (upd(suc(i), x, f), suc(i)) \quad \Leftarrow \quad s \equiv (f, i)$

$top(f, i) \equiv get(f, i)$

$pop(f, i) \equiv (f, i - 1)$

$s \sim s' \quad \Rightarrow \quad top(s) \equiv top(s') \wedge pop(s) \sim pop(s')$

$s \not\sim s' \quad \Leftarrow \quad top(s) \not\equiv top(s')$

$s \not\sim s' \quad \Leftarrow \quad pop(s) \not\sim pop(s')$

SP und SP' sind orthogonal und terminierend, also nach [14], Kor. 5.2.7 konfluent. Da beide Spezifikationen vollständig sind, sind sie nach [14], Kor. 5.1.15 auch konsistent, also funktional. Demzufolge sind nach [14], Satz 6.2.7 $\not\equiv^{Her(SP)}$ und $\not\equiv^{Her(SP')}$ die Komplemente von $\equiv^{Her(SP)}$ bzw. $\equiv^{Her(SP')}$. Die duale Version von [14], Satze 6.2.6, mit deren Hilfe man Komplemente von Coprädikaten als Prädikate spezifizieren kann, liefert die Komplementeigenschaft von $\not\sim^{Her(SP')}$ bzgl. $\sim^{Her(SP')}$.

Nach [14], Lemma 6.1.2(2) und Korollar 6.4.2 ist SP' eine Verfeinerung von SP entlang rep, wenn die folgenden rep-Bilder der Axiome von SP induktive Theoreme von SP' sind:

$$
\begin{array}{lll}
top(empty) \equiv () & & (1)\\
top(push(x,s)) \equiv just(x) & & (2)\\
pop(empty) \sim empty & & (3)\\
pop(push(x,s)) \sim s & & (4)\\
empty \not\sim push(x,s) & & (5)\\
push(x,s) \not\sim empty & & (6)\\
push(x,s) \not\sim push(y,s') & \Leftarrow \quad x \not\equiv y & (7)\\
push(x,s) \not\sim push(y,s') & \Leftarrow \quad s \not\sim s' & (8)\\
s \sim s & & (9)\\
s \sim s' & \Leftarrow \quad s' \sim s & (10)\\
s \sim s'' & \Leftarrow \quad s \sim s' \wedge s' \sim s'' & (11)\\
push(x,s) \sim push(x,s') & \Leftarrow \quad s \sim s' & (12)\\
top(s) \equiv top(s') & \Leftarrow \quad s \sim s' & (13)\\
pop(s) \sim pop(s') & \Leftarrow \quad s \sim s' & (14)
\end{array}
$$

Da die Relation $\sim$ im Herbrandmodell von SP' als größte Lösung ihres Axioms interpretiert wird und der Äquivalenzabschluss von $\sim^{Her(SP')}$ dieses Axiom ebenfalls erfüllt, gelten die Formeln (9)-(11) in $Her(SP')$.

☞ Führen Sie dieses Argument näher aus!

Die Konjunktion von (13) und (14) ist zum Axiom für $\sim$ äquivalent. Es bleiben also (1)-(8) und (12) zu zeigen. Die folgenden Narrowing-Ableitungen von (1)-(4) und (12) wurden mit Expander2 erstellt.

Die Ableitung von (4) verwendet das Lemma

$$upd(i, x, f), j) \sim (f, j) \quad \Leftarrow \quad i > j, \qquad (15)$$

dessen Beweis wir in [14], Beispiel 7.3.1 führen werden.

```
top(empty) = nothing                                (1)

Narrowing the preceding formula leads to

top((new,0)) = nothing

Narrowing the preceding formula leads to

top((new,0)) = index(0)

Simplifying the preceding formula leads to

top(new,0) = index(0)

Narrowing the preceding formula leads to

get(new,0) = index(0)

Narrowing the preceding formula leads to

index(0) = index(0)
```

```
Simplifying the preceding formula leads to

True
-----------------------------------------------------

top(push(x,s)) = just(x)                          (2)

Narrowing the preceding formula leads to

Any f i:(top((upd(suc(i),x,f),suc(i))) = just(x) & s = (f,i))

Narrowing the preceding formula leads to

Any f i:(top((upd(suc(i),x,f),suc(i))) = entry(x) & s = (f,i))

Simplifying the preceding formula leads to

Any f i:(top(upd(suc(i),x,f),suc(i)) = entry(x) & s = (f,i))

Narrowing the preceding formula leads to

Any f i:(get(upd(suc(i),x,f),suc(i)) = entry(x) & s = (f,i))

Narrowing the preceding formula leads to
```

```
Any f i:((entry(x) = entry(x) | get(f,suc(i)) = entry(x) & suc(i) =/= suc(i)) &
         s = (f,i))

Simplifying the preceding formula (7 steps) leads to

Any f i:(s = (f,i))

Applying the theorem

True ===> Any f i:(s = (f,i))

at position [] of the preceding formula leads to

All s0:(Any f2 i2:(s0 = (f2,i2))) ==> Any f i:(s = (f,i))

Simplifying the preceding formula leads to

True
----------------------------------------------------

pop(empty) ~ empty                                  (3)

Applying the axiom resp. theorem
```

```
empty = (new,0)

at positions [1],[0,0] of the preceding formula leads to

pop((new,0)) ~ (new,0)

Simplifying the preceding formula leads to

pop(new,0) ~ (new,0)

Applying the axiom resp. theorem

pop(f,i) = (f,pred(i))

at position [0] of the preceding formula leads to

(new,pred(0)) ~ (new,0)

Applying the axiom resp. theorem

pred(0) = 0

at position [0,1] of the preceding formula leads to

(new,0) ~ (new,0)
```

```
Simplifying the preceding formula leads to
                                -- Anwendung von (9)
True
---------------------------------------------------

pop(push(x,s)) ~ s                                (4)

Applying the axiom resp. theorem

push(x,(f,i)) = (upd(suc(i),x,f),suc(i))

at position [0,0] of the preceding formula leads to

Any f i:(pop((upd(suc(i),x,f),suc(i))) ~ (f,i) & s = (f,i))

Simplifying the preceding formula leads to

Any f i:(pop(upd(suc(i),x,f),suc(i)) ~ (f,i) & s = (f,i))

Applying the axiom resp. theorem

pop(f,i) = (f,pred(i))

at position [0,0,0] of the preceding formula leads to
```

```
Any f i:((upd(suc(i),x,f),pred(suc(i))) ~ (f,i) & s = (f,i))

Applying the axiom resp. theorem

pred(suc(i)) = i

at position [0,0,0,1] of the preceding formula leads to

Any f i:((upd(suc(i),x,f),i) ~ (f,i) & s = (f,i))

Applying the theorem

(upd(i,x,f),j) ~ (f,j) <=== i > j

at position [0,0] of the preceding formula leads to

Any f i:(suc(i) > i & s = (f,i))

Applying the axiom resp. theorem

suc(i) > i

at position [0,0] of the preceding formula leads to
```

```
Any f i:(True & s = (f,i))

Simplifying the preceding formula leads to

Any f i:(s = (f,i))

Applying the theorem

True ===> Any f i:(s = (f,i))

at position [] of the preceding formula leads to

All s0:(Any f2 i4:(s0 = (f2,i4))) ==> Any f i:(s = (f,i))

Simplifying the preceding formula leads to

True
-----------------------------------------------------

s ~ s' ==> push(x,s) ~ push(x,s')                (12)

Applying the axiom resp. theorem

s ~ s' ===> top(s) = top(s') & pop(s) ~ pop(s')
```

```
at position [1] of the preceding formula leads to

(s ~ s' ==> top(push(x,s)) = top(push(x,s'))) &
(s ~ s' ==> pop(push(x,s)) ~ pop(push(x,s')))

The reducts have been simplified.

Applying the axiom resp. theorem

top(push(x,s)) = entry(x)                        (2)

at position [0,1,0] of the preceding formula leads to

(s ~ s' ==> entry(x) = top(push(x,s'))) &
(s ~ s' ==> pop(push(x,s)) ~ pop(push(x,s')))

The reducts have been simplified.

Applying the axiom resp. theorem

top(push(x,s)) = entry(x)

at position [0,1,1] of the preceding formula leads to

s ~ s' ==> pop(push(x,s)) ~ pop(push(x,s'))
```

```
The reducts have been simplified.

A transitivity axiom at position [1] of the preceding formula leads to
                                                    -- Anwendung von (11)
s ~ s' ==> Any z0:(pop(push(x,s)) ~ z0 & z0 ~ pop(push(x,s')))

Applying the theorem

pop(push(x,s)) ~ s                                  (4)

at position [1,0,0] of the preceding formula leads to

s ~ s' ==> s ~ pop(push(x,s'))

The reducts have been simplified.

A transitivity axiom at position [1] of the preceding formula leads to
                                                    -- Anwendung von (11)
s ~ s' ==> Any z1:(s ~ z1 & z1 ~ pop(push(x,s')))

Applying the theorem

pop(push(x,s)) ~ s                                  (4)
```

```
at position [1,0,1] of the preceding formula leads to

True
```

☞ Zeigen Sie (5)-(11)!

☞ Wie ist der Abstraktionshomomorphismus dieser Verfeinerung definiert (siehe [14], Def. 6.4.1)?

[14], Beispiel 6.4.4: STACK implementiert SYMTAB

Hier geht es um die Implementierung von Symboltabellen, oder allgemeiner: blockstruktierten Abbildungen, durch Keller, deren Elemente Felder im Sinne von MAP (s.o.) sind. Im Unterschied zu Bsp. **??** besteht hier die Abstraktion von der implementierenden hin zur implementierten Spezifikation nicht in der Gleichsetzung von Daten, sondern in einer echten Restriktion des implementierenden Datenbereichs. Zumindest eins seiner Elemente, nämlich der leere Keller, implementiert nichts. Formal ausgedrückt, $Her(SP')_{rep}$ ist hier eine echte Unterstruktur von $Her(SP')|_{rep}$ (siehe Def. **??**, **??** und **??**). Zunächst die (parametrisierte) Symboltabellenspezifikation:

$SP = \text{SYMTAB}[\text{NEQ}(index),\text{NEQ}(entry)]$ `where` SYMTAB =

`sorts`	$symtab(index, entry)$
`constructs`	$() :\rightarrow 1 + entry$
	$just : entry \rightarrow 1 + entry$
	$init :\rightarrow symtab(index, entry)$
	$enter : symtab(index, entry) \rightarrow symtab(index, entry)$
	$add : index \times entry \times symtab(index, entry) \rightarrow symtab(index, entry)$
`defuncts`	$leave : symtab(index, entry) \rightarrow symtab(index, entry)$

$$
\begin{array}{lll}
 & retrieve : symtab(index, entry) \times index \to 1 + entry & \\
\texttt{preds} & \not\equiv : symtab(index, entry) \times symtab(index, entry) & \\
\texttt{vars} & i, j : index \quad x : entry \quad s : symtab(index, entry) & \\
\texttt{Horn axioms} & leave(init) \equiv init & \text{(A)} \\
 & leave(enter(s)) \equiv s & \text{(B)} \\
 & leave(add(i, x, s)) \equiv leave(s) & \\
 & retrieve(init, i) \equiv () & \\
 & retrieve(enter(s), i) \equiv retrieve(s, i) & \\
 & retrieve(add(i, x, s), i) \equiv just(x) & \\
 & retrieve(add(i, x, s), j) \equiv retrieve(s, j) \quad \Leftarrow \quad i \not\equiv j & \\
 & \text{Axiome für } \not\equiv \text{ gemäß [14], Satz 6.2.7} &
\end{array}
$$

Es folgt die Verfeinerung als Extension der Aktualisierung (des Parameters NEQ($entry$)) von STACK[NEQ($entry$)] durch MAP(NEQ($index$),NEQ($entry$)) (siehe [14], Beispiel 6.4.3):

$SP' = \text{STACK}_{map(index,entry)/entry}[\text{MAP}(\text{NEQ}(index),\text{NEQ}(entry))]$ `and`

$$
\begin{array}{ll}
\texttt{defuncts} & init :\to stack(map(index, entry)) \\
 & enter : stack(map(index, entry)) \to stack(map(index, entry)) \\
 & add : index \times entry \times stack \to stack(map(index, entry)) \\
 & leave : stack(map(index, entry)) \to stack(map(index, entry)) \\
 & retrieve : stack(map(index, entry)) \times index \to 1 + entry \\
\texttt{preds} & \not\equiv : stack(map(index, entry)) \times stack(map(index, entry)) \\
\texttt{vars} & i, j : index \quad x : entry \quad f : map(index, entry) \quad s : stack(map(index, entry)) \\
\texttt{axioms} & init \equiv push(new, empty) \\
 & enter(s) \equiv push(new, s) \\
 & add(i, x, empty) \equiv empty
\end{array}
$$

$$
\begin{aligned}
&add(i,x,push(f,s)) \equiv push(upd(i,x,f),s)\\
&leave(empty) \equiv empty\\
&leave(push(f,empty)) \equiv push(f,empty)\\
&leave(push(f,push(g,s))) \equiv push(g,s)\\
&retrieve(empty,i) \equiv ()\\
&retrieve(push(new,s),i) \equiv retrieve(s,i)\\
&retrieve(push(upd(i,x,f),s),i) \equiv just(x)\\
&retrieve(push(upd(i,x,f),s),j) \equiv retrieve(push(f,s),j) \quad \Leftarrow \quad i \not\equiv j\\
&\text{Axiome für } \not\equiv \text{ gemäß [14], Satz 6.2.7}
\end{aligned}
\tag{C}
$$

Wie im Beispiel [14], 6.4.3 sind SP und SP' sind orthogonal und terminierend, also nach Kor. 5.2.7 konfluent. Da beide Spezifikationen vollständig sind, sind sie nach Kor. 5.1.15 auch konsistent, also funktional. Demzufolge sind nach Satz 6.2.7 $\not\equiv^{Her(SP)}$ und $\not\equiv^{Her(SP')}$ die Komplemente von $\equiv^{Her(SP)}$ bzw. $\equiv^{Her(SP')}$.

Der Repräsentationsmorphismus *rep* bildet *symtab* auf *stack* und alle anderen SP-Symbole auf sich selbst ab. Im Gegensatz zu [14], Beispiel 6.4.3 ist hier das Refinement-Modell $Her(SP')_{rep}$ eine echte Restriktion des *rep*-Reduktes von $Her(SP')$: Jeder nur aus Funktionen von SP zusammengesetzte *stack*-Term ist SP'-äquivalent zu einer Normalform der Gestalt $push(f,s)$, repräsentiert also einen nichtleeren Keller. Damit ist nach [14], Lemma 6.1.2(2) und Korollar 6.4.2 SP' eine Verfeinerung von SP entlang *rep*, wenn alle mit *rep* übersetzten Instanzen von SYMTAB-Axiomen, die *stack*-Variablen durch Terme der Form $push(f,s)$ substituieren, induktive Theoreme von SP' sind.

☞ Zeigen Sie, dass $Her(SP')$ Axiom (A) erfüllt! Zeigen Sie, dass $Her(SP')$ Axiom (A) nicht erfüllt, wenn man Axiom (C) durch

$$leave(push(f,empty)) \equiv empty$$

ersetzt!

☞ Welche Terme von $Her(SP')$ enthält $Her(SP')_{rep}$ nicht?

☞ Zeigen Sie mithilfe von [14], Lemma 6.1.2(2), dass $Her(SP')_{rep}$, aber nicht $Her(SP')$, Axiom (B) erfüllt!

☞ Zeigen Sie mithilfe von [14], Lemma 6.1.2(2), dass $Her(SP')_{rep}$ alle Axiome von SP erfüllt!

[14], Beispiel 6.4.5: MAP implementiert HEAP

Ein Heap ist ein binärer Baum, der sich so als Feld darstellen lässt, dass alle Knoteneinträge auf benachbarte Feldelemente abgebildet werden. Die folgende Heapspezifikation stellt neben den Binärbaumkonstru die Funktion *heapify* zur Verfügung, die einen Heap in einen strukturgleichen, aber partiell geordneten Heap umwandelt (vgl. [14], Beispiel 7.4.4):

$SP = \text{HEAP}[\text{ORD}(entry)]$ `where` HEAP =

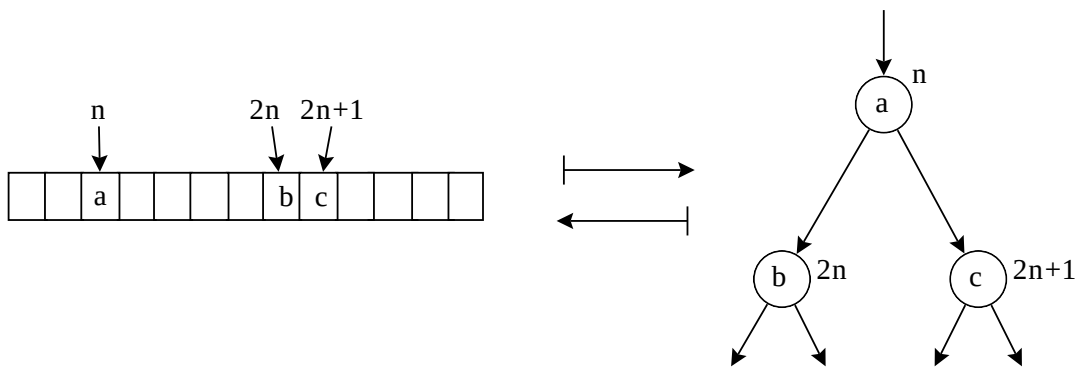
`sorts`	$bintree(entry)$
`constructs`	$mt :\rightarrow bintree(entry)$
	$\_\#\_\#\_ : bintree(entry) \times entry \times bintree(entry) \rightarrow bintree(entry)$
`defuncts`	$heapify, sift : bintree(entry) \rightarrow bintree(entry)$
`vars`	$x, y, z : entry \quad T1, T2, T3, T4 : bintree(entry)$
`Horn axioms`	$heapify(mt) \equiv mt$
	$heapify(T1\#x\#T2) \equiv sift(heapify(T1)\#x\#heapify(T2))$
	$sift(mt) \equiv mt$
	$sift(mt\#x\#mt) \equiv mt\#x\#mt$
	$sift((mt\#y\#mt)\#x\#mt) \equiv (mt\#y\#mt)\#x\#mt \quad \Leftarrow \quad x \leq y$

$$
\begin{aligned}
&\mathit{sift}((m\#y\#mt)\#x\#mt) \equiv (mt\#x\#mt)\#y\#mt \quad \Leftarrow \quad x > y \\
&\mathit{sift}((T1\#y\#T2)\#x\#(T3\#z\#T4)) \equiv (T1\#y\#T2)\#x\#(T3\#z\#T4) \\
&\qquad \Leftarrow \quad x \leq y \wedge x \leq z \\
&\mathit{sift}((T1\#y\#T2)\#x\#(T3\#z\#T4)) \equiv \mathit{sift}(T1\#x\#T2)\#y\#(T3\#z\#T4) \\
&\qquad \Leftarrow \quad x > y \wedge y \leq z \\
&\mathit{sift}((T1\#y\#T2)\#x\#(T3\#z\#T4)) \equiv (T1\#y\#T2)\#z\#\mathit{sift}(T3\#x\#T4) \\
&\qquad \Leftarrow \quad x > z \wedge y > z
\end{aligned}
$$

In den restlichen Fällen ist das Argument von *sift* kein Heap:

$$
\begin{aligned}
&\mathit{sift}(mt\#x\#(T1\#y\#T2)) \equiv mt \\
&\mathit{sift}(((T1\#x\#T2)\#y\#T3)\#z\#mt) \equiv mt \\
&\mathit{sift}((T1\#x\#(T2\#y\#T3))\#z\#mt) \equiv mt
\end{aligned}
$$

sift lässt den Eintrag x der Wurzel eines Heaps so lange in diesen "einsinken", bis er auf einen Knoten trifft, dessen direkter Nachfolger größer als x sind. *heapify*(T) wendet *sift* auf alle Teilbäume von T an und macht T so zum partiell geordneten Baum.



Vom Feld zum Heap und zurück.

Die Refinementspezifikation implementiert zunächst die Heapkonstruktoren durch Feldfunktionen. Die definierten Funktionen *heapify* und *sift* werden dann – modulo dieser Datenstrukturtransformation – durch die gleichen Algorithmen implementiert, die schon in HEAP zur Spezifikation von *heapify* bzw. *sift* verwendet wurden. Dazu benötigen wir noch die Rückübersetzung *heap2map* von Heaps in Felder.

$SP' = \mathrm{MAP}_{nat/index}[\mathrm{NAT}][\mathrm{ORD}(entry)]$ `and`

`defuncts` $mt : (\to nat, entry)map(nat, entry)$

$\_\#\_\#\_ : map(nat, entry) \times entry \times map(nat, entry) \to map(nat, entry)$

$heapify, sift : map(nat, entry) \to map(nat, entry)$

$shift : (nat \to nat) \times map(nat, entry) \to map(nat, entry)$

$\_ + \_ : map(nat, entry) \times map(nat, entry) \to map(nat, entry)$

$g_1, g_2, h_1, h_2 : nat \to nat$

`vars` $i, j, k : nat \quad x, y, z : entry \quad f, f_1, f_2, g : map(nat, entry) \quad h : nat \to nat$

`axioms` $mt \equiv new$

$f_1 \# x \# f_2 \equiv upd(1, x, shift(g_1, f_1) + shift(g_2, f_2))$

$shift(h, new) \equiv new$

$shift(h, upd(i, x, f)) \equiv upd(h(i), x, shift(h, f))$

$new + f \equiv f$

$upd(i, x, f) + g \equiv upd(i, x, f + g)$

$g_1(k) \equiv k + 2^i \quad \Leftarrow \quad k \equiv 2^i + j \ \wedge \ \forall r, s : (k \equiv 2^r + s \Rightarrow r \leq i)$[1]

$g_2(k) \equiv k + 2^{i+1} \quad \Leftarrow \quad k \equiv 2^i + j \ \wedge \ \forall r, s : (k \equiv 2^r + s \Rightarrow r \leq i)$

$heapify(new) \equiv new$

$heapify(upd(i, x, f)) \equiv sift(heapify(shift(h_1, f)) \# x \# heapify(shift(h_2, f)))$

$h_1(k) \equiv k - 2^i \quad \Leftarrow \quad k \equiv 2^{i+1} + j \ \wedge \ \forall r, s : (k \equiv 2^{r+1} + s \Rightarrow r \leq i)$

$h_2(k) \equiv k - 2^{i+1} \quad \Leftarrow \quad k \equiv 2^{i+1} + 2^i + j \ \wedge \ \forall r, s : (k \equiv 2^{r+1} + 2^r + s \Rightarrow r \leq i)$

$sift(new) \equiv new$

$$
\begin{aligned}
&sift(upd(i,x,new)) \equiv upd(i,x,new)\\
&sift(upd(i,x,f)) \equiv upd(i,x,f) \quad \Leftarrow \quad f \equiv upd(2*i,y,new) \wedge x \leq y\\
&sift(upd(i,x,f)) \equiv upd(i,y,upd(2*i,x,new)) \quad \Leftarrow \quad f \equiv upd(2*i,y,new) \wedge x > y\\
&sift(upd(i,x,f)) \equiv upd(i,x,f) \quad \Leftarrow \quad f \equiv upd(2*i,y,upd(2*i+1,z,g)) \wedge x \leq y \wedge x \leq z\\
&sift(upd(i,x,f)) \equiv upd(i,y,upd(2*i+1,z,sift(upd(2*i,x,f))))\\
&\qquad\qquad \Leftarrow \quad f \equiv upd(2*i,y,upd(2*i+1,z,g)) \wedge x > y \wedge y \leq z\\
&sift(upd(i,x,f)) \equiv upd(i,z,upd(2*i,y,sift(upd(2*i+1,x,f))))\\
&\qquad\qquad \Leftarrow \quad f \equiv upd(2*i,y,upd(2*i+1,z,g)) \wedge x > z \wedge y > z
\end{aligned}
$$

"Übersetzung der 3 Fälle, in denen das Argument von *sift* kein Heap ist:

$$
\begin{aligned}
&sift(upd(i,x,f)) \equiv new \quad \Leftarrow \quad get(f,2*i) \equiv index(j) \wedge f \equiv upd(2*i+1,y,g)\\
&sift(upd(i,z,f)) \equiv new\\
&\qquad\qquad \Leftarrow \quad get(f,2*i+1) \equiv index(j) \wedge f \equiv upd(2*i,x,upd(4*i,y,g))\\
&sift(upd(i,z,f)) \equiv new\\
&\qquad\qquad \Leftarrow \quad get(f,2*i+1) \equiv index(j) \wedge f \equiv upd(2*i,x,upd(4*i+1,y,g))
\end{aligned}
$$

Der Repräsentationsmorphismus *rep* bildet *bintree* auf *map* und alle anderen SP-Symbole auf sich selbst ab. Die Implementierungsaxiome wurden ähnlich den in Kapitel 8 abgeleiteten Programmen aus dem Beweis von Anforderungen an sie hergeleitet. Die Anforderungen sind hier die Axiome von HEAP. Sie müssen ja im Refinementmodell $Her(SP')_{rep}$ gelten, damit SP' eine Verfeinerung von HEAP ist.

Wie in [14], Beispiel 6.4.4 gehören auch hier nicht alle Elemente des SP'-Herbrandmodells zum Refinementmodell $Her(SP')_{rep}$: Jeder nur aus definierten Funktionen von SP' zusammengesetzte Grundterm der Sorte *map* ist zu einer Normalform der Gestalt

$$upd(1,x_1,upd(2,x_2,\ldots,upd(n,x_n,new)\ldots))$$

[1] Keine Hornformel, ließe sich aber unter Verwendung einer Hilfssuchfunktion in eine solche umwandeln.

SP'-äquivalent. Damit ist nach [14], Lemma 6.1.2(2) und Korollar 6.4.2 SP' eine Verfeinerung von SP entlang *rep*, wenn alle mit *rep* übersetzten Instanzen von HEAP-Axiomen, die *map*-Variablen solche Normalformen zuordnen, induktive Theoreme von SP' sind.

Der **Heapsort**-Algorithmus überführt zunächst ein Feld mithilfe von *heapify* in die Felddarstellung eines partiell geordneten Baums T. Da dessen Wurzel das kleinste Element unter allen Knoteneinträgen von T enthält, wird dieser Knoten aus T entfernt und zum ersten Element des sortierten Feldes. Das – bzgl. seiner Feldposition – größte Blatt von T wird zur neuen Wurzel von T und mithilfe von *sift* so lange "nach unten geschoben", bis sein Eintrag von Nachfolgereinträgen nicht mehr überschritten wird. Der Baum T hat jetzt einen Knoten weniger und seine Wurzel enthält das zweitkleinste Element des Ausgangsfeldes. Hat T n Knoten, dann terminiert der Algorithmus nach maximal n Iterationen des eben beschriebenen Schrittes mit einer sortierten Permutation des Ausgangsfeldes. Als Erweiterung von SP' lautet der Heapsort-Algorithmus wie folgt:

HEAPSORT $= SP'$ `and`

`constructs` $() :\to 1 + nat$

$just : nat \to 1 + nat$

`defuncts` $heapsort, loop : map(nat, entry) \to map(nat, entry)$

$remove : map(nat, entry) \times nat \to map(nat, entry)$

$maxindex : map(nat, entry) \to 1 + nat$

`axioms` $heapsort(f) \equiv loop(heapify(f))$

$loop(new) \equiv new$

$loop(upd(i, x, new)) \equiv upd(i, x, new)$

$loop(upd(i, x, f)) \equiv upd(i, x, loop(sift(upd(suc(i), y, remove(f, k)))))$

$\Leftarrow \quad maxindex(f) \equiv just(k) \wedge get(f, k) \equiv entry(y)$

$remove(new, i) \equiv new$

$$
\begin{aligned}
&remove(upd(i,x,f),i) \equiv f\\
&remove(upd(i,x,f),j) \equiv upd(i,x,remove(f,j)) \quad \Leftarrow \quad i \not\equiv j\\
&maxindex(new) \equiv ()\\
&maxindex(upd(i,x,f)) \equiv just(i) \quad \Leftarrow \quad maxindex(f) \equiv ()\\
&maxindex(upd(i,x,f)) \equiv just(max(i,k)) \quad \Leftarrow \quad maxindex(f) \equiv just(k)
\end{aligned}
$$

Obwohl *heapsort* hier vollständig spezifiziert ist, genügt es, die Korrektheit des Algorithmus', d.h. die Gültigkeit der Formel

$$sorted(heapsort(f)) \ \wedge \ heapsort(f) \text{ `}ist\ eine\ Permutation\ von\text{' } f$$

nur für die o.g. im Refinementmodell vorkommenden Normalforminstanzen von f zu zeigen. Spezifikationen des Sortiertheitsprädikates und der Permutationsrelation (auf Listen!) findet man in den Beispielen [14], 6.3.2 bzw. 7.2.6.

[14], §6.5: Ausnahmen und Nichtdeterminismus

Die Spezifikation MAP von Feldern (siehe Beispiel [14], 6.4.3) liefert ein Herbrand-Modell, in dem mehrere inäquivalente Grundnormalformen dasselbe Feld repräsentieren. So sind $upd(i,x,f)$ und $upd(i,x,upd(i,y,f))$ nicht MAP-äquivalent, obwohl beide Terme für dasselbe Feld stehen, weil die letzte Änderung von f an der Stelle i alle vorangegangenen Updates dieser Stelle überschreibt. In §6.1 haben wir Coprädikate eingeführt, zunächst um Komplementprädikate zu spezifizieren. Eine Axiomatisierung der MAP-Äquivalenz als Coprädikat löst auch das eben beschriebene Problem. Wir erweitern MAP um das Coprädikat $\sim$: $map \times map$ und das co-Hornaxiom

$$f \sim g \quad \Rightarrow \quad get(f,i) \equiv get(g,i).$$

Die Interpretation von $\sim$ im Herbrandmodell als größte Relation, die dieses Axiom erfüllt (siehe [14],

Def. 6.2.2), führt dazu, dass nicht nur die beiden o.g. Terme äquivalent sind, sondern auch alle anderen Grundnormalformen, die dasselbe Feld repräsentieren. Genauer gesagt, $Her(MAP)$ erfüllt folgende Äquivalenzen:

$$\begin{array}{lcl} upd(i,x,upd(i,y,f)) & \sim & upd(i,x,f) \\ upd(i,x,upd(j,y,f)) & \sim & upd(j,y,upd(i,x,f)) \quad \Leftarrow \quad i \not\equiv j \end{array}$$

Ein anderes Beispiel für eine Klasse von Äquivalenzrelationen, die sich einfach als Coprädikate spezifizieren läßt, sind *starke Äquivalenzen*, die die Identität der Elemente eines Summanden einer zweistelligen Summensorte erhalten, jedoch alle Elemente des anderen Summanden gleichsetzen soll (siehe [14], Def. 6.5.1). Die $index + entry$-Komponente des Abstraktionshomomorphismus in Beispiel [14], 6.4.3 hat z.B. eine starke Äquivalenz als Äquivalenzkern: Er bildet nämlich alle Grundnormalformen der Gestalt $index(t)$ auf $index(0)$ ab, diejenigen der Gestalt $entry(t)$ bleiben jedoch erhalten.

Gleichheitsprädikate werden verwendet, um Funktionen zu axiomatisieren. Umgekehrt werden in den co-Hornformeln, die eine Äquivalenzrelation axiomatisieren, Funktionen verwendet: *top* und *pop* in [14], Beispiel 6.4.3, *get* in MAP und *def* in den Axiomen für $\approx$. Dies ist neben der Komplementspezifizierbarkeit eine weitere Dualität zwischen Prädikaten und Coprädikaten.

[14], Beispiel 6.7.3

REPAL = NAT `and` LIST[TRIV(s)] `and` (siehe [14], 5.1.5 und 6.3.2)

`sorts`	$tree$
`constructs`	$leaf : nat \to tree$
	$\_\#\_ : tree \times tree \to tree$
`defuncts`	$rev : list(s) \to list(s)$
	$pal : list(s) \to bool$
	$rep : tree \times nat \to tree$

	$min{:}tree \to nat$
	$repByMin{:}tree \to tree$
`vars`	$x, y : entry \;\; L, L' : list(s) \;\; m, n : nat \;\; T, T' : tree$
`axioms`	$rev([]) \;\equiv\; []$
	$rev(x : L) \;\equiv\; rev(L) {+\!\!+} [x]$
(A)	$pal(L) \;\equiv\; eq(L, rev(L))$
	$min(leaf(n)) \;\equiv\; n$
	$min(T\#T') \;\equiv\; min(min(T), min(T'))$
	$rep(leaf(m), n) \;\equiv\; leaf(n)$
	$rep(T\#T', n) \;\equiv\; rep(T, n)\#rep(T', n)$
(B)	$repByMin(T) \;\equiv\; rep(T, min(T))$

$rev(L)$ revertiert die Liste L. Das Herbrandmodell von REPAL erfüllt $pal(L) \equiv true$ genau dann, wenn L ein **Palindrom** ist. $rep(T, n)$ ersetzt alle Blatteinträge von T durch n. $min(T)$ berechnet den minimalen Blatteintrag des Baums T. $repByMin(T)$ ersetzt alle Blatteinträge von T durch das Minimum der Blatteinträge von T. Die Axiome für pal bzw. $repByMin$ liefern recht ineffiziente Implementierungen der beiden Funktionen, weil jede Liste bzw. jeder Baum zweimal durchlaufen werden müssen, um das jeweilige Ergebnis zu berechnen. Mithilfe von λ-Abstraktionen lassen sie sich effizienter implementieren:

REPALA = NAT `and` LIST[TRIV(s)] `and`

`sorts`	$tree$
`constructs`	$leaf{:}nat \to tree$
	$\_\#\_{:}tree \times tree \to tree$
`defuncts`	$eq\&rev{:}list(s) \to ((list(s) \to bool) \times list(s))$
	$pal{:}list \to bool$

$$
\begin{array}{ll}
 & rep\&min{:}tree \to ((nat \to tree) \times nat) \\
 & repByMin{:}tree \to tree \\
\texttt{vars} & x, y : s \;\; L, L', L_1 : list(s) \;\; f : list(s) \to bool \;\; k, m, n : nat \;\; T, T' : tree \;\; g, h : nat \to tree \\
\texttt{axioms} & eq\&rev([]) \;\equiv\; (\lambda[].true \blacksquare \lambda x : L.false, []) \\
 & eq\&rev(x : L) \;\equiv\; (\lambda[].false \blacksquare \lambda y : L'.(eq(x, y) \; and \; f(L')), L_1 +\!\!+ [x]) \\
 & \qquad \Leftarrow \;\; eq\&rev(L) \equiv (f, L_1) \\
\text{(A')} & pal(L) \;\equiv\; f(L') \;\;\Leftarrow\;\; eq\&rev(L) \;\equiv\; (f, L') \\
 & rep\&min(leaf(n)) \;\equiv\; (leaf, n) \\
 & rep\&min(T\#T') \;\equiv\; (\lambda k.(g(k)\#h(k)), min(m, n)) \\
 & \qquad \Leftarrow \;\; rep\&min(T) \equiv (g, m) \wedge \; rep\&min(T') \equiv (h, n) \\
\text{(B')} & repByMin(T) \equiv g(n) \;\;\Leftarrow\;\; rep\&min(T) \equiv (g, n)
\end{array}
$$

[14], §7.2: Induktion

Mit Expander2 kann man auch Noethersche Induktionsbeweise wie folgt führen. Nach Auswahl der beweisenden Formel, i.a. eine Implikation $prem \Rightarrow conc$, werden Variablen $x_1, \ldots, x_n$ angeklickt, die damit als Induktionsvariablen deklariert sind. Ausserdem erzeugt das System zwei Lemmas, die Induktionshypothesen entsprechen:

$$conc' \;\Longleftarrow\; (x_1, \ldots, x_n) \gg (x'_1, \ldots, x'_n) \;\wedge\; prem' \tag{4}$$

$$prem' \;\Longrightarrow\; ((x_1, \ldots, x_n) \gg (x'_1, \ldots, x'_n) \;\Rightarrow\; conc') \tag{5}$$

$conc'$ und $prem'$ werden aus $conc$ bzw. $prem$ gebildet, indem jedes Vorkommen einer Induktionsvariablen x durch x' ersetzt wird. Die **Abstiegsbedingung** $(x_1, \ldots, x_n) \gg (x'_1, \ldots, x'_n)$ ist ein Atom, wobei $\gg$ die – ggf. noch unspezifizierte – Induktionsordnung bezeichnet, $x_1, \ldots, x_n$ die ursprünglichen

Induktionsvariablen sind und $x'_1, \ldots, x'_n$ eine Kopie davon, die bei Anwendung des Lemmas auf eine Instanz von *prem* bzw. *conc* instantiiert wird. Die Anwendung von (4) bzw. (5) kann erst dann als Induktionsschritt bezeichnet werden, wenn das Redukt der Anwendung, d.h. die entsprechende Instanz der Prämisse von (4) bzw. Konklusion von (5) bewiesen, also auf *True* reduziert worden ist. Dazu muss die Induktionsordnung $\gg$ axiomatisiert werden, natürlich so, dass ihre Interpretation im Herbrandmodell der jeweiligen Spezifikation wohlfundiert ist.

```
PARTITION = LIST[TRIV(entry)] and
            LIST{list(entry)/entry}[LIST[TRIV(entry)]] then
     defuncts  flatten:list(list(entry))->list(entry)
     preds     part:list*list(list(entry))
     vars      x,y:s  s,s':list(entry)  p:list(list(entry))
     axioms    flatten([]) = []
               flatten(s:p) = s++flatten(p)
               part([x],[[x]])
               part(x:y:s,[x]:p)  <==  part(y:s,p)
               part(x:y:s,(x:s'):p)  <==  part(y:s,s':p)
```

Die Korrektheit (der Axiome von) *part* wird durch folgende Hornformel beschrieben:

$$part(s,p) \quad \Rightarrow \quad s \equiv flatten(p). \tag{6}$$

Wir wählen s als Induktionsvariable (und markieren sie mit einem Ausrufungszeichen), werden also für einen Noetherschen Induktionsbeweis ein Prädikat $\gg: list(entry) \times list(entry)$ brauchen, dessen Interpretation im initialen Modell von PARTITION wohlfundiert ist. Die Induktionshypothesen (4) und (5) lauten hier demnach wie folgt:

```
s' = flatten(p')  <===  part(s',p') & !s >> s'
part(s',p')  ===>  (!s >> s' ==> s' = flatten(p'))
```

Das zweite wird an den Stellen (A) und (B) angewendet. Im jeweils folgenden Schritt wird die eben erzeugte Abstiegsbedingung durch Resolution mit dem (einzigen) Axiom $x : s \gg s$ für die Induktionsordnung sofort wieder eliminiert.

```
part(s,p) ==> s = flatten(p)

Selecting induction variables at position [0,0] of the preceding formula leads to

All p:(part(!s,p) ==> !s = flatten(p))

Applying the axioms

  flatten[] = []
& flatten(s:p) = s++flatten(p)
& part([x],[[x]])
& (part(x:y:s,[x]:p) <=== part(y:s,p))
& (part(x:y:s,x:s':p) <=== part(y:s,s':p))

at positions [0,1,1],[0,0] of the preceding formula leads to

All x:(!s = [x] ==> [] = flatten[]) &
All x y s p0:(!s = x:y:s & part(y:s,p0) ==> y:s = flatten(p0)) &
All x y s s' p0:(!s = x:y:s & part(y:s,s':p0) ==> y:s = s'++flatten(p0))
```

```
The reducts have been simplified.

Applying the axiom resp. theorem

flatten[] = []

at position [0,0,1,1] of the preceding formula leads to

All x y s p0:(!s = x:y:s & part(y:s,p0) ==> y:s = flatten(p0)) &
All x y s s' p0:(!s = x:y:s & part(y:s,s':p0) ==> y:s = s'++flatten(p0))

The reducts have been simplified.

Shifting subformulas at position [0,0,0,1] of the preceding formula leads to

All x y s p0:(!s = x:y:s ==> (part(y:s,p0) ==> y:s = flatten(p0))) &
All x y s s' p0:(!s = x:y:s & part(y:s,s':p0) ==> y:s = s'++flatten(p0))

Applying the induction hypothesis

part(s,p) ===> (!s >> s ==> s = flatten(p))                                (A)

at position [0,0,1,0] of the preceding formula leads to
```

```
All x y s p0:(!s = x:y:s & (x:y:s >> y:s ==> y:s = flatten(p0)) ==>
              y:s = flatten(p0)) &
All x y s s' p0:(!s = x:y:s & part(y:s,s':p0) ==> y:s = s'++flatten(p0))
```

The reducts have been simplified.

Applying the axioms

```
  x:s >> s
& (x >> y <=== x > y)
```

at position [0,0,0,1,0] of the preceding formula leads to

```
All x y s s' p0:(!s = x:y:s & part(y:s,s':p0) ==> y:s = s'++flatten(p0))
```

The reducts have been simplified.

Shifting subformulas at position [0,0,1] of the preceding formula leads to

```
All x y s s' p0:(!s = x:y:s ==> (part(y:s,s':p0) ==> y:s = s'++flatten(p0)))
```

Applying the induction hypothesis

```
part(s,p) ===> (!s >> s ==> s = flatten(p))                              (B)
```

```
at position [0,1,0] of the preceding formula leads to

All x y s s' p0:(!s = x:y:s & (x:y:s >> y:s ==> y:s = flatten(s':p0)) ==>
                 y:s = s'++flatten(p0))

The reducts have been simplified.

Applying the axioms

  x:s >> s
& (x >> y <=== x > y)

at position [0,0,1,0] of the preceding formula leads to

All x y s s' p0:(!s = x:y:s & y:s = flatten(s':p0) ==> y:s = s'++flatten(p0))

The reducts have been simplified.

Applying the axiom resp. theorem

s++flatten(p) = flatten(s:p)

at position [0,1,1] of the preceding formula leads to

True
```

Selbst um einfache Aussagen, deren induktive Gültigkeit schon beim ersten Hinsehen klar ist, formal zu beweisen, sind manchmal nichttriviale Induktionsordnungen erforderlich – wie das folgende Beispiel zeigt (siehe C.-P. Wirth, *Descente Infinie + Deduction*, Logic Journal of the IGPL 12, No. 1 (2004) 1-96, §3.2.2).

```
PQ = NAT and
     preds     P:nat
               Q:nat*nat
               `gt`:(nat*nat)*(nat*nat)        -- lexikographische Ordnung
               >>:(nat*nat*nat)*(nat*nat*nat   -- Induktionsordnung
               Nat:nat
     axioms    P(0)
               P(suc(x))  <==  P(x) /\ Q(x,suc(x))
               Q(x,0)
               Q(x,suc(y))  <==  P(x) /\ Q(x,y)
               (x,y) `gt` (x',y')  <==  x > x'
               (x,y) `gt` (x,y')  <==  y > y'
               (x,y,z) >> (x',y',z')  <==  x > x' /\ x > y'
               (x,y,z) >> (x',y',z')  <==  (y,z) `gt` (x',0) /\
                                           (y,z) `gt` (y',z')
               Nat(0)
               Nat(suc(x)) <== Nat(x)
```

Wir beweisen die Formel

$$Nat(x) \wedge Nat(y) \wedge Nat(z) \quad \Rightarrow \quad p(x) \wedge q(y,z)$$

durch Noethersche Induktion, wobei alle drei Variablen x, y, z als Induktionsvariablen deklariert werden. Das Prädikat Nat schränkt die Instanzen von x, y, z auf Grundterme ein, die zu Normalformen der Sorte *nat* reduziert werden können. Ohne diese Einschränkung wäre die Formel nicht beweisbar.

Die Induktionshypothesen (4) und (5) lauten hier demnach wie folgt:

```
  P(x') & Q(y',z')  <===  Nat(x') & Nat(y') & Nat(z')) & (!x,!y,!z) >> (x',y',z'))
  Nat(x') & Nat(y') & Nat(z')  ===>  (!x,!y,!z) >> (x',y',z') ==> (P(x') & Q(y',z'))
```

Die erste wird an den Stellen (A) und (B) angewendet.

```
Nat(x) & Nat(y) & Nat(z) ==> P(x) & Q(y,z)

Selecting induction variables at positions [0,0,0],[0,1,0],[0,2,0] of the preceding for

Nat(!x) & Nat(!y) & Nat(!z) ==> P(!x) & Q(!y,!z)

Applying the axioms

  Q(x,0)
& (Q(x,suc(y)) <=== P(x) & Q(x,y))
& P(0)
& (P(suc(x)) <=== P(x) & Q(x,suc(x)))
```

```
at positions [1,1],[1,0] of the preceding formula leads to

(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !x = 0 | Any x:(P(x) & Q(x,suc(x)) & !x = suc(x))) &
(Nat(!x) & Nat(!y) & Nat(!z) ==> !z = 0 | P(!y) & Any y:(Q(!y,y) & !z = suc(y)))

The reducts have been simplified.

Applying the induction hypothesis

P(x) & Q(y,z) <=== (Nat(x) & Nat(y) & Nat(z)) & (!x,!y,!z) >> (x,y,z)          (A)

at positions [0,1,1,0,0],[0,1,1,0,1] of the preceding formula leads to

(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !x = 0 |
 Any x:(!x = suc(x) & Nat(x) & Nat(suc(x)) & (suc(x),!y,!z) >> (x,x,suc(x)))) &
(Nat(!x) & Nat(!y) & Nat(!z) ==> !z = 0 | P(!y) & Any y:(Q(!y,y) & !z = suc(y)))

The reducts have been simplified.

Moving up quantifiers at position [1,1,1,1] of the preceding formula leads to

(Nat(!x) & Nat(!y) & Nat(!z) ==>
```

```
 !x = 0 |
 Any x:(!x = suc(x) & Nat(x) & Nat(suc(x)) & (suc(x),!y,!z) >> (x,x,suc(x)))) &
(Nat(!x) & Nat(!y) & Nat(!z) ==> !z = 0 | Any y:(P(!y) & Q(!y,y) & !z = suc(y)))
```

Applying the induction hypothesis

```
P(x) & Q(y,z) <=== (Nat(x) & Nat(y) & Nat(z)) & (!x,!y,!z) >> (x,y,z)          (B)
```

at positions [1,1,1,0,0],[1,1,1,0,1] of the preceding formula leads to

```
(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !x = 0 |
 Any x:(!x = suc(x) & Nat(x) & Nat(suc(x)) & (suc(x),!y,!z) >> (x,x,suc(x)))) &
(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !z = 0 | Nat(!y) & Any y:(!z = suc(y) & Nat(y) & (!x,!y,suc(y)) >> (!y,!y,y)))
```

The reducts have been simplified.

Applying the axiom resp. theorem

```
(x,y,z) >> (x',y',z') <=== x > x' & x > y'
```

at position [0,1,1,0,3] of the preceding formula leads to

```
(Nat(!x) & Nat(!y) & Nat(!z) ==>
```

```
 !x = 0 | Any x:(!x = suc(x) & Nat(x) & Nat(suc(x)) & suc(x) > x)) &
(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !z = 0 | Nat(!y) & Any y:(!z = suc(y) & Nat(y) & (!x,!y,suc(y)) >> (!y,!y,y)))
```

The reducts have been simplified.

Applying the axiom resp. theorem

```
(x,y,z) >> (x',y',z') <=== (y,z) `gt` (x',0) & (y,z) `gt` (y',z')
```

at position [1,1,1,1,0,2] of the preceding formula leads to

```
(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !x = 0 | Any x:(!x = suc(x) & Nat(x) & Nat(suc(x)) & suc(x) > x)) &
(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !z = 0 |
 Nat(!y) &
 Any y:(!z = suc(y) & Nat(y) & (!y,suc(y)) `gt` (!y,0) &
        (!y,suc(y)) `gt` (!y,y)))
```

The reducts have been simplified.

Applying the axioms

```
  ((x,y) `gt` (x',y') <=== x > x')
```

```
& ((x,y) `gt` (x,y') <=== y > y')

at positions [1,1,1,1,0,3],[1,1,1,1,0,2] of the preceding formula leads to

(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !x = 0 | Any x:(!x = suc(x) & Nat(x) & Nat(suc(x)) & suc(x) > x)) &
(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !z = 0 | Nat(!y) & Any y:(!z = suc(y) & Nat(y) & suc(y) > y))

The reducts have been simplified.

Applying the axiom resp. theorem

suc(x) > x

at position [0,1,1,0,3] of the preceding formula leads to

(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !x = 0 | Any x:(!x = suc(x) & Nat(x) & Nat(suc(x)))) &
(Nat(!x) & Nat(!y) & Nat(!z) ==>
 !z = 0 | Nat(!y) & Any y:(!z = suc(y) & Nat(y) & suc(y) > y))

The reducts have been simplified.

Applying the axioms
```

```
  (Nat(suc(x)) <=== Nat(x))
& Nat(0)

at positions [0,1,1,0,2],[0,0,0] of the preceding formula leads to

Nat(!x) & Nat(!y) & Nat(!z) ==>
!z = 0 | Nat(!y) & Any y:(!z = suc(y) & Nat(y) & suc(y) > y)

The reducts have been simplified.

Applying the axiom resp. theorem

suc(x) > x

at position [1,1,1,0,2] of the preceding formula leads to

Nat(!x) & Nat(!y) & Nat(!z) ==> !z = 0 | Nat(!y) & Any y:(!z = suc(y) & Nat(y))

The reducts have been simplified.

Applying the axioms

  Nat(0)
& (Nat(suc(x)) <=== Nat(x))
```

```
at position [0,2] of the preceding formula leads to

True
```

[14], Beispiel 7.2.6: Mergesort

Wir zeigen die Korrektheit eines funktionalen Sortierprogramms (*Sortieren durch Mischen*). Es wird als Extension von LIST'[ORD(entry)] spezifiziert. Die Beweise setzen voraus, dass die Relation $\sim$ ("ist Permutation von") eine Kongruenzrelation ist. Axiome für $\sim$ werden nicht verwendet, aber mehrere Lemmas.

```
MERGESORT = LIST'[ORD(entry)] and
     defuncts  sort:list(nat)->list(entry)
               merge:list(entry)*list(entry)->list(entry)
               split:list(entry)->(list(entry)*list(entry))
     preds     ~:list(entry)*list(entry)
     vars      x,y:s  s,s1,s2,s',s1',s2':list(entry)
     axioms    sort([]) = []
               sort([x]) = [x]
               sort(x:y:s) = merge(s1',s2')
                 <==  split(s) = (s1,s2) & sort(x:s1) = s1'
                                         & sort(y:s2) = s2'
               merge([],s) = s
               merge(s,[]) = s
               merge(x:s1,y:s2) = x:merge(s1,y:s2)  <==  x <= y
```

```
                 merge(x:s1,y:s2) = y:merge(x:s1,s2)  <==  x > y
                 split([]) = ([],[])
                 split([x]) = ([x],[])
                 split(x:y:s) = (x:s1,y:s2)  <==  split(s) = (s1,s2)
lemmas           sorted(s1) & sorted(s2)  ==>  sorted(merge(s1,s2))    (1)
                 split(s) = (s1,s2)  ==>  s ~ s1++s2                   (2)
                 s ~ merge(s1,s2) ==>  s ~ s1++s2                      (3)
                 x:y:(s++s') ~ (x:s)++(y:s')                           (4)
                 s'++x:s ~ x:s++s'                                     (5)
                 x > y ===> y <= x                                     (6)
```

Die Korrektheitsanforderungen an (die Axiome für) *sort* lauten:

$$\begin{aligned} sort(s) \equiv s' \quad &\Rightarrow \quad sorted(s'), \\ sort(s) \equiv s' \quad &\Rightarrow \quad s \sim s'. \end{aligned}$$

In beiden Teilen führt die Anwendung der Fixpunktinduktion zur Einführung eines Prädikats, das die jeweilige Konklusion $\varphi(x)$ (s.o.) wiedergibt. Die automatisch erzeugten (co-Horn-)Axiome (siehe §7.3) für diese Prädikate lauten demnach wie folgt:

```
sort0(s,s')  ===>  sorted(s')    bzw.   sort1(s,s')  ===>  s ~ s'
```

Hier sind die Beweise:

```
    sort(s) = s' ==> sorted(s')

Applying fixpoint induction w.r.t.
```

```
  sort[] = []
& sort[x] = [x]
& (sort(x:(y:s)) = merge(z0,z1)
      <===  split(s) = (s1,s2) & sort(x:s1) = z0 & sort(y:s2) = z1)
```

to the preceding tree leads to the formula

```
All x y s7 z0 z1 s1 s2:
(  sort0([],[])
 & sort0([x],[x])
 & (sort0(x:(y:s7),merge(z0,z1))
       <===  split(s7) = (s1,s2) & sort0(x:s1,z0) & sort0(y:s2,z1)))
```

Simplifying (4 steps) the preceding tree leads to new ones.
The current factor is given by

```
sort0([],[])
```

Narrowing the preceding factor leads to the factor

```
sorted[]
```

Narrowing the preceding factor leads to new ones. The current factor is given by

```
All x: (sort0([x],[x]))

Narrowing the preceding factor leads to the factor

All x2: (sorted[x2])

Narrowing the preceding factor leads to a new formula. The current formula is given by

All x y s7 z0 z1 s1 s2:
(split(s7) = (s1,s2) & sort0(x:s1,z0) & sort0(y:s2,z1)
 ==> sort0(x:(y:s7),merge(z0,z1)))

Applying the axiom

     sort0(s,s') ===> sorted(s')

at positions [0,1],[0,0,2],[0,0,1] of the preceding tree leads to the formula

All x y s7 z0 z1 s1 s2:
(split(s7) = (s1,s2) & sorted(z0) & sorted(z1)  ==>  sorted(merge(z0,z1)))

Applying the theorem

     sorted(s) & sorted(s') ===>  sorted(merge(s,s'))
```

```
at position [0,0] of the preceding tree leads to the formula

All x y s7 z0 z1 s1 s2:
(sorted(merge(z0,z1)) & split(s7) = (s1,s2)  ==>  sorted(merge(z0,z1)))

Simplifying (3 steps) the preceding tree leads to the formula

True

*****************************************************************************

    sort(s) = s' ==> s ~ s'

Applying fixpoint induction w.r.t.

  sort[] = []
& sort[x] = [x]
& (sort(x:(y:s)) = merge(z3,z4)
      <=== split(s) = (s1,s2) & sort(x:s1) = z3 & sort(y:s2) = z4)

to the preceding tree leads to the formula

All x y s7 z3 z4 s1 s2:
(  sort1([],[])
 & sort1([x],[x])
```

```
 & (sort1(x:(y:s7),merge(z3,z4))
       <===  split(s7) = (s1,s2) & sort1(x:s1,z3) & sort1(y:s2,z4)))
```

Applying the axiom

```
     sort1(s,s') ===> s ~ s'
```

at positions [0,2,1,2],[0,2,1,1],[0,2,0],[0,1],[0,0]
of the preceding tree leads to the formula

```
All x y s7 z3 z4 s1 s2:
(  [] ~ []
 & [x] ~ [x]
 & (x:(y:s7) ~ merge(z3,z4)  <===  split(s7) = (s1,s2) & x:s1 ~ z3 & y:s2 ~ z4))
```

Simplifying (13 steps) the preceding tree leads to the formula

```
All x y s7 s1 s2: (split(s7) = (s1,s2) ==> x:(y:s7) ~ merge(x:s1,y:s2))
```

Applying the theorem

```
     split(s) = (s1,s2) ===> s ~ s1++s2
```

at position [0,0] of the preceding tree leads to the formula

```
All x y s7 s8 s9: (s7 ~ s8++s9 ==> x:(y:s7) ~ merge(x:s8,y:s9))

Applying the theorem

     s ~ merge(s1,s2) <=== s ~ s1++s2

at position [0,1] of the preceding tree leads to the formula

All x y s7 s8 s9: (s7 ~ s8++s9 ==> x:(y:s7) ~ (x:s8)++(y:s9))

Simplifying the preceding tree leads to the formula

All x y s7 s8 s9: (s7 ~ s8++s9 ==> x:(y:s7) ~ x:(s8++(y:s9)))

Applying the theorem

x:y:(s1++s2) ~ x:(s1++(y:s2))

at position [0,1] of the preceding tree leads to the formula

All x3 y2 s7 s8 s9: (s7 ~ s8++s9 ==> True)

Simplifying (3 steps) the preceding tree leads to the formula

True
```

☞ *sorted* spezifiert aufsteigende Sortierung. Geben Sie Hornaxiome für ein Prädikat *sortedB* an, das im Herbrandmodell genau dann gilt, wenn die Argumentliste absteigend sortiert ist. Zeigen Sie

$$reverse(L) \equiv L' \quad \Rightarrow \quad (sorted(L) \Rightarrow sortedB(L'))$$

mit Fixpunktinduktion über *reverse* (siehe Bsp. [14], 6.3.2)!

[14], **Beispiel 7.2.7: Quick- und Bubblesort** Zeigen Sie die Korrektheit von *quicksort* und *bubblesort* bzgl. folgender Spezifikationen:

```
QUICKSORT = LIST'[ORD(entry)] and
     defuncts  sort:list(entry)->list(entry)
               smaller,greater:nat->(entry->bool)
     vars      x,y:nat  s,s1,s2:list(entry)
     axioms    sort([]) = []
               sort(x:s) = sort(s1)++x:sort(s2)
                     <==  filter(smaller(x),s) = s1 /\ filter(greater(x),s)
               smaller(x)(y) = true  <==  x <= y
               smaller(x)(y) = false  <==  x > y
               greater(x)(y) = not(smaller(x)(y))

BUBBLESORT = LIST'[ORD(entry)] then
     defuncts  sort:list(entry)->list(entry)
               bubble:list(entry)->list(entry)
     vars      x,y:entry  s,s':list(entry)
     axioms    sort(s) = s'  <==  bubble(s) = s' /\ s = s'
               sort(s) = sort(s')  <==  bubble(s) = s' /\ s =/= s'
```

```
bubble([]) = []
bubble([x]) = [x]
bubble(x:y:s) = x:bubble(y:s)  <==  x <= y
bubble(x:y:s) = y:bubble(x:s)  <==  x > y
```

☞ Optimieren Sie BUBBLESORT durch Erweiterung des Wertebereiches von *bubble* um einen Booleschen Wert (*flag*), der angibt, ob der jeweilige Aufruf von *bubble* die Argumentliste verändert hat! Anstatt die Bedingung $s = s'$ bzw. $s \neq s'$ zu überprüfen, braucht *sort* dann nur den Wert des flags abzufragen. Zeigen Sie, dass Ihr optimiertes *bubble* zum *bubble* von BUBBLESORT äquivalent ist!

[14], Beispiel 7.2.8: Rucksackproblem

Gegeben sind ein Rucksack der Größe b und $n > 0$ Objekte mit Größen $a_1, \ldots, a_n$ und Nutzwerten $c_1, \ldots, c_n$. Gesucht ist eine Teilmenge M von $\{1, \ldots, n\}$ mit $\sum_{i=1}^{n} a_i x_i \leq b$ und maximalem Gesamtnutzen $c(\vec{x}) =_{def} \sum_{i=1}^{n} c_i x_i$. $\vec{x} \in \{0,1\}$ ist der charakteristische Vektor von M, d.h. $x_i = 1 \Leftrightarrow i \in \{0,1\}$. Der Aufruf $knap(n, b)$ des folgenden, leider exponentiellen Haskell-Programms *knap* berechnet $\vec{x}$ und $c(\vec{x})$ rekursiv. Vektoren werden hier als Listen implementiert.

```
knap(1,b) = if a(1) > b then ([0],0) else ([1],c(1))
knap(n,b) = if d1 > d2' then (v1++[0],d1) else (v2++[1],d2')
            where (v1,d1) = knap(n-1,b)
                  (v2,d2) = knap(n-1,b-a(n))
                  d2' = d2+c(n)
```

☞ Zeigen Sie die Korrektheit von *knap* durch Fixpunktinduktion! Die zu beweisende Behauptung lautet als Formel wie folgt:

$$knap(k, a) = \vec{x} \implies (\sum_{i=1}^{n} a_i x_i \leq b \wedge \forall \vec{y} : (\sum_{i=1}^{n} a_i y_i \leq b \implies \sum_{i=1}^{n} c_i y_i \leq \sum_{i=1}^{n} c_i x_i)).$$

☞ Zeigen Sie $unsorted([1, 2, 3, 5, 4])$ erstens mit Coinduktion über *unsorted* und zweitens durch Widerlegung, d.h. durch Ableitung der Formel $sorted([1, 2, 3, 5, 4]) \Rightarrow False$ nach $True$!

[14], Beispiel 7.3.1: MAP implementiert STACK, cont.

```
i > j ==> (upd(i,x,f),j) ~ (f,j)

Applying coinduction w.r.t.

s ~ s' ===> top(s) = top(s') & pop(s) ~ pop(s')

at position [] of the preceding formula leads to

All i j x f:(i > j ==> top(upd(i,x,f),j) = top(f,j)) &
All i j x f:(i > j ==>
             Any i0 j0 x0 f0:(i0 > j0 & pop(upd(i,x,f),j) = (upd(i0,x0,f0),j0) &
                              pop(f,j) = (f0,j0)))

Reducts have been simplified.

Applying the axioms

  pop(f,i) = (f,pred(i))
& top(f,i) = get(f,i)

at positions [1,0,1,0,2,0],[1,0,1,0,1,0],[0,0,1,1],[0,0,1,0]
of the preceding formula leads to
```

```
All i j:(i > j ==> i > pred(j)) &
All i j x f:(i > j ==> get(upd(i,x,f),j) = get(f,j))

Reducts have been simplified.

Applying the axioms

  get(upd(i,x,f),i) = entry(x)
& (get(upd(i,x,f),j) = get(f,j) <=== i =/= j)

at position [1,0,1,0] of the preceding formula leads to

All i j:(i > j ==> i > pred(j))

Reducts have been simplified.

Applying the theorem

i > j ===> i > pred(j)

at position [0,0] of the preceding formula leads to

True
```

[14], Beispiel 7.4.1: Merge

Wir zeigen:

$$sorted(s_1) \wedge sorted(s_2) \quad \Rightarrow \quad sorted(merge(s_1, s_2)). \tag{1}$$

(1) beschreibt eine typische **Invarianzbedingung**: Wird *merge* auf sortierte Listen angewendet, dann ist auch die Ergebnisliste sortiert. Zum Beweis mit Fixpunktinduktion muss (1) generalisiert werden:

$$merge(s_1, s_2) = s \wedge sorted(s_1) \wedge sorted(s_2) \quad \Rightarrow \quad sorted(s) \wedge s \sim s1 + +s2. \tag{2}$$

(2) ist äquivalent zur Konjunktion von Lemma (1) und Lemma (3) aus Beispiel 7.2.6.

```
merge(s1,s2) = s & sorted(s1) & sorted(s2) ==> sorted(s) & s ~ s1++s2

Shifting subformulas at positions [0,1],[0,2] of the preceding formula leads to

merge(s1,s2) = s ==> (sorted(s1) & sorted(s2) ==> sorted(s) & s ~ s1++s2)

Applying fixpoint induction w.r.t.

merge([],s) = s & merge(s,[]) = s &
(merge(x:s,y:s') = x:z3 <=== x <= y & merge(s,y:s') = z3) &
(merge(x:s,y:s') = y:z3 <=== x > y & merge(x:s,s') = z3)

at position [] of the preceding formula leads to

All s x y s' z3:((sorted[] & sorted(s) ==> sorted(s) & s ~ []++s) &
                 (sorted(s) & sorted[] ==> sorted(s) & s ~ s++[]) &
```

```
                 ((sorted(x:s) & sorted(y:s') ==>
                   sorted(x:z3) & x:z3 ~ x:s++y:s') <===
                  x <= y &
                  (sorted(s) & sorted(y:s') ==> sorted(z3) & z3 ~ s++y:s')) &
                 ((sorted(x:s) & sorted(y:s') ==>
                   sorted(y:z3) & y:z3 ~ x:s++y:s') <===
                  x > y &
                  (sorted(x:s) & sorted(s') ==> sorted(z3) & z3 ~ x:s++s')))
```

Simplifying the preceding formula (33 steps) leads to

```
All s x y s' z3:(x <= y & (sorted(s) & sorted(y:s') ==> sorted(z3)) &
                 (sorted(s) & sorted(y:s') ==> z3 ~ s++y:s') & sorted(x:s) &
                 sorted(y:s') ==>
                 sorted(x:z3)) &
All s x y s' z3:(x <= y & (sorted(s) & sorted(y:s') ==> sorted(z3)) &
                 (sorted(s) & sorted(y:s') ==> z3 ~ s++y:s') & sorted(x:s) &
                 sorted(y:s') ==>
                 x:z3 ~ x:s++y:s') &
All s x y s' z3:(x > y & (sorted(x:s) & sorted(s') ==> sorted(z3)) &
                 (sorted(x:s) & sorted(s') ==> z3 ~ x:s++s') & sorted(x:s) &
                 sorted(y:s') ==>
                 sorted(y:z3)) &
All s x y s' z3:(x > y & (sorted(x:s) & sorted(s') ==> sorted(z3)) &
                 (sorted(x:s) & sorted(s') ==> z3 ~ x:s++s') & sorted(x:s) &
```

```
                 sorted(y:s') ==>
                 y:z3 ~ x:s++y:s')
```

Applying the axiom resp. theorem

```
sorted(s) <=== sorted(x:s)
```

at positions [0,0,2,0,0],[0,0,1,0,0] of the preceding formula leads to 4 factors.
The current factor is given by

```
All s x y s' z3:(x <= y & (Any x0:(sorted(x0:s)) & sorted(y:s') ==> sorted(z3)) &
                 (Any x1:(sorted(x1:s)) & sorted(y:s') ==> z3 ~ s++y:s') &
                 sorted(x:s) & sorted(y:s') ==>
                 sorted(x:z3))
```

Simplifying the preceding factors (2 steps) leads to the factor

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & x <= y & sorted(z3) &
                 z3 ~ s++y:s' ==>
                 sorted(x:z3))
```

Applying the axiom resp. theorem

```
sorted(x:s1) <===
sorted(x:s) & sorted(y:s') & x <= y & sorted(s1) & s1 ~ s++y:s'
```

```
at position [0,1] of the preceding factors leads to the factor

All s x y s' z3:(sorted(x:s) & sorted(y:s') & x <= y & sorted(z3) &
                 z3 ~ s++y:s' ==>
                 Any s9 y0 s'0:(sorted(x:s9) & sorted(y0:s'0) & x <= y0 &
                                sorted(z3) & z3 ~ s9++y0:s'0))

Simplifying the preceding factors (2 steps) leads to 3 factors.
The current factor is given by

All s x y s' z3:(x <= y & (sorted(s) & sorted(y:s') ==> sorted(z3)) &
                 (sorted(s) & sorted(y:s') ==> z3 ~ s++y:s') & sorted(x:s) &
                 sorted(y:s') ==>
                 x:z3 ~ x:s++y:s')

Applying the axiom resp. theorem

sorted(s) <=== sorted(x:s)

at position [0,0,2,0,0] of the preceding factors leads to the factor

All s x y s' z3:(x <= y & (sorted(s) & sorted(y:s') ==> sorted(z3)) &
                 (Any x3:(sorted(x3:s)) & sorted(y:s') ==> z3 ~ s++y:s') &
                 sorted(x:s) & sorted(y:s') ==>
```

```
                x:z3 ~ x:s++y:s')
```

Simplifying the preceding factors leads to the factor

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & x <= y &
                (sorted(s) & sorted(y:s') ==> sorted(z3)) & z3 ~ s++y:s' ==>
                x:z3 ~ x:s++y:s')
```

Decomposing the atom at position [0,1] of the preceding factors leads to the factor

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & x <= y &
                (sorted(s) & sorted(y:s') ==> sorted(z3)) & z3 ~ s++y:s' ==>
                z3 ~ s++y:s')
```

Simplifying the preceding factors (2 steps) leads to 2 factors.
The current factor is given by

```
All s x y s' z3:(x > y & (sorted(x:s) & sorted(s') ==> sorted(z3)) &
                (sorted(x:s) & sorted(s') ==> z3 ~ x:s++s') & sorted(x:s) &
                sorted(y:s') ==>
                sorted(y:z3))
```

Applying the theorem

```
x > y ===> y <= x
```

```
at position [0,0,0] of the preceding factors leads to the factor

All s x y s' z3:(y <= x & (sorted(x:s) & sorted(s') ==> sorted(z3)) &
                 (sorted(x:s) & sorted(s') ==> z3 ~ x:s++s') & sorted(x:s) &
                 sorted(y:s') ==>
                 sorted(y:z3))

Applying the axiom resp. theorem

sorted(s) <=== sorted(x:s)

at positions [0,0,2,0,1],[0,0,1,0,1] of the preceding factors leads to the factor

All s x y s' z3:(y <= x & (sorted(x:s) & Any x5:(sorted(x5:s')) ==> sorted(z3)) &
                 (sorted(x:s) & Any x6:(sorted(x6:s')) ==> z3 ~ x:s++s') &
                 sorted(x:s) & sorted(y:s') ==>
                 sorted(y:z3))

Simplifying the preceding factors (2 steps) leads to the factor

All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x & sorted(z3) &
                 z3 ~ x:s++s' ==>
                 sorted(y:z3))
```

Applying the axiom resp. theorem

```
sorted(x:s1) <===
sorted(x:s) & sorted(y:s') & x <= y & sorted(s1) & s1 ~ s++y:s'
```

at position [0,1] of the preceding factors leads to the factor

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x & sorted(z3) &
                 z3 ~ x:s++s' ==>
                 Any s13 y2 s'1:(sorted(y:s13) & sorted(y2:s'1) & y <= y2 &
                                 sorted(z3) & z3 ~ s13++y2:s'1))
```

Simplifying the preceding factors (3 steps) leads to the factor

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x & sorted(z3) &
                 z3 ~ x:s++s' ==>
                 Any s13 y2 s'1:(sorted(y:s13) & z3 ~ s13++y2:s'1 &
                                 sorted(y2:s'1) & y <= y2))
```

Applying the theorem

```
s'++x:s ~ x:s++s'
```

at position [0,0,4] of the preceding factors leads to the factor

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x & sorted(z3) &
                 z3 = s'++x:s ==>
                 Any s13 y2 s'1:(sorted(y:s13) & z3 ~ s13++y2:s'1 &
                                 sorted(y2:s'1) & y <= y2))
```

Simplifying the preceding factors (2 steps) leads to the factor

```
All s x y s':(sorted(x:s) & sorted(y:s') & y <= x & sorted(s'++x:s) ==>
              Any s13 y2 s'1:(sorted(y:s13) & s'++x:s ~ s13++y2:s'1 &
                              sorted(y2:s'1) & y <= y2))
```

Substituting s' for s13 at position [0,1,0,1,0,0] of the preceding factors leads to the factor

```
All s x y s':(sorted(x:s) & sorted(y:s') & y <= x & sorted(s'++x:s) ==>
              Any s13 y2 s'1:(sorted(y:s') & s'++x:s ~ s'++y2:s'1 &
                              sorted(y2:s'1) & y <= y2))
```

Substituting x for y2 at position [0,1,0,1,0,1,0] of the preceding factors leads to the factor

```
All s x y s':(sorted(x:s) & sorted(y:s') & y <= x & sorted(s'++x:s) ==>
              Any s13 y2 s'1:(sorted(y:s') & s'++x:s ~ s'++x:s'1 &
                              sorted(x:s'1) & y <= x))
```

Substituting s for s'1 at position [0,1,0,1,0,1,1] of the preceding factors leads to the factor

```
All s x y s':(sorted(x:s) & sorted(y:s') & y <= x & sorted(s'++x:s) ==>
              Any s13 y2 s'1:(sorted(y:s') & s'++x:s ~ s'++x:s & sorted(x:s) &
                              y <= x))
```

Simplifying the preceding factors (6 steps) leads to a single formula, which is given by

```
All s x y s' z3:(x > y & (sorted(x:s) & sorted(s') ==> sorted(z3)) &
                 (sorted(x:s) & sorted(s') ==> z3 ~ x:s++s') & sorted(x:s) &
                 sorted(y:s') ==>
                 y:z3 ~ x:s++y:s')
```

Applying the theorem

```
x > y ===> y <= x
```

at position [0,0,0] of the preceding formula leads to

```
All s x y s' z3:(y <= x & (sorted(x:s) & sorted(s') ==> sorted(z3)) &
                 (sorted(x:s) & sorted(s') ==> z3 ~ x:s++s') & sorted(x:s) &
                 sorted(y:s') ==>
                 y:z3 ~ x:s++y:s')
```

Applying the axiom resp. theorem

```
sorted(s) <=== sorted(x:s)
```

at position [0,0,2,0,1] of the preceding formula leads to

```
All s x y s' z3:(y <= x & (sorted(x:s) & sorted(s') ==> sorted(z3)) &
                 (sorted(x:s) & Any x10:(sorted(x10:s')) ==> z3 ~ x:s++s') &
                 sorted(x:s) & sorted(y:s') ==>
                 y:z3 ~ x:s++y:s')
```

Simplifying the preceding formula leads to

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x &
                 (sorted(x:s) & sorted(s') ==> sorted(z3)) & z3 ~ x:s++s' ==>
                 y:z3 ~ x:s++y:s')
```

A transitivity axiom at position [0,1] of the preceding formula leads to

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x &
                 (sorted(x:s) & sorted(s') ==> sorted(z3)) & z3 ~ x:s++s' ==>
                 Any z4:(y:z3 ~ z4 & z4 ~ x:s++y:s'))
```

Applying the theorem

```
y:x:s++s' ~ x:s++y:s'
```

at position [0,1,0,1] of the preceding formula leads to

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x &
                 (sorted(x:s) & sorted(s') ==> sorted(z3)) & z3 ~ x:s++s' ==>
                 Any z4:(y:z3 ~ z4 & z4 = y:x:s++s'))
```

Substituting y:x:s++s' for z4 at position [0,1,0,1,1] of the preceding formula leads to

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x &
                 (sorted(x:s) & sorted(s') ==> sorted(z3)) & z3 ~ x:s++s' ==>
                 Any z4:(y:z3 ~ y:x:s++s' & y:x:s++s' = y:x:s++s'))
```

Simplifying the preceding formula (6 steps) leads to

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x &
                 (sorted(x:s) & sorted(s') ==> sorted(z3)) & z3 ~ x:s++s' ==>
                 y:z3 ~ y:x:s++s')
```

Decomposing the atom at position [0,1] of the preceding formula leads to

```
All s x y s' z3:(sorted(x:s) & sorted(y:s') & y <= x &
```

```
                (sorted(x:s) & sorted(s') ==> sorted(z3)) & z3 ~ x:s++s' ==>
                z3 ~ x:s++s')

Simplifying the preceding formula (2 steps) leads to

True

Number of proof steps: 31
```

[14], **Beispiel 7.4.2: 2-3-Bäume** *insert* trägt ein Element in einen balancierten binären Baum ein und rebalanciert anschließend den Ergebnisbaum.

```
2-3-TREE[ORD(s)] and NAT then
    sorts      tree
    constructs  mt:->tree                                    - leerer Baum
                _#_#_:tree*s*tree->tree                      - Wurzel hat 2 Nachfolger
                _#_#_#_:tree*s*tree*s*tree->tree             - Wurzel hat 3 Nachfolger
    defuncts    insert:s*tree->(tree*nat)
                h:tree->nat                                  - Baumhöhe
    preds       balanced:tree
    vars        x,y,z,z':s  T1,T2,T3,T4,T5:tree
    axioms      insert(x,mt) = (mt#x#mt,1)
                insert(x,T1#x#T2) = (T1#x#T2,0)
                insert(x,T1#x#T2#y#T3) = (T1#x#T2#y#T3,0)
                insert(x,T1#y#T2#x#T3) = (T1#y#T2#x#T3,0)
                insert(x,T1#y#T2) = (T3#y#T2,0)  ⇐  x < y ∧ insert(x,T1) = (T3,0)
```

```
insert(x,T1#y#T2) = (T1#y#T3,0)  ⇐  y < x ∧ insert(x,T2) = (T3,0)
insert(x,T1#y#T2#z#T3) = (T4#y#T2#z#T3,0)
                  ⇐  x < y ∧ insert(x,T1) = (T4,0)
insert(x,T1#y#T2#z#T3) = (T4#y#T2#z#T3,0)
                  ⇐  y < x < z ∧ insert(x,T2) = (T4,0)
insert(x,T1#y#T2#z#T3) = (T1#y#T2#z#T4,0)
                  ⇐  z < x ∧ insert(x,T3) = (T4,0)
insert(x,T1#y#T2) = (T3#z#T4#y#T2,0)
                  ⇐  x < y ∧ insert(x,T1) = (T3#z#T4,1)
insert(x,T1#y#T2) = (T1#y#T3#z#T4,0)
                  ⇐  y < x ∧ insert(x,T2) = (T3#z#T4,1)
insert(x,T1#y#T2#z#T3) = (T4#y#(T2#z#T3),1)
                  ⇐  x < y ∧ insert(x,T1) = (T4#z'#T5,1)
insert(x,T1#y#T2#z#T3) = ((T1#y#T4)#z'#(T5#z#T3),1)
                  ⇐  y < x < z ∧ insert(x,T2) = (T4#z'#T5,1)
insert(x,T1#y#T2#z#T3) = ((T1#y#T2)#z#T4,1)
                  ⇐  z < x ∧ insert(x,T3) = (T4#z'#T5,1)
h(mt) = 0
h(T1#x#T2) = max(h(T1),h(T2))+1
h(T1#x#T2#y#T3) = max(h(T1),h(T2),h(T3))+1
balanced(mt)
balanced(T1#x#T2) ⇐  h(T1) = h(T2) ∧ balanced(T1) ∧ balanced(T2)
balanced(T1#x#T2#y#T3)
        ⇐  h(T1) = h(T2) = h(T3) ∧ balanced(T1) ∧ balanced(T2)
           ∧ balanced(T3)
```

☞ Zeigen Sie folgende Invarianzbedingungen durch Fixpunktinduktion über *insert*:

$$\begin{array}{lcl} insert(x,T) \equiv (T',0) \wedge balanced(T) & \Rightarrow & h(T) \equiv h(T') \wedge balanced(T') \\ insert(x,T) \equiv (T',1) \wedge balanced(T) & \Rightarrow & h(T)+1 \equiv h(T') \wedge balanced(T'). \end{array}$$

[14], **Beispiel 7.4.3: AVL-Bäume** *insert* trägt ein Element in einen balancierten binären Baum ein und rebalanciert anschließend den Ergebnisbaum.

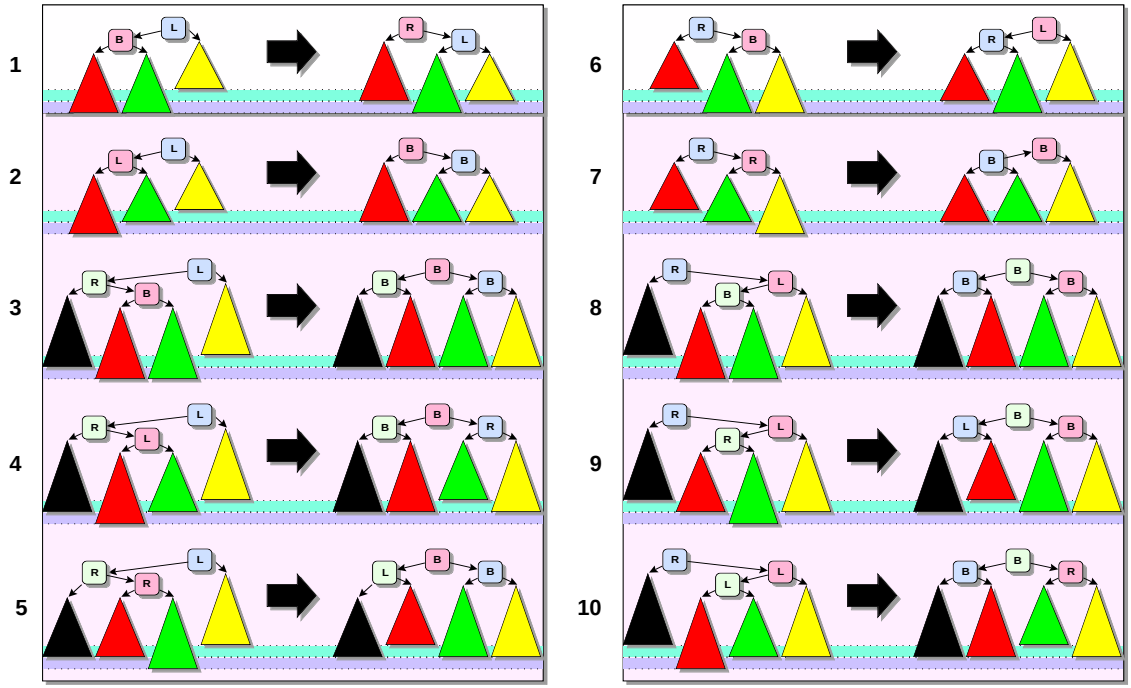
```
AVL-TREE[ORD(s)] = NAT then
    sorts        bintree   nonempty
  constructs     mt:->tree                                  - leerer Baum
                 _:nonempty->tree                           - nichtleerer Baum
                 _#_#_:tree*s*tree->nonempty                - ausgeglichener Baum
                 [_]#_#_:nonempty*s*tree->nonempty          - linkslastiger Baum
                 _#_#[_]:tree*s*nonempty->nonempty          - rechtslastiger Baum
    defuncts     insert:s*tree->nonempty
                 [_]#_#_:nonempty*s*tree->nonempty          - Rotation von links her
                 _#_#[_]:tree*s*nonempty->nonempty          - Rotation von rechts her
                 h:tree->nat                                - Baumhöhe
    preds        balanced:tree
    vars         x,y,z:s  T1,T2,T3,T4:tree
    axioms       insert(x,mt) = mt#x#mt
                 insert(x,T1#x#T2) = T1#x#T2
                 insert(x,[T1]#x#T2) = [T1]#x#T2
                 insert(x,T1#x#[T2]) = T1#x#[T2]
                 insert(x,T1#y#T2) = T3#y#T2  ⇐  x < y ∧ insert(x,T1) = T3
```

$$\begin{array}{lcl}
 & & \wedge\ \mathtt{h(T1)} = \mathtt{h(T3)} \\
\mathtt{insert(x,T1\#y\#T2)} = \mathtt{T1\#y\#T3} & \Leftarrow & \mathtt{y} < \mathtt{x} \wedge \mathtt{insert(x,T2)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} = \mathtt{h(T3)} \\
\mathtt{insert(x,}\boxed{\mathtt{T1}}\mathtt{\#y\#T2)} = \boxed{\mathtt{T3}}\mathtt{\#y\#T2} & \Leftarrow & \mathtt{x} < \mathtt{y} \wedge \mathtt{insert(x,T1)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} = \mathtt{h(T3)} \\
\mathtt{insert(x,}\boxed{\mathtt{T1}}\mathtt{\#y\#T2)} = \boxed{\mathtt{T1}}\mathtt{\#y\#T3} & \Leftarrow & \mathtt{y} < \mathtt{x} \wedge \mathtt{insert(x,T2)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} = \mathtt{h(T3)} \\
\mathtt{insert(x,T1\#y\#}\boxed{\mathtt{T2}}\mathtt{)} = \mathtt{T3\#y\#}\boxed{\mathtt{T2}} & \Leftarrow & \mathtt{x} < \mathtt{y} \wedge \mathtt{insert(x,T1)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} = \mathtt{h(T3)} \\
\mathtt{insert(x,T1\#y\#}\boxed{\mathtt{T2}}\mathtt{)} = \mathtt{T1\#y\#}\boxed{\mathtt{T3}} & \Leftarrow & \mathtt{y} < \mathtt{x} \wedge \mathtt{insert(x,T2)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} = \mathtt{h(T3)} \\
\mathtt{insert(x,T1\#y\#T2)} = \boxed{\mathtt{T3}}\mathtt{\#y\#T2} & \Leftarrow & \mathtt{x} < \mathtt{y} \wedge \mathtt{insert(x,T1)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} \neq \mathtt{h(T3)} \\
\mathtt{insert(x,T1\#y\#T2)} = \mathtt{T1\#y\#}\boxed{\mathtt{T3}} & \Leftarrow & \mathtt{y} < \mathtt{x} \wedge \mathtt{insert(x,T2)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T2)} \neq \mathtt{h(T3)} \\
\mathtt{insert(x,}\boxed{\mathtt{T1}}\mathtt{\#y\#T2)} = \boxed{\mathtt{T3}}\mathtt{\#y\#T2} & \Leftarrow & \mathtt{x} < \mathtt{y} \wedge \mathtt{insert(x,T1)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} \neq \mathtt{h(T3)} \\
\mathtt{insert(x,}\boxed{\mathtt{T1}}\mathtt{\#y\#T2)} = \mathtt{T1\#y\#T3} & \Leftarrow & \mathtt{y} < \mathtt{x} \wedge \mathtt{insert(x,T2)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T2)} \neq \mathtt{h(T3)} \\
\mathtt{insert(x,T1\#y\#}\boxed{\mathtt{T2}}\mathtt{)} = \mathtt{T3\#y\#T2} & \Leftarrow & \mathtt{x} < \mathtt{y} \wedge \mathtt{insert(x,T1)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T1)} \neq \mathtt{h(T3)} \\
\mathtt{insert(x,T1\#y\#}\boxed{\mathtt{T2}}\mathtt{)} = \mathtt{T1\#y\#}\boxed{\mathtt{T3}} & \Leftarrow & \mathtt{y} < \mathtt{x} \wedge \mathtt{insert(x,T2)} = \mathtt{T3} \\
 & & \wedge\ \mathtt{h(T2)} \neq \mathtt{h(T3)}
\end{array}$$

(1) $\boxed{\mathtt{T1\#x\#T2}}\mathtt{\#y\#T3} = \mathtt{T1\#x\#}\boxed{\boxed{\mathtt{T2}}\mathtt{\#y\#T3}}$

(2) $\boxed{\boxed{\mathtt{T1}}\mathtt{\#x\#T2}}\mathtt{\#y\#T3} = \mathtt{T1\#x\#(T2\#y\#T3)}$

```
(3)   T1#x#T2#y#T3#z#T4 = (T1#x#T2)#y#(T3#z#T4)
(4)   T1#x#T2#y#T3#z#T4 = (T1#x#T2)#y#(T3#z#T4)
(5)   T1#x#T2#y#T3#z#T4 = (T1#x#T2)#y#(T3#z#T4)
(6)   T1#x#T2#y#T3 = T1#x#T2#y#T3
(7)   T1#x#T2#y#T3 = (T1#x#T2)#y#T3
(8)   T1#x#T2#y#T3#z#T4 = (T1#x#T2)#y#(T3#z#T4)
(9)   T1#x#T2#y#T3#z#T4 = (T1#x#T2)#y#(T3#z#T4)
(10)  T1#x#T2#y#T3#z#T4 = (T1#x#T2)#y#(T3#z#T4)
      h(mt) = 0
      h(T1#x#T2) = max(h(T1),h(T2))+1
      h(T1#x#T2) = max(h(T1),h(T2))+1
      h(T1#x#T2) = max(h(T1),h(T2))+1
      balanced(mt)
      balanced(T1#x#T2) ⇐ h(T1) = h(T2) ∧ balanced(T1) ∧ balanced(T2)
      balanced(T1#x#T2) ⇐ h(T1) = h(T2)+1 ∧ balanced(T1) ∧ balanced(T2)
      balanced(T1#x#T2) ⇐ h(T1)+1 = h(T2) ∧ balanced(T1) ∧ balanced(T2)
```



Die 10 Rotationsfälle (nach A. Hallmann)

Wieder ist die wesentliche Anforderung an *insert* eine Invarianzbedingung:

$$balanced(T) \quad \Rightarrow \quad balanced(insert(x, T)).$$

☞ Zeigen Sie durch Fixpunktinduktion über *insert* und die beiden Rotationsfunktionen:

$$\begin{array}{rcl}
insert(x,T) \equiv T' & \Rightarrow & (balanced(T) \quad \Rightarrow \quad balanced(T')) \\
\boxed{\text{T1}}\#x\#T2 \equiv T & \Rightarrow & ((balanced(T1) \wedge balanced(T2) \wedge h(T1) \equiv h(T2)+2) \quad \Rightarrow \quad balanced(T)) \\
T1\#x\#\boxed{\text{T2}} \equiv T & \Rightarrow & ((balanced(T1) \wedge balanced(T2) \wedge h(T1)+2 \equiv h(T2)) \quad \Rightarrow \quad balanced(T)).
\end{array}$$

☞ Optimieren Sie – ähnlich der Einfügeoperation bei 2-3-Bäumen (s. Bsp. **??**) – *insert* durch Erweiterung des Wertebereiches der Funktion um einen Booleschen Wert (*flag*), der angibt, ob der jeweilige Aufruf von *insert* die Höhe des Argumentbaumes verändert hat! Anstatt die Bedingungen $h(T1) = h(T3)$, $h(T1) \neq h(T3)$, etc., zu überprüfen, braucht *insert* dann nur den Wert des flags abzufragen. Zeigen Sie, dass Ihr optimiertes *insert* zum *insert* von AVL-TREE äquivalent ist!

☞ Spezifizieren Sie eine Operation $remove\colon s \times nonempty \to tree$, die Einträge entfernt und wie *insert* die Balanciertheit von Bäumen erhält! Da eine Löschoperation die Höhe eines Baumes verringern kann, müssen ggf. Teilbäume mehrfach rotiert werden, was zu rekursiven Aufrufen der Rotationsfunktionen führt.

[14], Beispiel 7.4.4: Baumordnungen

Infixordnung (auch Suchbaumeigenschaft genannt) und partielle Ordnung sind zwei Arten, die Struktur eines binären Baumes auszunutzen, um Einträge zu sortieren und damit die Zugriffszeit zu verkürzen.[2] Für Suchbäume bieten sich Konstruktoren an, die den leeren Baum ausschließen.

```
TREE-ORDERS[ORD(s)] = AVL-TREE[ORD(s)] then
      sorts      searchtree
      constructs leaf:s -> searchtree
                 _&_&_:searchtree*s*searchtree -> searchtree
      defuncts   minT,maxT:searchtree -> s
                 bin:searchtree -> tree
      preds      infix:searchtree
                 parti:tree
```

[2] Alle Axiome mit dem Konstruktor _#_#_ gibt es implizit auch für [_]#_#_ und _#_#[_].

```
axioms     infix(leaf(x))
           infix(T&x&T')  <==  infix(T) /\ maxT(T) <= x /\ x <= minT(T') /\ in
           minT(leaf(x)) = x
           minT(T&x&T') = (min(minT(T),min(x,minT(T'))))
           maxT(leaf(x)) = x
           maxT(T&x&T') = (max(maxT(T),max(x,maxT(T'))))
           bin(leaf(x)) = mt#x#mt
           bin(T&x&T')  = bin(T)#x#bin(T')
           parti(mt)
           parti(mt#x#(T1#y#T2))          <==  x <= y /\ parti(T1#x#T2)
           parti((T1#x#T2)#y#mt)          <==  y <= x /\ parti(T1#x#T2)
           parti((T1#x#T2)#y#(T3#z#T4))   <==  y <= x /\ parti(T1#x#T2) /\
                                               y <= z /\ parti(T3#z#T4)
```

☞ Zeigen Sie, dass die Invarianzbedingung

$$\mathit{infix}(T) \wedge \mathit{insert}(x, \mathit{bin}(T)) \equiv \mathit{bin}(T') \quad \Rightarrow \quad \mathit{infix}(T')$$

ein induktives Theorem von TREE-ORDERS ist, dass also *insert* die Infixordnung erhält!

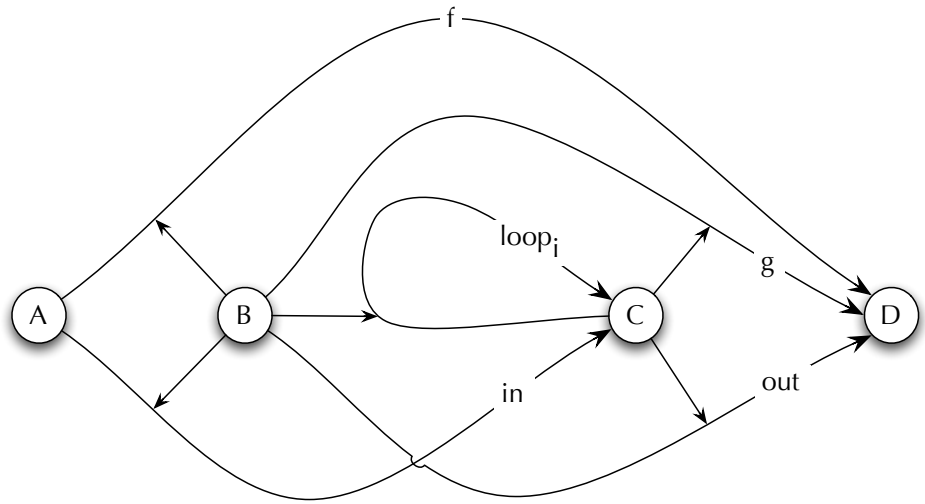
☞ Zeigen Sie, dass auch Ihre spezifizierte Löschfunktion *remove* Suchbäume in Suchbäume überführt!

☞ Zeigen Sie, dass die Invarianzbedingung

$$\mathit{parti}(T1) \wedge \mathit{parti}(T2) \quad \Rightarrow \quad \mathit{parti}(\mathit{sift}(T1\#x\#T2))$$

ein induktives Theorem von HEAP ist (siehe [14], Beispiel 6.4.5)! Folgern Sie daraus die Gültigkeit von $\mathit{parti}(\mathit{heapify}(T))$!

[14], §7.5: Schleifeninvarianten



Die Funktionen eines iterativen Programms: die Ausgangsfunktion f, deren Schleifenfunktion g, die Initialisierung in der Schleifenvariablen, deren mögliche Modifikationen $loop_i$ und die Projektion vom Schleifenbereich C in den Ausgabebereich D. Eine bestimmte Struktur der Axiome für eine Funktion oder ein Prädikat nennt man auch **Programmschema**. Eins der gebräuchlichsten ist das für iterative Programme charakteristische, das aus vier Sorten A, B, C und D, einer Funktion $f : A \times B \to D$, einer **Schleifenfunktion** $g : B \times C \to D$, Hilfsfunktionen $in : A \times B \to C$, $out : B \times C \to D$ und

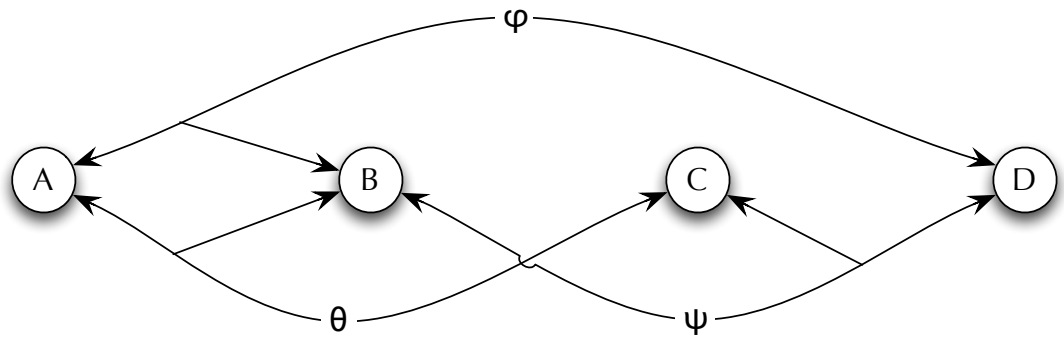
$loop_1, \ldots, loop_n : B \times C \to C$ sowie einer Axiomenmenge der Form

$$\boxed{\begin{array}{l} f(a,b) \equiv g(b, in(a,b)) \\ g(b,c) \equiv g(b, loop_1(b,c)) \;\Leftarrow\; \delta_1(b,c) \\ \ldots \\ g(b,c) \equiv g(b, loop_n(b,c)) \;\Leftarrow\; \delta_n(b,c) \\ g(b,c) \equiv out(b,c) \;\Leftarrow\; \delta(b,c) \end{array}} \tag{1}$$

besteht. Dass (1) tatsächlich einer iterativen Definition entspricht, zeigt die folgende Implementierung der Axiome als Funktionsprozedur mit Schleifenrumpf:

```
function f(a:A,b:B):D;
    begin c:C;
          c:=in(a,b);
          while δ_1(b,c) ∨ ··· ∨ δ_n(b,c)
          do if δ_1(b,c) then c:=loop_1(b,c);
             ...
             if δ_n(b,c) then c:=loop_n(b,c)
          od;
          if δ(b,c) then return out(b,c)
    end
```

Welche Bedingung muss an die Spezifikation, in die (1) eingebettet ist, gestellt werden, damit diese Implementierung von (1) korrekt ist?



Die Relationen eines iterativen Programms: die (gewünschte) Ein/Ausgaberelation φ von f, die Hoare-Invariante θ und die Subgoal-Invariante ψ, die der (gewünschten) Ein/Ausgaberelation der Schleifenfunktion g entspricht.

Der Verifikation von Schleifenprogrammen dienen zwei komplementäre Verfahren: **Hoare-Induktion** und **Subgoal-Induktion**.

Subgoal-Induktion ist Fixpunktinduktion über f (Beweis der E/A-Relation $\varphi(a,b,d)$ von f) mit anschließender Fixpunktinduktion über g (Beweis der **Subgoal-Invarianten**). Die Subgoal-Invariante $\psi(b,c,d)$ setzt Werte der **Schleifenvariablen** b,c mit Werten der **Ausgabevariable** d in Beziehung. Die an ψ gestellten Bedingungen bilden die Konklusion folgender Regel:

Subgoal-Invarianten-Erzeugung

$$\frac{f(a,b) \equiv d \;\Rightarrow\; \varphi(a,b,d)}{\begin{array}{l}\psi(b, in(a,b), d) \;\Rightarrow\; \varphi(a,b,d) \\ \wedge\;\; g(b,c) \equiv d \;\Rightarrow\; \psi(b,c,d)\end{array}} \Uparrow$$

Der zweite Faktor der Konklusion lässt sich dann durch Fixpunktinduktion über g beweisen (siehe §7.2). Die Axiome für g sind durch (1) gegeben.

Dual dazu beschreibt eine **Hoare-Invariante** $\theta(a,b,c)$ einen Zusammenhang zwischen Werten der **Eingabevariablen** a,b und Werten der Schleifenvariable c. Eine Hoare-Invariante θ lässt sich folgendermaßen aus einer Subgoal-Invarianten ψ konstruieren:

$$\theta(a,b,c) \iff_{def} \forall d : (\psi(b,c,d) \Rightarrow \varphi(a,b,d))$$

Umgekehrt liefert jede Hoare-Invariante θ eine Subgoal-Invariante ψ:

$$\psi(b,c,d) \iff_{def} \forall a : (\theta(a,b,c) \Rightarrow \varphi(a,b,d))$$

Mit dieser Beziehung erhält man die Korrektheit Regel zur Erzeugung von Hoare-Invarianten aus der Korrektheit der o.g. Regel zur Erzeugung von Subgoal-Invarianten:

Hoare-Invarianten-Erzeugung

$$\frac{f(a,b) \equiv d \;\Rightarrow\; \varphi(a,b,d)}{\begin{array}{l}\theta(a,b,in(a,b)) \\ \wedge\;\; g(b,c) \equiv d \;\Rightarrow\; (\theta(a,b,c) \Rightarrow \varphi(a,b,d))\end{array}} \Uparrow$$

Der zweite Faktor der Konklusion lässt sich wieder durch Fixpunktinduktion über g beweisen.

Beide Regeln sind natürlich auch dann korrekt, wenn die Axiome für g nicht dem iterativen Schema (1) folgen. f muss allerdings wie in (1) durch das Axiom $f(a,b) \equiv g(b, in(a,b))$ definiert sein.

Beispiel Wir beweisen die Korrektheit eines iterativen Programms zur Zerlegung einer Liste in Teillisten fester Länge mit Hoare-Induktion.

```
PARTN = PARTITION and NAT then
     defuncts  partn:nat*list->list(list)
               g:nat*list*nat*list*list(list)->list(list)
               length:list->nat
```

```
preds      lgOk:nat*list(list)
vars       k,n:nat  x:s  s,s':list  p,p':list(list)
           partn(s,n) = g(n,s,0,[],[])
           g(n,[],k,s,p) = p++[s]
           g(n,x:s,k,s',p) = g(n,s,k+1,s'++[x],p) <=== k < n
           g(n,x:s,k,s',p) = g(n,x:s,0,[],p++[s']) <=== k > n
           length[] = 0
           length(x:s) = length(s)+1
           lgOk(n,[])
           lgOk(n,[s])  <===  length(s) <= n
           lgOk(n,s:s':p)  <===  length(s) = n /\ lgOk(n,s':p)
theorems:  flatten(p++p') = flatten(p)++flatten(p')
           lgOk(n,p++[s]) <=== lgOk(n,p) /\ length(s) <= n
conjects:  partn(s,n) = p ==> s = flatten(p)
           partn(s,n) = p ==> lgOk(n,p)
```

Die beiden letzten Formeln sind die Korrektheitsbedingungen an *partn*. Aus den Argumentsorten von *partn* und g folgt, dass eine Hoare-Invariante den Typ

$$nat \times list \times list \times nat \times list \times list(list)$$

haben muss. Zum Beweis von (3) definieren wir sie durch das Axiom:

$$INV(s, n, s_1, k, s_2, p) \quad \Leftarrow \quad s \equiv flatten(p + +[s_2 + +s_1]).$$

`partn(s,n) = p ==> s = flatten(p)`

```
Hoare invariant creation for the iterative program

partn(s,n) = g(n,s,0,[],[])

at position [] of the preceding formula leads to

INV(s,n,s,0,[],[]) &
(g(n,z3,z4,z5,z6) = p & INV(s,n,z3,z4,z5,z6) ==> s = flatten(p))

Narrowing at position [0] of the preceding formula (3 steps) leads to

True & (g(n,z3,z4,z5,z6) = p & INV(s,n,z3,z4,z5,z6) ==> s = flatten(p))

Reducts have been simplified.

Simplifying the preceding formula leads to

g(n,z3,z4,z5,z6) = p & INV(s,n,z3,z4,z5,z6) ==> s = flatten(p)

Shifting subformulas at position [0,1] of the preceding formula leads to

g(n,z3,z4,z5,z6) = p ==> (INV(s,n,z3,z4,z5,z6) ==> s = flatten(p))

Applying fixpoint induction w.r.t.
```

```
g(n,[],k,s,p) = p++[s] &
(g(n,x:s,k,s',p) = z7 <=== k < n & g(n,s,k+1,s'++[x],p) = z7) &
(g(n,x:s,k,s',p) = z7 <=== k > n & g(n,x:s,0,[],p++[s']) = z7)
```

at position [] of the preceding formula leads to

```
All n k s p x s' z7:(All s6:(INV(s6,n,[],k,s,p) ==> s6 = flatten(p++[s])) &
                     (All s7:(INV(s7,n,x:s,k,s',p) ==> s7 = flatten(z7)) <===
                      k < n &
                      All s7:(INV(s7,n,s,k+1,s'++[x],p) ==> s7 = flatten(z7))) &
                     (All s8:(INV(s8,n,x:s,k,s',p) ==> s8 = flatten(z7)) <===
                      k > n &
                      All s8:(INV(s8,n,x:s,0,[],p++[s']) ==> s8 = flatten(z7))))
```

Applying the axiom resp. theorem

```
INV(s,n,s1,k,s2,p) <=== s = flatten(p++[s2++s1])
```

at positions [0,2,1,1,0,0],[0,2,0,0,0],[0,1,1,1,0,0],[0,1,0,0,0],[0,0,0,0] of the prece
leads to

```
All n k s p x s' z7:(All s6:(s6 = flatten(p++[s++[]]) ==> s6 = flatten(p++[s])) &
                     (All s7:(s7 = flatten(p++[s'++x:s]) ==> s7 = flatten(z7)) <===
                      k < n &
                      All s7:(s7 = flatten(p++[s'++[x]++s]) ==>
```

```
                        s7 = flatten(z7))) &
                    (All s8:(s8 = flatten(p++[s'++x:s]) ==> s8 = flatten(z7)) <===
                     k > n &
                     All s8:(s8 = flatten(p++[s']++[[]++x:s]) ==>
                        s8 = flatten(z7))))
```

Simplifying the preceding formula (29 steps) leads to

```
All n k s p x s' z7:(k > n & flatten(p++[s',x:s]) = flatten(z7) ==>
                    flatten(p++[s'++x:s]) = flatten(z7))
```

Applying the axiom resp. theorem

```
flatten(p++p') = flatten(p)++flatten(p')
```

at positions [0,1],[0,0,1] of the preceding formula leads to

```
All n k s p x s' z7:(k > n & flatten(p)++flatten[s',x:s] = flatten(z7) ==>
                    flatten(p)++flatten[s'++x:s] = flatten(z7))
```

Applying the axiom resp. theorem

```
flatten(s:p) = s++flatten(p)
```

at positions [0,1],[0,0,1] of the preceding formula leads to

```
All n k s p x s' z7:(k > n & flatten(p)++s'++flatten[x:s] = flatten(z7) ==>
                     flatten(p)++s'++x:s++flatten[] = flatten(z7))
```

Applying the axioms

```
  flatten[] = []
& flatten(s:p) = s++flatten(p)
```

at positions [0,1],[0,0,1] of the preceding formula leads to

```
All n k s p x s' z7:(k > n & flatten(p)++s'++x:s++flatten[] = flatten(z7) ==>
                     flatten(p)++s'++x:s++[] = flatten(z7))
```

Applying the axiom resp. theorem

```
flatten[] = []
```

at position [0,0,1] of the preceding formula leads to

```
All n k s p x s' z7:(k > n & flatten(p)++s'++x:s++[] = flatten(z7) ==>
                     flatten(p)++s'++x:s++[] = flatten(z7))
```

Simplifying the preceding formula (4 steps) leads to

```
True

Number of proof steps: 12
```

Zum Beweis der zweiten Bedingung an *partn* definieren wir eine Hoare-Invariante durch das Axiom:

$$INV(s, n, s_1, k, s_2, p) \quad \Leftarrow \quad lgOk(n, p) \wedge k = length(s_2) \wedge k <= n$$

```
partn(s,n) = p ==> lgOk(n,p)

Hoare invariant creation for the iterative program

partn(s,n) = g(n,s,0,[],[])

at position [] of the preceding formula leads to

INV(s,n,s,0,[],[]) &
(g(n,z3,z4,z5,z6) = p & INV(s,n,z3,z4,z5,z6) ==> lgOk(n,p))

Narrowing at position [0] of the preceding formula (2 steps) leads to

True & (g(n,z3,z4,z5,z6) = p & INV(s,n,z3,z4,z5,z6) ==> lgOk(n,p))

Reducts have been simplified.

Simplifying the preceding formula leads to
```

```
g(n,z3,z4,z5,z6) = p & INV(s,n,z3,z4,z5,z6) ==> lgOk(n,p)
```

Shifting subformulas at position [0,1] of the preceding formula leads to

```
g(n,z3,z4,z5,z6) = p ==> (INV(s,n,z3,z4,z5,z6) ==> lgOk(n,p))
```

Applying fixpoint induction w.r.t.

```
g(n,[],k,s,p) = p++[s] &
(g(n,x:s,k,s',p) = z7 <=== k < n & g(n,s,k+1,s'++[x],p) = z7) &
(g(n,x:s,k,s',p) = z7 <=== k > n & g(n,x:s,0,[],p++[s']) = z7)
```

at position [] of the preceding formula leads to

```
All n k s p s4:(INV(s4,n,[],k,s,p) ==> lgOk(n,p++[s])) &
All n k s p x s' z7 s5:(k < n &
                        All s5:(INV(s5,n,s,k+1,s'++[x],p) ==> lgOk(n,z7)) &
                        INV(s5,n,x:s,k,s',p) ==>
                        lgOk(n,z7)) &
All n k s p x s' z7 s6:(k > n &
                        All s6:(INV(s6,n,x:s,0,[],p++[s']) ==> lgOk(n,z7)) &
                        INV(s6,n,x:s,k,s',p) ==>
                        lgOk(n,z7))
```

```
Reducts have been simplified.

Applying the axiom resp. theorem

INV(s,n,s1,k,s2,p) <=== lgOk(n,p) & k = length(s2) & k <= n

at positions [2,0,0,2],[2,0,0,1,0,0],[1,0,0,2],[1,0,0,1,0,0],[0,0,0]
of the preceding formula leads to

All n s p:(length(s) <= n & lgOk(n,p) ==> lgOk(n,p++[s])) &
All n p s' z7:(length(s') <= n & length(s') < n &
               (lgOk(n,p) & length(s')+1 <= n ==> lgOk(n,z7)) & lgOk(n,p) ==>
               lgOk(n,z7))

Reducts have been simplified.

Applying the axiom resp. theorem

lgOk(n,p++[s]) <=== lgOk(n,p) & length(s) <= n

at position [0,0,1] of the preceding formula leads to

All n p s' z7:(length(s') <= n & length(s') < n &
               (lgOk(n,p) & length(s')+1 <= n ==> lgOk(n,z7)) & lgOk(n,p) ==>
               lgOk(n,z7))
```

```
Applying the theorem

k < n ===> k+1 <= n

at position [0,0,1] of the preceding formula leads to

True

Reducts have been simplified.

Number of proof steps: 8
```

Beispiel Wir beweisen die Korrektheit eines iterativen Programms zur Berechnung von Quotient und Rest der Division natürlicher Zahlen.

```
DIVLOOP = NAT then
     defuncts  div:nat->nat*nat
               loop:nat*nat*nat->nat*nat
     preds     INV:nat*nat*nat*nat           -- Hoare-Invariante
     vars      x,y,q,r:nat
     axioms    div(x,y) = loop(y,0,x)
               loop(y,q,r) = (q,r) <=== r < y
               loop(y,q,r) = loop(y,q+1,r-y) <=== r >= y
               INV(x,y,q,r) <=== x = (y*q)+r
```

```
div(x,y) = (q,r) ==> x = (y*q)+r & r < y

Hoare invariant creation for the iterative program

div(x,y) = loop(y,0,x)

at position [] of the preceding formula leads to

INV(x,y,0,x) &
(loop(y,z1,z2) = z0 & INV(x,y,z1,z2) ==> (z0 = (q,r) ==> x = (y*q)+r & r < y))

Narrowing at position [0] of the preceding formula leads to

True &
(loop(y,z1,z2) = z0 & INV(x,y,z1,z2) ==> (z0 = (q,r) ==> x = (y*q)+r & r < y))

Reducts have been simplified.

Simplifying the preceding formula leads to

loop(y,z1,z2) = z0 & INV(x,y,z1,z2) ==> (z0 = (q,r) ==> x = (y*q)+r & r < y)

Shifting subformulas at position [0,1] of the preceding formula leads to

loop(y,z1,z2) = z0 ==>
```

```
(INV(x,y,z1,z2) ==> (z0 = (q,r) ==> x = (y*q)+r & r < y))

Applying fixpoint induction w.r.t.

(loop(y,q,r) = (q,r) <=== r < y) &
(loop(y,q,r) = z3 <=== r >= y & loop(y,q+1,r-y) = z3)

at position [] of the preceding formula leads to

All y q r z3:((All x q0 r0:(INV(x,y,q,r) ==>
                            ((q,r) = (q0,r0) ==> x = (y*q0)+r0 & r0 < y)) <===
               r < y) &
              (All x q1 r1:(INV(x,y,q,r) ==>
                            (z3 = (q1,r1) ==> x = (y*q1)+r1 & r1 < y)) <===
               r >= y &
               All x q1 r1:(INV(x,y,q+1,r-y) ==>
                            (z3 = (q1,r1) ==> x = (y*q1)+r1 & r1 < y))))

Applying the axiom resp. theorem

INV(x,y,q,r) <=== x = (y*q)+r

at positions [0,1,1,1,0,0],[0,1,0,0,0],[0,0,0,0,0] of the preceding formula leads to

All y q r z3:((All x q0 r0:(x = (y*q)+r ==>
```

```
                    ((q,r) = (q0,r0) ==> x = (y*q0)+r0 & r0 < y)) <===
          r < y) &
         (All x q1 r1:(x = (y*q)+r ==>
                    (z3 = (q1,r1) ==> x = (y*q1)+r1 & r1 < y)) <===
          r >= y &
          All x q1 r1:(x = (y*(q+1))+(r-y) ==>
                    (z3 = (q1,r1) ==> x = (y*q1)+r1 & r1 < y))))
```

Simplifying the preceding formula (57 steps) leads to

```
All y q r q1 r1:(r >= y & All q1 r1:((y*(q+1))+(r-y) = (y*q1)+r1) &
                 All r1:(r1 < y) ==>
                 (y*q)+r = (y*q1)+r1)
```

Applying the axiom resp. theorem

```
(x*(y+1))+(z-x) = (x*y)+z
```

at position [0,0,1,0] of the preceding formula leads to

```
All y q r q1 r1:(r >= y & All q1 r1:((y*q)+r = (y*q1)+r1) & All r1:(r1 < y) ==>
                 (y*q)+r = (y*q1)+r1)
```

Simplifying the preceding formula (2 steps) leads to

```
True

Number of proof steps: 9
```

Beispiele

Es folgen einige z.T. nichttriviale iterative Programme mit ihrer jeweiligen Korrektheitsbedingung φ und Hoare-Invarianten θ. Hilfsfunktionen werden nicht explizit spezifiziert. Jede vom Programm nicht veränderte gemeinsame Komponente von Eingabe- und Schleifenvariable taucht nur einmal als Argument von θ auf.

Summe

```
sum(x) = sum1(x,0)
sum1(y,z) = if y > 0 then sum1(y-1,y+z) else z
```

$\varphi(x,z) \iff_{def} z = \sum_{i=1}^{x} i.$

$\theta(x,y,z) \iff_{def} \sum_{i=1}^{x} i = (\sum_{i=1}^{y} i) + z.$

Fakultät

```
fact(x) = fact1(x,1)
fact1(y,z) = if y > 0 then fact1(y-1,y*z) else z
```

$\varphi(x,z) \iff_{def} z = x!.$

$\theta(x,y,z) \iff_{def} x! = y! * z.$

Fibonacci-Zahlen

```
fib(x) = fib1(x,0,1)
fibl1(x,y,z) = if x > 0 then fib1(x-1,z,y+z) else y
```

$\varphi(x,z) \iff_{def}$ z ist die x-te Fibonacci-Zahl.

$\theta(n,x,y,z) \iff_{def}$ $n \geq x$ und y ist die $(n-x)$-te und z die $(n-x+1)$-te Fibonacci-Zahl.

Ganzzahlige Wurzel

```
root(x) = root1(x,0,1)
root1(x,y,z) = if x >= z then root1(x,y+1,z+2y+3) else y
```

$\varphi(x,z) \iff_{def} z = \lceil\sqrt{x}\rceil$.

$\theta(x,y,z) \iff_{def} x \geq y^2 \wedge z = (y+1)^2$.

Größter gemeinsamer Teiler

```
gcd(x,y) = gcd1(x,y)
gcd1(x,y) = if x > y then gcd1(x-y,y) else if x < y then gcd1(x,y-x) else y
```

$\varphi(x,y,z) \iff_{def} z = ggT(x,y)$.

$\theta(m,n,x,y) \iff_{def} ggT(m,n) = ggT(x,y)$.

Quotient und Rest (Beispiel **??**)

```
div(x,y) = div1(y,0,x)
div1(y,q,r) = if r >= y then div1(y,q+1,r-y) else (q,r)
```

$\varphi(x,y,q,r) \iff_{def} x = y * q + r \wedge r < y$.

$\theta(x,y,q,r) \iff_{def} x = y * q + r$.

Kürzeste Wege in Graphen des Typs nat $\times$ nat $\rightarrow$ nat nach Floyd und Warshall (vgl. [12], §6.2)

```
minGraph(G) = f(G,1)
f(G,i) = if i <= N then f(λ(j,k).min(G(j,k),G(j,i)+G(i,k)),i+1) else G
```

$\varphi(G,G') \iff_{def}$ G und G' sind markierte gerichtete Graphen mit der Knotenmenge $\{1,\ldots,N\}$. Jede Kante (j,k) von G' ist markiert mit der Länge des kürzesten Weges, der in G von j nach k führt.

$\theta(G,G',i) \iff_{def}$ G und G' sind markierte gerichtete Graphen mit der Knotenmenge $\{1,\ldots,N\}$. Jede Kante (j,k) von G' ist markiert mit der Länge des kürzesten Weges, der in G von j nach k führt und nur Knoten der Menge $\{1,\ldots,i-1\}$ passiert.

Tiefensuche in Graphen des Typs $s \to list(s)$ (vgl. [12], §6.3)

```
search(G,L) = search1(G,L,[])
search1(G,[],V) = V
search1(G,x:L,V) = if x in V then search1(G,L,V) else search1(G,G(x)++L,x:V)
```

$\varphi(G, L, V) \iff_{def}$ V ist eine Liste aller (von einem Knoten) von L aus erreichbaren Knoten vo

$\theta(G, L, L', V) \iff_{def}$ Jeder von L aus erreichbare Knoten von G liegt in V oder ist von L' aus au einem Weg erreichbar, der keinen Knoten von V passiert.

Eine Subgoal-Invariante ist hier leichter zu finden:

$\psi(G, L, V, V') \iff_{def}$ V' besteht aus den Knoten von V und den von $L \setminus V$ aus erreichbaren Knoten von G.

Minimaler Spannbaum nach Kruskal

```
KRUSKAL = LIST{nat/s}[NAT] and ORD(label) and LIST{edge/s}[NAT]
                           and LIST{list(nat)/s}[LIST{nat/s}[NAT]] then
   sorts      edge = nat*label*nat
              graph = list(edge)
              partition = list(list(nat))
   defuncts   maxnode:->nat
              firstPart:->partition
              spanTree:graph->graph
              spanTree1:graph*partition*graph->graph
   preds      _<=_:edge*edge
   vars       i,j,m,n:nat  x,y:label  G,T:graph  L,L',L1,L2:list(nat)  P:partition
   axioms     (i,x,j) <= (m,y,n)  <==  x <= y
```

```
firstPart = [[1],...,[maxnode]]
spanTree(G) = spanTree1(sort(G),firstPart,[])
spanTree1([],L:L':P,T) = []     -- Hier ist G ist unzusammenhaengend.
spanTree1(G,[L],T) = T
spanTree1((i,x,j):G,L:L':P,T)
 = if L1 = L2 then spanTree1(G,L:L':P,T)
              else spanTree1(G,(L1++L2):remove(L1,remove(L2,L:L':P)),(
   <==  i in L1 /\ j in L2
```

$\varphi(G,T) \quad \Longleftrightarrow_{def}$ G ist ein markierter Graph mit der Knotenmenge $\{1,\ldots,N\}$ und T ist ein minimaler Spannbaum von G.

$\theta(G,G',P,T) \quad \Longleftrightarrow_{def}$ G und G' sind markierte Graphen mit der Knotenmenge $\{1,\ldots,N\}$ und P enthält alle Knoten von G. Jede Kante von $G \setminus G'$ verbindet zwei Knoten der Liste von P.
Für alle Knotenlisten L von P ist der von L erzeugte Teilgraph von T ein min Spannbaum des von L erzeugten Teilgraphen von G.

Optimaler Binärcode nach Huffman

```
BINTREE = LIST[TRIV(s)] then
  sorts       tree
  constructs  leaf:list(s)->tree
              _#_#_:tree*list(s)*tree->tree
  defuncts    root:tree->list(s)
  vars        T,T':tree  cL:list(s)
  axioms      root(T#cL#T') = cL
```

```
HUFFMAN = LIST{tree/s}[BINTREE] and NAT then
  defuncts    minOf2:(s->nat)*tree*tree->tree
              minOfList:(s->nat)*list(tree)->tree
              firstList:list(s)->list(tree)
              newList:list(s)*list(tree)->list(tree)
              Huffman:(s->nat)*list(s)->tree
              Huffman1:(s->nat)*list(tree)->tree
  vars        T,T',T1,T2:tree  fr:s->nat  c:s  cL:list(s)  TL:list(tree)
  axioms      minOf2(fr,T,T') = if sum(map(fr)(root(T))) <= sum(map(fr)(root(T')))
                                 then T else T'
              minOfList(fr,T:TL) = if TL = [] then T
                                      else minOf2(fr,T,minOfList(fr,TL))
              firstList([]) = []
              firstList(c:cL) = [c]:firstList(cL)
              newList(fr,T:TL) = (T1#(root(T1)++root(T2))#T2):remove(T2,TL')
                                <==  minOfList(fr,T:TL) = T1 /\
                                     remove(T1,T:TL) = TL' /\
                                     minOfList(fr,TL') = T2
              Huffman(fr,cL) = Huffman1(fr,firstList(cL))
              Huffman1(fr,T:TL) = if TL = [] then T
                                        else Huffman1(fr,newList(fr,T:TL))
```

$\varphi(fr, cL, T) \iff_{def}$ T ist ein Codebaum so, dass für alle Zeichen $c, c' \in cL$ und Wege w, w' von der Wurzel von T zu dem Blatt,

das mit c bzw. c' markiert ist, gilt:
$fr(c) < fr(c') \;\Rightarrow\; length(w) \geq length(w')$.

$\theta(fr, cL, TL) \iff_{def}$ TL ist eine Codebaumliste derart, dass für alle Zeichen $c, c' \in cL$ und Wege w, w' von der Wurzel der Bäume T, T' zu dem Blatt, das mit c bzw. c' markiert ist, gilt:
$fr(c) < fr(c') \Rightarrow\; length(w) \geq length(w')$,
$fr(c) < fr(c') \wedge length(w) = length(w') \wedge T \neq T'$
$\Rightarrow\; \sum_{c'' \in root(T)} fr(c'') \;<\; \sum_{c'' \in root(T')} fr(c'')$.

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