An Invitation to General Algebra and Universal Constructions

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Berkeley comments: Copies may be purchased at $5 discount from my office.
There are also two copies in the library, 100 Evans Hall, call number QA251 .B47 1998.

Note on the PostScript files below: When a chapter begins on an even-numbered page, the file below begins with a blank page, for convenience in 2-sided printing. So if you display a file and see a blank page, just page forward to see the text.

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(*Further topics, some of which I hope to add in the future:* The adjoint tower. Many-sorted algebras. Ultraproducts? Normal forms, and the diamond lemma? quasivarieties? subdirect products? profiniteness? reflective and coreflective subcategories? various sorts of duality? ... ?)

Chapter 0. About the course, and these notes.

0.1. Aims and prerequisites. This course will develop some concepts and results which occur repeatedly throughout the various areas of algebra, and sometimes in other fields of mathematics as well, and which can provide valuable tools and perspectives to those working in these fields. There will be a strong emphasis on examples and instructive exercises.

I will assume only an elementary background in algebra, corresponding to an honors undergraduate algebra course or one semester of graduate algebra, plus a moderate level of mathematical sophistication. A student whose background included a presentation of free groups, but who isn't sure he or she thoroughly understood that construction, would be in an ideal position to begin. On the other hand, anyone conversant with fewer than three ways of proving the existence of free groups has something to learn from Chapters 1-2.

As a general rule, we will pay attention to petty details when they first come up, but take them for granted later on. So students who find the beginning sections devoted too much to ‘‘trivia’’ should be patient!

In this first published version of these course notes, I have not removed remarks about homework, course procedures etc. (mostly in this introductory chapter) aimed at students taking the course from me. This is largely because there are some nonstandard aspects to the way I run the course, which I thought would be of interest to others. Anyone teaching from this text should, of course, let his or her students know which, if any, of these instructions apply to them. In later revisions I may delete or reword such remarks. In the mean time, I hope you find this aspect of the book more quaint than annoying.

0.2. Approach. Since I took my first graduate course, it has seemed to me that there is something wrong with our method of teaching. Why, for an hour at a time, should an instructor write notes on a blackboard and students copy them into their notebooks – often too busy with the copying to pay attention to the content – when this work could be done just as well by a photocopying machine? If this is all that happens in the classroom, why not assign a text or distribute duplicated notes, and run most courses as reading courses?

One answer is that this is not all that happens in a classroom. Students ask questions about confusing points and the instructor answers them. Solutions to exercises are discussed. Sometimes a result is developed by the Socratic method through discussion with the class. Often an instructor gives motivation, or explains an idea in an intuitive fashion he or she would not put into a written text.

As for this last point, I think one should not be embarrassed to put motivation and intuitive discussion into a text, and I will include a great deal of both in these notes. In particular, I shall often first approach general results through very particular cases. The other items – answering questions, discussing solutions to exercises, etc., which seem to me to contain the essential human value of class contact – are what classroom time will be spent on in this course, while these duplicated notes will replace the mechanical copying of notes from the board.

Such a system is not assured of success. Some students may be in the habit of learning material through the process of copying it, and may not get the same benefit by reading it. I advise such students to read these notes with a pad of paper in their hands, and try to anticipate details, work out examples, summarize essential points, etc., as they go.

0.3. A question a day. To help the system described above work effectively, I require every student taking this course to hand in, on each day of class, one question concerning the reading for that day. I strongly encourage you to get your questions to me, either by e-mail or my mailbox, by an hour before class. If you do, I will try to work the answer into my lecture for that day. Questions handed in at the start of the class hour will generally be answered at the next class. At times, I will answer a question with a note to you.

On the sheet on which you submit a question, you should put your name, the point in these notes that the question refers to, and the word ‘‘urgent’’, ‘‘important’’, ‘‘unimportant’’ or ‘‘pro forma’’ – the first three to indicate how important it is to you to have the question answered, the last if there was nothing in the reading that you really felt needed clarification. In that case, your ‘‘pro forma’’ question should still be one that some reader might be puzzled by; perhaps something that puzzled you at first, but that you then resolved. If you give a ‘‘pro forma’’ question, you must indicate the answer along with it!

You may ask more than one question; you may ask, in addition to your question on the current reading, questions relating to earlier readings, and you are encouraged to ask questions in class as well. But you must always submit in writing at least one question related to the reading for the day.

I will show the format in which questions should be submitted in more detail on the first-day handout.

0.4. Homework. These notes contain a large number of exercises. I would like you to hand in solutions to an average of one or two problems of medium difficulty per week, or a correspondingly smaller number of harder problems, or a larger number of easier problems. Choose problems that are interesting to you. But please, look at *all* the exercises, and at least think about how you would approach them. A minimum of 5 minutes of thought per exercise, except those for which you can see a solution sooner, is a good rule of thumb. We will discuss many of them in class. The exercises are interspersed through the text; you may sometimes prefer to think about them as you come to them, at other times, to come back to them after you finish the section.

Grades will be based largely on homework. The amount of homework suggested above, reasonably well done, will give an A. I will give partial credit for partial results, as long as you show you realize that they are partial. I would also welcome your bringing to the attention of the class related problems that you think of, or that you find in other sources.

It should hardly need saying that a solution to a homework exercise in general requires a proof. If a problem asks you to find an object with a certain property, it is not sufficient to give a description and say, ‘‘This is the desired object’’; you must prove that it has the property, unless this is completely obvious. If a problem asks whether a calculation can be done without a certain axiom, it is not enough to say, ‘‘No, the axiom is used in the calculation’’; you must prove that no calculation not using that axiom can lead to the result in question. If a problem asks whether something is true in all cases, and the answer is no, then to establish this you must, in general, give a counterexample.

I am worried that the amount of ‘‘handwaving’’ (informal discussion) in these notes may lead some students to think handwaving is an acceptable substitute for proof. If you read these notes attentively, you will see that handwaving does not *replace* proofs. I use it to guide us to proofs, to communicate my understanding of what is behind some proofs, and at times to abbreviate a proof which is similar to one we have already seen; but in cases of the last sort there is a tacit challenge to you, to think through whether you can indeed fill in the steps. Homework is meant to develop and demonstrate *your* mastery of the material and methods, so it is not a place for you to follow this model by challenging the instructor to fill in steps!

Of course, there is a limit to the amount of detail you can and should show. Most nontrivial mathematical proofs would be unreadable if we tried to give every step of every step. So truly obvious steps can be skipped, and familiar methods can be abbreviated. But more students err in the direction of incomplete proofs than of excessive detail. If you have doubts whether to abbreviate a step, think out (perhaps with the help of a scratch-pad) what would be involved in a more complete argument. If you find that the ‘‘step’’ is more complicated than you had thought, then it should *not* be omitted! But bear in mind that ‘‘to show or not to show’’ a messy step may not be the only alternatives – be on the lookout for a simpler argument, that will avoid the messiness.

I will try to be informative in my comments on your homework. If you are still in doubt as to how much detail to supply, come to my office and discuss it. If possible, come with a specific proof in mind for which you have thought out the details, but need to know how much should be written down.

There are occasional exceptions to the principle that every exercise requires a proof. Sometimes I give problems of a different sort, such as ‘‘Write down precisely the definition of ...’’, or ‘‘How would one motivate ... ?’’ Sometimes, once an object with a given property has been found, the verification of that property is truly obvious. However, if direct verification of the property would involve 32 cases each comprising a 12-step calculation, you should if at all possible find some argument that simplifies or unifies these calculations.

Exercises frequently consist of several successive parts, and you may hand in some parts without doing others (though when one part is used in another, you should if possible do the former if you are going to do the latter). The parts of an exercise may or may not be of similar difficulty – one part may be an easy verification, leading up to a more difficult part, or an exercise of moderate difficulty may introduce an open question. (Open questions, when given, are indicated as such.)

Homework should be legible and well-organized. If a solution you have figured out is complicated, or your conception of it is still fuzzy, outline it first on scratch paper, and revise the outline until it is clean and elegant, before writing up the version to hand in.

If you hand in a proof that is incorrect, I will point this out, and it is up to you whether to try to find and hand in a better proof. If, instead, I find the proof poorly presented, I may request that you redo it.

If you want to see the solution to an exercise that we haven’t gone over, ask in class. But I may postpone answering, or just give a hint, if other people still want to work on it. In the case of an exercise that asks you to supply details for the proof of a result in the text, if you cannot see how to do it you should ask to see it done.

You may also ask for a hint on a problem. If possible, do so in class rather than in my office, so that everyone has the benefit of the hint.

If two or more of you solve a problem together and feel you have contributed approximately equal amounts to the solution, you may hand it in as joint work. If you turn in a homework solution which is inspired by results you have seen in another text or course, indicate this, so that credit can be adjusted to indicate your contribution.

0.5. The name of the game. The general theory of algebraic structures has long been called Universal Algebra, but in recent decades, many workers in the field have come to dislike this term, feeling that ''it promises too much'', and/or that it suggests an emphasis on universal constructions, which, though a major theme of this course, is not all that the field is about.

The most popular replacement term is General Algebra, and I have used it in the title of these notes; but it has the disadvantage that in some contexts it may not be understood as referring to a specific area. Below, I mostly say ''General Algebra'', but occasionally refer to the older term.

0.6. Other reading. Aside from these notes, there is no recommended reading for the course, but I will mention here some items in the list of references at the end that you might like to look at. The books [**3**], [**5**], [**9**], [**15**] and [**17**] are other general texts in General (a.k.a. Universal) Algebra. Of these, [**9**] is the most technical and encyclopedic. [**15**] and [**17**] are both, like these notes, aimed at students not necessarily having great prior mathematical background; however [**17**] differs from this course in emphasizing *partial* algebras. [**5**] has in common with this course the viewpoint that this subject is an important tool for algebraists of all sorts, and it gives some interesting applications to groups, skew fields, etc..

[**26**] and [**28**] are standard texts for Berkeley's basic graduate algebra course. Though we will not assume the full material of such a course, you may find them useful references. [**28**] is more complete and rigorous; [**26**] is sometimes better on motivation. [**20**]-[**22**] include similar material. (Incidentally, some subset of Chapters 1-6 of these notes can be useful supplementary reading for students taking such a course.) A presentation of the core material of such a course at approximately an honors undergraduate level, with detailed explanations and examples, is [**23**].

Each of [**3**], [**5**], [**9**], [**15**], [**17**] and [**21**] gives a little of the theory of *lattices*, introduced in Chapter 5 of these notes; a thorough treatment of this subject may be found in [**4**].

Chapter 6 of these notes introduces *category theory*. [**6**] is the paper that created that discipline, and still very stimulating reading; [**14**] is a general text on the subject. [**7**] deals with an important area of category theory that our course leaves out. For the thought-provoking paper from which the ideas we develop in Chapter 9 come, see [**8**].

An amusing parody of some of the material we shall be seeing in Chapters 4-9 is [**13**].

0.7. Numeration; advice; web access; request for corrections. These notes are divided into Chapters, and each Chapter into Sections. In each Section, I use two numbering systems: one that embraces lemmas, theorems, definitions, numbered displays, etc., and one for exercises. The number of each item begins with the chapter-and-section numbers; this is followed by ''.'' and the number of the result, or '':'' and the number of the exercise. For instance, in §*m.n*, i.e., Section *n* of Chapter *m*, we might have display (*m.n*.1), followed by Definition *m.n*.2, followed by Theorem *m.n*.3; while interspersed among these will be Exercises *m.n*:1, *m.n*:2, *m.n*:3, etc.. The reason for using a common numbering system for results, definitions, and displays is that it is easier to find Proposition 3.2.**5** if it is between Lemma 3.2.**4** and display (3.2.**6**) than it would be if it were Proposition 3.2.**3**, located between Lemma 3.2.**1** and display (3.2.**1**). The exercises form a separate system so that you can easily keep track of which you have and haven't done. They are listed in the ''Index of Exercises'' at the end of these notes, along with telegraphic descriptions of their subjects.

To other instructors who may teach from these notes (and myself, in case I forget), I recommend moving fast through the early, easy, material and more slowly through the later parts, which include more concepts new to the students and more nontrivial proofs. Roughly speaking, this hard material begins with Chapter 7. (A finer description of the hard parts might be: the last three sections of Chapter 6, §7.3, the latter halves of Chapters 7 and 8, and all of Chapter 9.) However, this judgement is based on the last time I taught the course, Spring 1995, when a large fraction of the students had relatively advanced backgrounds. For students who have not seen ordinals or categories before (the kind of students I had in mind in writing these notes), the latter halves of Chapters 4 and 6 might also be places to move slowly.

The last two sections of each of Chapters 6, 7 and 8 are sketchy (to varying degrees), so students should be expected either to read them mainly for the ideas, or to put in extra effort to work out details.

After many years of editing, reworking, and extending these notes, I know one reason why the lecture-and-copy system has not been generally replaced by the distribution of the same material in written form: A good set of notes takes an *enormous* amount of time to develop. But I think that it is worth the effort.

I am grateful to the many students who have pointed out errata in these notes – in particular, in the recent years, Arturo Magidin and David Wasserman.

Comments and suggestions on any aspect of these notes – organizational, mathematical or other, including indications of typographical errors – are welcomed! They can be sent to me by e-mail at the address `gbergman@math.berkeley.edu`, or by regular mail at Department of Mathematics, University of California, Berkeley, CA 94720-3840.

I presently have PostScript files of these notes accessible through my web page, `http://math.berkeley.edu/~gbergman`. I am not sure whether I will keep them there, but as soon as I start accumulating important errata regarding these notes, I will put up a web-page of these.

Part I. Motivation and examples.

In the next three chapters, we shall look at particular cases of algebraic structures and universal constructions involving them, so as to get some sense of the general results we will want to prove in the chapters that follow.

The construction of free groups will be our first example. We will prepare for it in Chapter 1 by making precise some concepts such as that of a group-theoretic expression in a set of symbols; then, in Chapter 2, we will construct free groups by several mutually complementary approaches. Finally, in Chapter 3 we shall look at a large number of other constructions – from group theory, semigroup theory, ring theory, etc. – which have, to greater or lesser degrees, the same spirit as the free group construction, and, for a bit of variety, two examples from topology as well.

Chapter 1. Making some things precise.

1.1. Generalities. Most notation will be explained as it is introduced. We will assume familiarity with basic set-theoretic and logical notation: $\forall$ for ‘‘for all’’ (universal quantification), $\exists$ for ‘‘there exists’’ (existential quantification), $\wedge$ for ‘‘and’’, and $\vee$ for ‘‘or’’. I will write $=_{\mathrm{def}}$ for ‘‘equals by definition’’. Functions will be indicated by arrows $\to$, while their behavior on elements will be shown by flat-tailed arrows, $\mapsto$; that is, if a function $X \to Y$ carries an element x to an element y, this may be symbolized $x \mapsto y$ (‘‘x goes to y’’).

We will (with rare exceptions, which will be noted) write functions on the left of their arguments, i.e., $f(x)$ rather than xf, and understand composite functions fg to be defined so that $(fg)(x) = f(g(x))$.

1.2. What is a group? Loosely speaking, a group is a set G given with a *composition* (or *multiplication*, or *group operation*) $\mu: G \times G \to G$, an *inverse* operation $\iota: G \to G$, and a *neutral* element $e \in G$, satisfying certain well-known laws. (We will make a practice of saying ‘‘neutral element’’ rather than ‘‘identity element’’ to avoid confusion with the other sense of ‘‘identity’’, meaning an equation that holds identically, which will be of great importance to us.)

The most convenient way to make precise this idea of a set ‘‘given with’’ three operations is to define the group to be, not the set G, but the 4-tuple (G, μ, ι, e). In fact, from now on, a letter such as G representing a group will stand for such a 4-tuple, and the first component, called the ‘‘underlying set’’ of the group, will be written $|G|$. Thus

$$G = (|G|, \mu, \iota, e).$$

For simplicity, many mathematicians ignore this formal distinction, and use a letter such as G to represent both a group and its underlying set, writing $x \in G$, for instance, where they mean $x \in |G|$. This is okay, as long as one always understands what ‘‘precise’’ statement such a shorthand statement stands for. Note that to be entirely precise, if G and H are two groups, we should use different symbols, say μ_G and μ_H, ι_G and ι_H, e_G and e_H, for the operations of G and H. How precise and formal one needs to be depends on the situation. Since the aim of this course is to abstract the concept of algebraic structure and study what makes these things tick, we shall be somewhat more precise here than in an ordinary algebra course.

(Many workers in General Algebra use a special type-font, e.g., German or boldface, to represent algebraic objects, and regular type for their underlying sets. Thus, where we will write $G = (|G|, \mu, \iota, e)$, they would write something like $\mathsf{G} = (G, \mu, \iota, e)$.)

Perhaps the easiest exercise in the course is:

Exercise 1.2:1. Give a precise definition of a homomorphism from a group G to a group H, distinguishing between the operations of G and the operations of H.

We will often refer to a homomorphism $f: G \to H$ as a ‘‘map’’ from G to H. That is, unless the contrary is mentioned, ‘‘maps’’ between mathematical objects mean maps between their underlying sets which respect their structure. Note that if we wish to refer to a set map not assumed to respect the group operations, we can call this ‘‘a map from $|G|$ to $|H|$’’.

The use of letters (μ and ι) for the operations of a group, and the functional notation $\mu(x, y)$, $\iota(z)$ which this entails, are desirable for precisely stating results in a form which will *generalize* to a wide class of other sorts of structures. But when actually working with elements of a group, we will generally use conventional notation, writing $x\cdot y$ (or xy, or sometimes in abelian groups $x+y$) for $\mu(x,y)$, and z^{-1} (or $-z$) for $\iota(z)$. When we do this, we either may continue to write $G = (|G|, \mu, \iota, e)$, or we may write $G = (|G|, \cdot, ^{-1}, e)$.

Let us now recall the conditions which must be satisfied by a 4-tuple $G = (|G|, \cdot, ^{-1}, e)$, where $|G|$ is a set, $\cdot$ is a map $|G| \times |G| \to |G|$, ‘‘$^{-1}$’’ is a map $|G| \to |G|$, and e is an element of $|G|$, in order for G to be called a group:

$$\begin{gathered}(\forall\ x, y, z \in |G|)\ \ (x\cdot y)\cdot z = x\cdot(y\cdot z),\\ (\forall\ x \in |G|)\ \ e\cdot x = x = x\cdot e,\\ (\forall\ x \in |G|)\ \ x^{-1}\cdot x = e = x\cdot x^{-1}.\end{gathered} \tag{1.2.1}$$

There is another definition of group that you have probably also seen: In effect, a group is defined to be a pair $(|G|, \cdot)$, such that $|G|$ is a set, and $\cdot$ is a map $|G| \times |G| \to |G|$ satisfying

$$\begin{gathered}(\forall x, y, z \in |G|)\ \ (x\cdot y)\cdot z = x\cdot(y\cdot z),\\ (\exists e \in |G|)\ \ ((\forall x \in |G|)\ e\cdot x = x = x\cdot e) \wedge ((\forall x \in |G|)(\exists y \in |G|)\ y\cdot x = e = x\cdot y).\end{gathered} \tag{1.2.2}$$

It is easy to see that given $(|G|, \cdot)$ satisfying (1.2.2), there exist a *unique* operation $^{-1}$ and element e, such that $(|G|, \cdot, ^{-1}, e)$ satisfies (1.2.1). (Remember the lemmas saying that neutral elements and 2-sided inverses are unique when they exist.) Thus, these two versions of the concept of group provide equivalent information. Our description in terms of 4-tuples may seem ‘‘uneconomical’’ compared with one using pairs, but we will stick with it. We shall eventually see that, more important than the number of terms in the tuple, is the fact that condition (1.2.1) consists only of identities, i.e., universally quantified equations, while (1.2.2) does not. But we will at times acknowledge the idea of the second definition; for instance when we ask (imprecisely) whether some semigroup ‘‘is a group’’.

Exercise 1.2:2. If G is a group, let us define an operation δ_G on $|G|$ by $\delta_G(x,y) = x\cdot y^{-1}$. Does the pair $G' = (|G|, \delta_G)$ determine the group $(|G|, \cdot, ^{-1}, e)$? (I.e., if G_1 and G_2 yield the same pair, $G'_1 = G'_2$, must $G_1 = G_2$?)

Suppose $|X|$ is any set and $\delta\colon |X| \times |X| \to |X|$ any map. Can you write down a set of axioms for the pair $X = (|X|, \delta)$, which will be necessary and sufficient for it to arise from a group G in the manner described above? (That is, given a set $|X|$ and a map $\delta\colon |X| \times |X| \to |X|$ try to find necessary and sufficient conditions for there to exist a group G such that $G' = (X, \delta)$.)

If you get such a set of axioms, then try to see how brief and simple you can make it.

Exercise 1.2:3. Again let G be a group, and now define $\sigma_G(x,y) = x\cdot y^{-1}\cdot x$. Consider the same questions for $(|G|, \sigma_G)$ that were raised for $(|G|, \delta_G)$ in the preceding exercise.

My point in discussing the distinction between a group and its underlying set, and between groups described using (1.2.1) and using (1.2.2), was not to be petty, but to make us conscious of the various ways we use mathematical language – so that we can use it without its leading us astray. At times we will bow to convenience rather than trying to be consistent. For instance, since we distinguish between a group and its underlying set, we should logically distinguish between the *set* of integers, the *additive group* of integers, the *multiplicative semigroup* of integers, the *ring* of integers, etc.; but we shall in fact call all of these ‘‘**Z**’’ unless there is a real danger of ambiguity, or a need to emphasize a distinction. When there is such a need, we can write

$(\mathbf{Z}, +, -, 0) = \mathbf{Z}_{\mathrm{add}}$, $(\mathbf{Z}, \cdot, 1) = \mathbf{Z}_{\mathrm{mult}}$, $(\mathbf{Z}, +, \cdot, -, 0, 1) = \mathbf{Z}_{\mathrm{ring}}$, etc.. Likewise, we may use ''ready made'' symbols for other objects, such as $\{e\}$ for the trivial subgroup of a group G, rather than interrupting a discussion to set up a notation that distinguishes this subgroup from its underlying set.

The approach of regarding ''sets with operations'' as tuples, whose first member is the set and whose other members are the operations, applies, as we have just seen, to other algebraic structures than groups – to semigroups, rings, lattices, and the more exotic beasties we will meet on our travels. To discuss the general case, we need to be clear about what we mean by such concepts as ''n-tuple of elements'' and ''n-ary operation''. We shall review these in the next two sections.

1.3. Indexed sets. If I and X are sets, an *I-tuple of elements of* X, or a *family of elements of* X *indexed by* I, will be defined formally as a function from I to X, but we shall write it $(x_i)_{i\in I}$ rather than $f: I \to X$. The difference is one of viewpoint. We think of such families as arrays of elements of X, which we keep track of with the help of an index set I, while when we write $f: A \to B$ we are most often interested in some properties relating an element of A and its image in B. But the distinction is not sharp. Sometimes there *is* an interesting functional relation between the indices i and the values x_i; sometimes typographical or other reasons will dictate the use of $x(i)$ rather than x_i.

There will be a minor formal exception to the above definition when we speak of an *n-tuple* of elements of X $(n \geq 0)$ in these beginning chapters. I will take this to mean a function from $\{1, \dots, n\}$ to X, written $(x_1, \dots, x_n)$ or $(x_i)_{i=1,\dots,n}$, despite the fact that set theorists define the natural number n inductively to be the set $\{0, \dots, n-1\}$. Most set theorists, for consistency with that definition, write their n-tuples $(x_0, \dots, x_{n-1})$; and we shall switch to that notation after reviewing the set theorist's approach to the natural numbers in Chapter 4.

If I and X are sets, then the set of *all* functions from I to X, equivalently, of all I-tuples of members of X, is written X^I. Note that X^n will denote the set of n-tuples of elements of X, defined as above.

1.4. Arity. An *n-ary* operation on a set S means a map $f: S^n \to S$. For $n = 1, 2, 3$ the words are *unary, binary*, and *ternary* respectively. If f is an n-ary operation, we call n the *arity* of f. More generally, given any set I, an I-ary operation on S is defined as a map $S^I \to S$.

Thus, the definition of a group involves one binary operation, one unary operation, and one distinguished element, or ''constant'', e. Likewise, a ring can be described as a 6-tuple $R = (|R|, +, \cdot, -, 0, 1)$, where $+$ and $\cdot$ are binary operations on $|R|$, ''$-$'' is a unary operation, and $0, 1$ are distinguished elements, all satisfying certain identities.

One may make these descriptions formally more homogeneous by considering ''distinguished elements'' as 0*-ary operations* of our algebraic structures. Indeed, since an n-ary operation on S is something that turns out a value in S when we feed in n arguments in S, it makes sense that a 0-ary operation should be something that gives a value in S without our feeding in anything. Or, looking at it formally, S^0 is the set of all maps from the empty set to S, of which there is exactly one; so S^0 is a one-element set, so a map $S^0 \to S$ determines, and is determined by, a single element of S.

We note also that constants show the right *numerical* behavior to be called ''zeroary operations''. Indeed, if f and g are an m-ary and an n-ary operation on S, and i a positive integer $\leq m$, then on inserting g in the ith place of f, we get an operation $f(-, \dots, -, g(-, \dots, -), -, \dots, -)$ of arity $m+n-1$. Now if, instead, g is a fixed element of S, then when we put it into the ith place of f we get $f(-, \ldots, -, g, -, \ldots, -)$, an $(m-1)$-ary operation, as we should if g is to be thought of as a ''zeroary operation''.

Strictly speaking, distinguished elements and zeroary operations are in one-to-one correspondence, but are not the same thing: One can distinguish between a map $S^0 \to S$, and its (unique) value in S. But we shall find it safe to ignore this difference in describing structures on a set.

So we shall henceforth treat ''distinguished elements'' in the definition of groups, rings, etc., as zeroary operations, and we will find that they can be handled essentially like the other operations. I say ''essentially'' because there are some minor ways in which zeroary operations differ from operations of positive arity. Most notably, on the empty set $X = \varnothing$ there is a *unique* n-ary operation for each positive n, but *no* zeroary operation. Sometimes this trivial fact will make a difference in an argument.

1.5. Group-theoretic terms. One is often interested in talking about what *relations* hold among certain elements of a group or other algebraic system. For example, *every* pair of elements (ξ, η) of a group satisfies the relation $(\xi\cdot\eta)^{-1} = \eta^{-1}\cdot\xi^{-1}$. Some *particular* pair (ξ, η) of elements of some group may satisfy the relation $\xi\cdot\eta = \eta\cdot\xi^2$.

In general, a group-theoretic relation in a family of elements $(\xi_i)_I$ of a group G means an equation $p(\xi_i) = q(\xi_i)$ holding in G, where p and q are *expressions* formed from an I-tuple of symbols using formal group operations $\cdot$, $^{-1}$ and e. So to study relations in groups, we need to define the set of all ''formal expressions'' in the elements of a set X under symbolic operations of multiplication, inverse and neutral element.

The technical word for such a formal expression is a ''term''. Intuitively, a *group-theoretic term* is a set of instructions on how to apply the group-operations to a family of elements. E.g., starting with a set of three symbols, $X = \{x, y, z\}$, an example of a group-theoretic term in X is the symbol $(y\cdot x)\cdot(y^{-1})$; or we might write it $\mu(\mu(y, x), \iota(y))$. Whichever way we write it, the idea is: ''apply the operation μ to the pair (y, x), apply the operation ι to the element y, and then apply the operation μ to the pair of elements so obtained, taken in that order''. The idea can be ''realized'' when we are given a map f of the set X into the underlying set $|G|$ of a group $G = (|G|, \mu_G, \iota_G, e_G)$, say $x \mapsto \xi$, $y \mapsto \eta$, $z \mapsto \zeta$ $(\xi, \eta, \zeta \in |G|)$. We can then define the result of ''evaluating the term $\mu(\mu(y, x), \iota(y))$ using the map f'' as the *element* $\mu_G(\mu_G(\eta, \xi), \iota_G(\eta)) \in |G|$, that is, $(\eta\cdot\xi)\cdot(\eta^{-1})$.

Let us try to make the concept of group-theoretic term precise. ''The set of all terms in the elements of X, under formal operations $\cdot$, $^{-1}$ and e'' should be a set $T = T_{X,\cdot,^{-1},e}$ with the following properties:

(a_X) For every $x \in X$, T contains a symbol representing x.

($\mathrm{a}_\cdot$) For every $s, t \in T$, T contains a ''symbolic combination of s and t under $\cdot$''.

(a_{-1}) For every $s \in T$, T contains an element gotten by ''symbolic application of $^{-1}$ to s''.

(a_e) T contains an element symbolizing e.

(b) Each element of T can be written in *one and only one way* as one and only one of the following:

- (b_X) The symbol representing an element of X.
- ($\mathrm{b}_\cdot$) The symbolic combination of two members of T under $\cdot$.
- (b_{-1}) The symbol representing the result of applying $^{-1}$ to an element of T.

(b$_e$) The symbol representing e.

(c) Every element of T can be obtained from the elements of X via the given symbolic operations. That is, T contains no proper subset satisfying (a$_X$) - (a$_e$).

In functional language, (a$_X$) says that we are to be given a function $X \to T$ (the ‘‘symbol for x’’ function); (a$_\cdot$) says we have another function, which we call ‘‘formal product’’, from $T \times T$ to T; (a$_{-1}$) posits a function $T \to T$, the ‘‘formal inverse’’, and (a$_e$) a distinguished element of T. Translating our definition into this language, we get

Definition 1.5.1. *By ‘‘the set of all terms in the elements of X under the formal group operations μ, ι, e’’ we shall mean a set T which is:*

(a) *given with functions*

$$\mathrm{symb}_T\colon X \to T, \qquad \mu_T\colon T^2 \to T, \qquad \iota_T\colon T \to T, \qquad \textit{and} \qquad e_T\colon T^0 \to T,$$

such that

(b) *each of these maps is one-to-one, their images are disjoint, and T is the union of these images, and*

(c) *T is generated by* $\mathrm{symb}_T(X)$, *under the operations μ_T, ι_T, and e_T.*

The next exercise justifies the use of the word ‘‘the’’ in the above definition.

Exercise 1.5:1. Assuming T and T' are two sets given with functions that satisfy Definition 1.5.1, establish a natural one-to-one correspondence between the elements of T and T'. (You must, of course, *show* that the correspondence you set up is well-defined, and is a bijection.)

Exercise 1.5:2. Is condition (c) of Definition 1.5.1 a consequence of (a) and (b)?

How can we obtain a set T with the properties of the above definition? One approach is to construct elements of T as finite strings of symbols from some alphabet which contains symbols representing the elements of X, additional symbols μ (or $\cdot$), ι (or $^{-1}$), and e, and perhaps some symbols of punctuation. But we need to be careful. For instance, if we defined μ_T to take a string of symbols s and a string of symbols t to the string of symbols $s\cdot t$, and ι_T to take a string of symbols s to the string of symbols s^{-1}, then condition (b) would not be satisfied! For a string of symbols of the form $x\cdot y\cdot z$ (where $x, y, z \in X$) could be obtained by formal multiplication either of x and $y\cdot z$, or of $x\cdot y$ and z. In other words, μ_T takes the pairs $(x, y\cdot z)$ and $(x\cdot y, z)$ to the same string of symbols, so it is not one-to-one. Likewise, the expression $x\cdot y^{-1}$ could be obtained either as $\mu_T(x, y^{-1})$ or as $\iota_T(x\cdot y)$, so the images of μ_T and ι_T are not disjoint. (It happens that in the first case, the two interpretations of $x\cdot y\cdot z$ come to the same thing in any group, because of the associative law, while in the second, the two interpretations do not: $\xi\cdot(\eta^{-1})$ and $(\xi\cdot\eta)^{-1}$ are generally distinct for elements ξ, η of a group G. But the point is that in both cases condition (b) fails, making these expressions ambiguous as *instructions* for applying group operations. Note that a notational system in which ‘‘$x\cdot y\cdot z$’’ was ambiguous in the above way could never be used in *writing down* the associative law; and writing down identities is one of the uses we will want to make of these expressions.)

On the other hand, it is not hard to show that by introducing parentheses among our symbols, and letting $\mu_T(s, t)$ be the string of symbols $(s\cdot t)$, and $\iota_T(s)$ the string of symbols (s^{-1}), we can get a set of expressions satisfying the conditions of our definition.

Exercise 1.5:3. Verify the above assertion. (How, precisely, will you define T? What assumptions must you make on the set of symbols representing elements of X? Can we allow some of these ''symbols'' to be strings of other symbols?)

Another symbolism that will work is to define the value of μ_T at s and t to be the string of symbols $\mu(s,t)$, and the value of ι_T at s to be the string of symbols $\iota(s)$.

Exercise 1.5:4. Suppose we define the value of μ_T on s and t to be the symbol μst, and the value of ι_T at s to be the symbol ιs. Will the resulting set of strings of symbols satisfy Definition 1.5.1?

A disadvantage of the strings-of-symbols approach is that, though it can be extended to other kinds of algebras with finitary operations (such as rings, lattices, etc.), one cannot use it for algebras with operations of *infinite* arities, because, even if one allows infinite strings of symbols, one cannot string several infinite strings together. One can, however, for an infinite set I, create I-tuples which have I-tuples among their members, and this leads to the more versatile *set-theoretic* approach. Let us show it for the case of group-theoretic terms.

Choose any set of four elements, which will be denoted $*$, $\cdot$, $^{-1}$ and e. For each $x\in X$, define $\mathrm{symb}_T(x)$ to be the ordered pair $(*, x)$; for $s,\ t\in T$, define $\mu_T(s,t)$ to be the ordered 3-tuple $(\cdot,\ s,\ t)$; for $s\in T$ define $\iota_T(s)$ to be the ordered pair $(^{-1},\ s)$, and finally, define e_T to be the 1-tuple (e). Let T be the smallest set closed under the above operations.

Now it is a basic lemma of set theory that no element can be written as an n-tuple in more than one way; i.e., if $(x_1, \dots, x_n) = (x'_1, \dots, x'_{n'})$, then $n' = n$ and $x_i = x'_i$ $(i=1, \dots, n)$. It is easy to deduce from this that the above construction will satisfy the conditions of Definition 1.5.1.

Exercise 1.5:5. Would there have been anything wrong with defining $\mathrm{symb}_T(x) = x$ instead of $(*, x)$? If so, can you find a way to modify the definitions of μ_T etc., so that the definition of $\mathrm{symb}_T(x) = x$ can always be used?

I leave it to you to decide (or not to decide) which construction for group-theoretic terms you prefer to assume during these introductory chapters. We shall only need the *properties* given in Definition 1.5.1. From now on, we shall often use conventional notation for such terms, e.g., $(x\cdot y)\cdot(x^{-1})$. In particular, we shall often identify X with its image $\mathrm{symb}_T(X)$. We will use the more formal notation of Definition 1.5.1 mainly when we want to emphasize particular distinctions, such as that between the formal operations μ_T etc., and the operations μ_G etc. of a particular group.

1.6. Evaluation. Now suppose G is a group, and $f\colon X \to |G|$ a set map, in other words, an X-tuple of elements of G. We wish to say how to *evaluate* a term in an X-tuple of symbols,

$$s \in T \ = \ T_{X,\,\cdot\,,\,^{-1},\,e}$$

at this family f of elements, so as to get a value $s_f \in |G|$. We shall do this inductively (or more precisely, ''recursively''; we will learn the distinction in §4.3).

If $s = \mathrm{symb}_T(x)$ for some $x\in X$ we define $s_f = f(x)$. If $s = \mu_T(t,u)$, then assuming inductively that we have already defined $t_f,\ u_f \in |G|$, we define $s_f = \mu_G(t_f,\ u_f)$. Likewise, if $s = \iota_T(t)$, we assume t_f defined and define $s_f = \iota_G(t_f)$. Finally, for $s = e_T$ we define $s_f = e_G$. Since every element $s\in T$ is obtained from $\mathrm{symb}_T(X)$ by the operations $\mu_T,\ \iota_T,\ e_T$, and in a unique manner, this construction will give one, and only one, value s_f for each s.

We have not discussed the general principles that allow one to make recursive definitions like the above. We shall develop these in Chapter 4, in preparation for Chapter 8 where we will do rigorously and in full generality what we are sketching here. Some students might want to look into this question for themselves at this point, so I will make this:

Exercise 1.6:1. Show rigorously that the procedure loosely described above yields a unique well-defined map $T \to |G|$. (Suggestion: If you are familiar with Zorn's Lemma, consider *partial* maps from T to $|G|$ satisfying appropriate conditions, and apply that Lemma to get a maximal one.)

In the above discussion of evaluation, we fixed $f \in |G|^X$, and got a function $T \to G$. If we vary f as well as T, we get an ''evaluation map'',

$$(T_{X,\cdot,{}^{-1},e}) \times |G|^X \to |G|$$

taking each pair (s, f) to s_f. Still another viewpoint is to fix an $s \in T$, and define a map $s_G \colon |G|^X \to |G|$ by $s_G(f) = s_f$ $(f \in |G|^X)$; this represents ''substitution into s''. For example, suppose $X = \{x, y, z\}$, let us identify $|G|^X$ with $|G|^3$, and let s be the term $(y\cdot x)\cdot(y^{-1})$. Then for each group G, s_G is the operation taking each 3-tuple (ξ, η, ζ) to the element $(\eta\xi)\eta^{-1}$. Such operations will be of importance to us, so we give them a name.

Definition 1.6.1. *Let G be a group and n a nonnegative integer. Let $T = T_{n,{}^{-1},\cdot,e}$ denote the set of group-theoretic terms in n symbols. Then for each $s \in T$, we will let $s_G \colon |G|^n \to |G|$ denote the map taking each n-tuple $f \in |G|^n$ to the element $s_f \in |G|$. The n-ary operations s_G obtained in this way from terms $s \in T$ will be called the* derived n-ary operations *of G.* (*Some authors call these* term operations.)

Note that *distinct terms* can induce the same *derived operation*. E.g., the associative law for groups says that for any group G, the derived ternary operations induced by the terms $(x\cdot y)\cdot z$ and $x\cdot(y\cdot z)$ are the same. As another example, in the particular group S_3 (the symmetric group on three elements), the derived binary operations induced by the terms $(x\cdot x)\cdot(y\cdot y)$ and $(y\cdot y)\cdot(x\cdot x)$ are the same, though this is not true in all groups. (It is true in all dihedral groups.)

Some more examples of derived operations on groups are the binary operation of *conjugation*, commonly written $\xi^\eta = \eta^{-1}\xi\eta$ (induced by the term $y^{-1}\cdot(x\cdot y)$), the binary *commutator* operation, $[\xi, \eta] = \xi^{-1}\eta^{-1}\xi\eta$, the unary operation of *squaring*, $\xi^2 = \xi\cdot\xi$, and the two binary operations δ and σ of Exercises 1.2:2 and 1.2:3. Some trivial examples are also worth noting: the *primitive* group operations – group multiplication, inverse, and neutral element – are by definition also *derived* operations; and finally, one has very trivial derived operations such as the ternary ''second component'' function, $p_{3,2}(\xi, \eta, \zeta) = \eta$, induced by $y \in T_{\{x,y,z\},{}^{-1},\cdot,e}$.

1.7. Terms in other families of operations. These concepts can be applied to more general sorts of algebraic structures. Let Ω be an ordered pair $(|\Omega|, \mathrm{ari})$, where $|\Omega|$ is a set of symbols (thought of as representing operations), and ari a function associating to each $\alpha \in |\Omega|$ a nonnegative integer $\mathrm{ari}(\alpha)$, the intended *arity* of α (§1.4). (For instance, in the group case which we have been considering, we would take $|\Omega| = \{\mu, \iota, e\}$, $\mathrm{ari}(\mu) = 2$, $\mathrm{ari}(\iota) = 1$, $\mathrm{ari}(e) = 0$. Incidentally, the commonest symbol, among specialists, for the arity of an operation α is $n(\alpha)$, but I will use $\mathrm{ari}(\alpha)$ to avoid confusion with other uses of the letter n.) Then an Ω-*algebra* will mean a system $A = (|A|, (\alpha_A)_{\alpha\in|\Omega|})$, where $|A|$ is a set, and for each $\alpha \in |\Omega|$, α_A is some $\mathrm{ari}(\alpha)$-ary operation on $|A|$:

$$\alpha_A \colon |A|^{\mathrm{ari}(\alpha)} \to |A|.$$

For any set X, we can mimic the preceding ideas to get a set $T = T_{X,\Omega}$, the set of ‘‘terms in elements of X under the operations of Ω’’; and given any Ω-algebra A, we can get substitution and evaluation maps as before, and so define *derived operations* of A.

The eventual aim of this course will be to study such general systems A. In order to discover what kinds of results we want to prove about them, we shall devote Chapters 2 and 3 to looking at specific situations involving familiar sorts of algebraic objects.

However, let me give here a few exercises based on these general concepts.

Exercise 1.7:1. On the set $\{0,1\}$, let M_3 denote the ternary ‘‘majority vote’’ operation; i.e., for $a,\ b,\ c\in\{0,1\}$, $M_3(a,b,c)$ is 0 if two or more of $a,\ b$ and c are 0, and 1 if two or more of them are 1. One can form various terms in a symbolic operation M_3 (e.g., $p(w,x,y,z) = M_3(x,\ M_3(z,w,y),\ z))$ and then evaluate these in the algebra $(\{0,1\},\ M_3)$ to get operations on $\{0,1\}$ ‘‘derived from’’ M_3.

General problem: Determine which operations (of arbitrary arity) on $\{0,1\}$ can be expressed as derived operations of this algebra.

As steps toward such a result, you might try to determine whether each of the following can or cannot be so expressed:

(a) The 5-ary majority vote function M_5: $\{0,1\}^5 \to \{0,1\}$, defined in the obvious manner.

(b) The binary operation sup. (I.e., $\sup(a,b) = 0$ if $a = b = 0$; otherwise $\sup(a,b) = 1$.)

(c) The unary ‘‘reversal’’ operation r, defined by $r(0) = 1$, $r(1) = 0$.

(d) The 4-ary operation N_4, described as ‘‘the majority vote function, where the first voter has extra tie-breaking power’’; i.e., $N_4(a,b,c,d) =$ the majority value among $a,\ b,\ c,\ d$ if there is one, while if $a+b+c+d = 2$ we set $N_4(a,b,c,d) = a$.

Advice: (i) If you succeed in proving that some operation is *not* derivable from M_3, try to formulate and prove this as a general result to the effect that all operations derived from M_3 must have a certain property. (ii) A mistake that many students make is to think that a formula such as $s(\xi,\eta) = M_3(0,\xi,\eta)$ defines a derived operation. In fact, since our system $(\{0,1\},\ M_3)$ does not include the *zeroary operation* 0 (nor 1), ‘‘$M_3(0,x,y)$’’ is not a term.

Exercise 1.7:2. (Question raised by Jan Mycielski, letter dated 1/17/83.) Let $\mathbf{C}$ denote the set of complex numbers, and exp the exponential function $\exp(x) = e^x$, a unary operation on $\mathbf{C}$. Does the algebra $(\mathbf{C},\ +,\ \cdot,\ \exp)$ have any automorphisms other than the identity and complex conjugation? (An *automorphism* means a bijection of the underlying set with itself, which respects the operations.) I don’t know the answer to this question.

It is not hard to prove using the theory of transcendence bases of fields ([**26**, §VI.1], [**28**, §VIII.1]) that $(\mathbf{C},\ +,\ \cdot)$ has infinite automorphism group (cf. [**26**, Exercise VI.6(b)], [**28**, Exercise VIII.1]). A couple of easy results in the opposite direction, which you may prove and hand in, are (i) that this structure has no *continuous* automorphisms other than those mentioned, and (ii) that if we write ‘‘cj’’ for the unary operation of complex conjugation, then the algebra $(\mathbf{C},\ +,\ \cdot,\ \mathrm{cj})$ has no automorphisms other than id and cj. You will also find it easy to verify that (iii) a map $\mathbf{C} \to \mathbf{C}$ is an automorphism of $(\mathbf{C},\ +,\ \cdot,\ \exp)$ if and only if it is an automorphism of $(\mathbf{C},\ +,\ \exp)$.

Exercise 1.7:3. Given operations $\alpha_1, \ldots, \alpha_r$ (of various arities) on a *finite* set S, and another operation β on S, describe a test that will determine in a finite number of steps whether β is a derived operation of $\alpha_1, \ldots, \alpha_r$.

The arities considered so far have been finite; the next exercise will deal with terms in operations of possibly *infinite* arities. To make this reasonable, let me give some naturally arising examples of operations of countably infinite arity on familiar sets:

On the real unit interval $[0,1]$:

(a) the operation lim sup (‘‘limit superior’’, defined by $\limsup x_i = \lim_{i\to\infty} \sup_{j\geq i} x_j$),

(b) the operation defined by $s(a_1, a_2, ...) = \Sigma\, 2^{-i} a_i$.

On the set of positive real numbers:

(c) the continued fraction operation $c(a_1, a_2, ...) = a_1 + 1/(a_2 + 1/(...))$.

On the class of subsets of the set of all integers:

(d) the operation $\bigcup a_i$,

(e) the operation $\bigcap a_i$.

Exercise 1.7:4. Suppose Ω is a pair $(|\Omega|, \mathrm{ari})$, where $|\Omega|$ is again a set of operation symbols, but where the arities $\mathrm{ari}(\alpha)$ may now be finite or infinite cardinals; and let X be a set of variable symbols. Suppose we can form a set T of expressions satisfying the analogs of conditions (a)-(c) (§1.5). For $s, t \in T$, let us write $s \gg t$ if t is ''immediately involved'' in s, that is, if s has the form $\alpha(u_1, u_2, ...)$ where $\alpha \in |\Omega|$, and for *some* i, $u_i = t$.

(i) Show that if all the arities $\mathrm{ari}(\alpha)$ are *finite*, then for each s we can find a finite bound $B(s)$ on the lengths n of sequences $s_1, ..., s_n \in T$ such that $s = s_1 \gg ... \gg s_n$.

(ii) If not all $\mathrm{ari}(\alpha)$ are finite, and X is nonempty, show that there exists s for which no such finite bound exists.

(iii) In the situation of (ii), is it possible to have an infinite chain $s = s_1 \gg ... \gg s_n \gg ...$ in T?

(iv) Show that one cannot have a ''cycle'' $s_1 \gg ... \gg s_n \gg s_1$ in T.

Chapter 2. Free groups.

In this chapter, we introduce the idea of universal constructions through the particular case of free groups. We shall first motivate the free group concept, then develop three ways of constructing such groups.

2.1. Motivation. Suppose G is a group and we take (say) three elements $a,\ b,\ c\in|G|$, and consider what group-theoretic relations these satisfy. That is, letting T be the set of all group-theoretic terms in three symbols $x,\ y$ and z, we look at pairs of elements $p(x,y,z)$, $q(x,y,z)\in T$, and if $p_G(a,b,c)=q_G(a,b,c)$ in $|G|$, we say that (a,b,c) satisfies the relation $p=q$. We note:

Lemma 2.1.1. *Suppose F and G are groups, $a,\ b,\ c\in|F|$ are three elements* generating F, *and α,β,γ are* any *three elements of G. Then the following conditions are equivalent:*

(i) *Every group-theoretic relation $p=q$ satisfied by (a,b,c) in F is also satisfied by (α,β,γ) in G.*

(ii) *There exists a group homomorphism $h\colon F\to G$ under which $a\mapsto\alpha,\ b\mapsto\beta,\ c\mapsto\gamma$.*

Further, when these conditions hold, the homomorphism h of (ii) *is unique.*

If the assumption that $a,\ b$ and c generate F is dropped, one still has (ii)$\Rightarrow$(i).

Proof. Suppose (ii) is satisfied. Then it is easy to see by induction on p that for all $p\in T$,

$$h(p_F(a,b,c)) \;=\; p_G(\alpha,\beta,\gamma).$$

Statement (i) follows. If, also, $a,\ b,\ c$ generate F, then every element of $|F|$ can be written $p_F(a,b,c)$ for some p, so the above formula shows that an h as in (ii) is uniquely determined on all of F.

On the other hand, suppose $\{a,b,c\}$ generates F, and (i) holds. For each $g=p_F(a,b,c)\in|F|$, define $h(g)=p_G(\alpha,\beta,\gamma)$. To show that this gives a well-defined map from $|F|$ to $|G|$, note that if we have two ways of writing an element $g\in|F|$, $p_F(a,b,c)=g=q_F(a,b,c)$, then the relation $p=q$ is satisfied by (a,b,c) in F, hence by (i), it is satisfied by (α,β,γ) in G, hence the two values our definition would prescribe for $h(g)$, namely $p_G(\alpha,\beta,\gamma)$ and $q_G(\alpha,\beta,\gamma)$, are the same.

That this set map is a homomorphism follows from the way evaluation of group-theoretic terms at (α,β,γ) is defined. For instance, given $g\in|F|$, suppose we want to show that $h(g^{-1})=h(g)^{-1}$. We write $g=p_F(a,b,c)$. Then $(\iota_T(p))_F(a,b,c)=g^{-1}$, so $h(g^{-1})=_{\text{def}}(\iota_T(p))_G(\alpha,\beta,\gamma)=_{\text{def}}p_G(\alpha,\beta,\gamma)^{-1}=h(g)^{-1}$. The same method works for products and for the neutral element. □

Exercise 2.1:1. Show by example that if $\{a,b,c\}$ does not generate F, then condition (i) of the above lemma can hold and (ii) fail, and also that (ii) can hold but h not be unique. (You may replace (a,b,c) with a smaller family, (a,b) or (a), if you like.)

Lemma 2.1.1 leads one to wonder: Among all groups F generated by 3-tuples of elements (a,b,c), is there one such group in which these three elements satisfy the *smallest* possible set of relations? We note what the above lemma would imply for such a group:

Corollary 2.1.2. *Let F be a group, and $a,\ b,\ c\in|F|$. Then the following conditions are equivalent:*

(i) *$a,\ b,\ c$ generate F, and the only relations satisfied by $a,\ b,\ c$ in F are those relations satisfied by every 3-tuple (α,β,γ) of elements in every group G.*

(ii) *For every group G, and every 3-tuple of elements (α,β,γ) in G, there exists a unique homomorphism $h\colon F\to G$ such that $h(a)=\alpha,\ h(b)=\beta,\ h(c)=\gamma$.* □

Only one point in the deduction of this corollary from Lemma 2.1.1 is not completely obvious; I will make it an exercise:

Exercise 2.1:2. In the situation of the above corollary, show that (ii) implies that $\{a,b,c\}$ generates F. (Hint: Let G be the subgroup of F generated by those three elements.)

I've been speaking of 3-tuples of elements for concreteness; the same observations are valid for n-tuples for any n, and generally, for X-tuples for any set X. An X-tuple of elements of F means a set map $X\to|F|$, so in this general context, condition (ii) above takes the form given by the next definition. (Though making this definition does not answer the question of whether such objects exist!)

Definition 2.1.3. *Let X be a set. By a* free group *F on the set X, we shall mean a pair $(F,\ u)$, where F is a group, and u a set map $X\to|F|$, having the following* universal property:

For any group G, and any set map $v\colon X\to|G|$, there exists a unique homomorphism $h\colon F\to G$ such that $v=hu$; i.e., making the diagram below commute.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, "\forall v"'] & {|F|} \arrow[d] & F \arrow[d, "\exists 1\ h"] \\
 & {|G|} & G
\end{tikzcd}$$

(In the above diagram, the first vertical arrow also represents the homomorphism h, there regarded as a map on the underlying sets of the groups.)

Corollary 2.1.2 (as generalized to X-tuples) says that $(F,\ u)$ is a free group on X if and only if the elements $u(x)\ (x\in X)$ generate F, and satisfy *no* relations *except* those that must hold in any group. In this situation, one says that these elements ''freely'' generate F, hence the term *free group*. Note that if such an F exists, then every X-tuple of members of any group G can be obtained, in a unique way, as the image under a group homomorphism $F\to G$ of the particular X-tuple given by u. Hence that X-tuple can be thought of as a *universal X-tuple* of group elements, and so the property characterizing it is called a *universal property*.

We note a few elementary facts and conventions about such objects. If $(F,\ u)$ is a free group on X, then the map $u\colon X\to|F|$ is one-to-one. (This is easy to prove from the universal property, plus the well-known fact that there exist groups with more than one element. The student who has not seen free groups developed before should think this argument through.) Hence given a free group, it is easy to get from it one such that the map u is actually an inclusion $X\subseteq|F|$. Hence for notational convenience, one frequently assumes that this is so; or, what is approximately the same thing, one often uses the same symbol for an element of X and its image in $|F|$.

If $(F,\ u)$ and $(F',\ u')$ are both free groups on the same set X, there is a unique isomorphism between them *as* free groups, i.e., respecting the maps u and u'. (Cf. diagram below.)

$$\begin{tikzcd}
 & {|F|} \arrow[dd, shift right] & F \arrow[dd, shift right] \\
X \arrow[ur, "u"] \arrow[dr, "u'"'] & & \\
 & {|F'|} \arrow[uu, shift right] & F' \arrow[uu, shift right]
\end{tikzcd}$$

(If you haven't seen this result before, again see whether you can work out the details. For the technique you might refer to the proof of Proposition 3.3.3 below.) As any two free groups on X are thus ''essentially'' the same, one sometimes speaks of *the* free group on X.

One also often says that a group F ''is free'' to mean ''there exists some set X and some map $u: X \to |F|$ such that (F, u) is a free group on X''. (Here u can indeed be taken to be the inclusion of a subset $X \subseteq |F|$.)

But it is time we proved that free groups exist! We will show three different ways of constructing them in the next three sections.

Exercise 2.1:3. Suppose one replaces the word ''group'' by ''finite group'' throughout Definition 2.1.3. Show that for any nonempty set X, *no* finite group will exist having the stated universal property.

2.2. The logician's approach: construction from group-theoretic terms. We know from Corollary 2.1.2 that if a free group F on three generators a, b, c exists, then each of its elements can be written $p_F(a,b,c)$ for some group-theoretic term p, and that *two* such elements, $p_F(a,b,c)$ and $q_F(a,b,c)$ are equal if and only if the equation ''$p = q$'' holds for every three elements of every group, i.e., follows from the group axioms. This suggests that we may be able to construct such a group by taking the set of all group-theoretic terms in three variables, constructing an equivalence relation ''$p \sim q$'' on this set which means ''the equality of p and q is a consequence of the group axioms'', taking for $|F|$ the quotient of our set of terms by this relation, and defining operations $\cdot$, ${}^{-1}$ and e on $|F|$ in some natural manner. This we shall now do!

Let X be any set, and $T = T_{X,\,\cdot,\,^{-1},\,e}$ the set of all group-theoretic terms in the elements of X. What conditions must a relation ''$\sim$'' satisfy for $p \sim q$ to be the condition ''$p_v = q_v$'' for *some* map v of X into *some* group G? Well, the group axioms tell us that it must satisfy:

(2.2.1) $$(\forall\, p, q, r \in T)\ ((p\cdot q)\cdot r) \sim (p\cdot(q\cdot r)),$$

(2.2.2) $$(\forall\, p \in T)\ \ (p\cdot e) \sim p\ \ \wedge\ \ (e\cdot p) \sim p,$$

(2.2.3) $$(\forall\, p \in T)\ \ (p\cdot(p^{-1})) \sim e\ \ \wedge\ \ ((p^{-1})\cdot p) \sim e.$$

Also, just the well-definedness of the operations of G tells us that:

(2.2.4) $$(\forall\, p, p', q \in T)\ \ (p \sim p') \Rightarrow ((p\cdot q) \sim (p'\cdot q)\ \wedge\ (q\cdot p) \sim (q\cdot p')),$$

(2.2.5) $$(\forall\, p, p' \in T)\ \ (p \sim p') \Rightarrow (p^{-1}) \sim (p'^{-1}).$$

Finally, of course, $\sim$ must be an equivalence relation:

(2.2.6) $(\forall\, p \in T)\quad p \sim p,$

(2.2.7) $(\forall\, p,\, q \in T)\ \ (p \sim q) \Rightarrow (q \sim p),$

(2.2.8) $(\forall\, p,\, q,\, r \in T)\ \ (p \sim q \wedge q \sim r) \Rightarrow (p \sim r).$

So let us take for ''$\sim$'' the *least* binary relation on T satisfying conditions (2.2.1-8).

Let us note what this means, and why it exists: Recall that a binary relation on a set T is formally a subset $R \subseteq T \times T$; when we write $p \sim q$, this is understood to be an abbreviation for $(p, q) \in R$. ''Least'' means smallest with respect to set-theoretic inclusion. Our conditions (2.2.1-8) are in the nature of closure conditions, and, as with all sets defined by closure conditions, the existence of a least set satisfying them can be established in two ways:

We may capture this set ''from above'' by forming the *intersection* of all binary relations on T satisfying (2.2.1-8) – the set-theoretic intersection of these relations as subsets of $T \times T$. (Note, incidentally, that if we think of such relations as predicates rather than as sets, this intersection $\bigcap$ becomes a (generally infinite) conjunction $\bigwedge$.) The key point to observe is that these conditions are such that an intersection of relations satisfying them again satisfies them. Hence the intersection of *all* relations satisfying them will be the least such relation.

Or we can ''build it up from below''. Let R_0 denote the empty relation $\varnothing \subseteq T \times T$, and recursively construct the i+1st relation R_{i+1} from the ith, by adding to R_i those elements that conditions (2.2.1-8) say must *also* be in R, given that the elements of R_i are there. Precisely:

$$\begin{array}{lll} R_{i+1} =_{\text{def}} & R_i & \text{(elements already constructed)} \\ & \cup\ \{((p\cdot q)\cdot r,\ p\cdot(q\cdot r)) \mid p, q, r \in T\} & \text{(elements arising by (2.2.1))} \\ & \cup\ \ldots & \ldots \\ & \cup\ \{(p, r) \mid (\exists\, q)\ (p, q) \in R_i \wedge (q, r) \in R_i\}. & \text{(elements arising by (2.2.8))} \end{array}$$

We now define $R = \bigcup_i R_i$. It is straightforward to show that R satisfies (2.2.1-8), and that any subset of $T \times T$ satisfying (2.2.1-8) must contain R; so R, looked at as a binary relation $\sim$ on T, is the desired least relation.

By (2.2.6-8), $\sim$ is an equivalence relation; so define $|F| = T/\!\sim$. We shall denote the typical element of $|F|$, the equivalence class of an element $p \in T$, by $[p]$. We now map X into $|F|$ by setting

$$u(x) = [x]$$

(or, if we do not identify $\mathrm{symb}_T(x)$ with x in our construction of T, then $u(x) = [\mathrm{symb}_T(x)]$). We define operations $\cdot$, ${}^{-1}$ and e on $|F|$ by

$$[p]\cdot[q] = [p\cdot q],$$
$$[p]^{-1} = [p^{-1}],$$
$$e = [e].$$

That the first two of these are *well-defined* follows from the properties (2.2.4) and (2.2.5) of $\sim$! (With the third there is no problem.) Also, from properties (2.2.1-3) of $\sim$, it follows that $(|F|,\ \cdot,\ {}^{-1},\ e)$ satisfies the group axioms. E.g., given $[p], [q], [r] \in |F|$, if we evaluate $([p]\cdot[q])\cdot[r]$ and $[p]\cdot([q]\cdot[r])$ in $|F|$, we get $[((p\cdot q)\cdot r)]$ and $[(p\cdot(q\cdot r))]$ respectively, which are equal by (2.2.1). Finally, writing F for the group $(|F|, \cdot, {}^{-1}, e)$, it is clear from our construction of $\sim$ that the only relations satisfied by the images in F of the elements of X are relations that follow logically from the group axioms; so by Corollary 2.1.2 (stated there for 3-tuples, but now generalized to X-tuples), F has the desired universal property.

To see this universal property more directly, suppose v is any map $X \to |G|$, where G is a group. Write $p \sim_v q$ to mean $p_v = q_v$ in G. Then clearly the relation $\sim_v$ satisfies conditions (2.2.1-8), hence it contains the least such relation, our $\sim$. So a well-defined map $h\colon |F| \to |G|$ is given by $h([p]) = p_v \in |G|$, and it follows immediately from the way the operations of F, and the evaluation of terms in G with respect to v, are defined, that h is a homomorphism, and is the unique homomorphism such that $hu = v$.

Thus we have proven

Proposition 2.2.9. (F, u), *constructed as above, is a free group on the given set* X. □

So a free group on every set X does indeed exist!

Further notes:

2.2.10. There is a viewpoint that goes along with this construction, which will be helpful in thinking about universal constructions in general. We are given a set X. Suppose we know that G is a group, with a map $v\colon X \to |G|$. How much can we ''say about'' G from this fact alone? We can *name* certain elements of G, namely the $v(x)$ $(x \in X)$ and all the elements that can be obtained from these by the group operations of G (e.g., $((v(x)\cdot v(y))^{-1})\cdot((v(y)^{-1}\cdot e)^{-1}\cdot v(z))$ $(x, y, z \in X)$). A particular G may contain more elements than those obtained in such ways, but we have no way of getting our hands on them from the given information. We can also derive from the identities for groups certain relations that these elements satisfy, (e.g., $(v(x)\cdot v(y))^{-1} = v(y)^{-1}\cdot v(x)^{-1}$). The elements $v(x)$ may, in particular cases, satisfy more relations than these, but again we have no way of deducing these additional relations. If we now gather together this limited ''data'' that we have about such a group G – labels for certain elements, modulo certain identifications among these – we find that this collection of ''data'' itself forms a group with a map of X into it; and is, in fact, a universal such group!

2.2.11. At the beginning of this section, I motivated our construction by saying that ''$\sim$'' should mean ''equality that follows from the group axioms''. I then wrote down a series of eight rules, (2.2.1-8), all of which are clearly valid procedures for deducing equations which must hold in all groups. What was not obvious was whether they would be sufficient to yield *all* such equations. But they were – the proof of the pudding being that $(T/\sim, \cdot, ^{-1}, e)$ could be shown to be a group.

This is an example of a very general type of situation in mathematics: Some class, in this case, a class of pairs of formal group-theoretic terms, is described ''from above'', i.e., is defined as the class of all elements satisfying certain restrictions (in this case, those pairs $(p, q) \in T \times T$ such that the relation $p = q$ holds on all X-tuples of elements of all groups). We seek a way of describing it ''from below'', i.e., of *constructing* or *generating* all members of the class. Some procedure which produces members of the set is found, and one seeks to show that this procedure yields the whole set – or, if it does not, one seeks to extend it to a procedure that does.

The converse situation is equally important, where we are given a construction which ''builds up'' a set, and seek a convenient way of characterizing the elements that result. Exercise 1.7:1 was of that form. You will see more examples of both situations throughout this course, and in fact, in most every mathematics course you take.

Exercise 2.2:1. Show directly from (2.2.1-8) that for $x, y \in X$, $((x \cdot y)^{-1}) \sim ((y^{-1}) \cdot (x^{-1}))$.

Exercise 2.2:2. Does the relation of the preceding exercise follow from (2.2.1-3) and (2.2.6-8) alone?

Note that in our recursive construction of the set R (that is, the relation $\sim$), repeated application of (2.2.1-3) was really unnecessary; these conditions give the same elements of R each time they are applied, so we might as well just have applied them the first time and only applied (2.2.4-8) from then on. Less obvious is:

Exercise 2.2:3. (A. Tourubaroff) Can the construction of R be done in three stages: First take the set P of elements given by (2.2.1-3), then form the closure Q of this set under applications of (2.2.4-5) (as before, by recursion or as an intersection), and finally, obtain R as the closure of Q under applications of (2.2.6-8) (another recursion or intersection)? This procedure will yield some subset of $T \times T$; the question is whether it is the R we want.

What if we do things in a different order – first (2.2.1-3), then (2.2.6-8), then (2.2.4-5)?

2.3. Free groups as subgroups of big enough direct products. Another way of getting a group in which some X-tuple of elements satisfies the smallest possible set of relations is suggested by the following observation. Let G_1 and G_2 be two groups, and let

$$\alpha_1, \beta_1, \gamma_1 \in |G_1|, \qquad \alpha_2, \beta_2, \gamma_2 \in |G_2|.$$

In the direct product group $G = G_1 \times G_2$, define the three elements

$$\alpha = (\alpha_1, \alpha_2), \qquad \beta = (\beta_1, \beta_2), \qquad \gamma = (\gamma_1, \gamma_2).$$

Then the set of relations satisfied by α, β, γ in G will be precisely the *intersection* of the sets of relations satisfied by $\alpha_1, \beta_1, \gamma_1$ in G_1 and by $\alpha_2, \beta_2, \gamma_2$ in G_2. This may be seen from the fact that for any $s \in T$,

$$s_G(\alpha, \beta, \gamma) = (s_{G_1}(\alpha_1, \beta_1, \gamma_1),\ s_{G_2}(\alpha_2, \beta_2, \gamma_2)),$$

as is easily verified by induction.

More generally, if we take an arbitrary family of groups $(G_i)_{i \in I}$, and in each of them three elements $\alpha_i, \beta_i, \gamma_i$, then in the product group $G = \prod G_i$ we can define the three elements

$$\alpha = (\alpha_i)_{i \in I}, \quad \beta = (\beta_i)_{i \in I}, \quad \gamma = (\gamma_i)_{i \in I},$$

and the relations that these satisfy will be just those relations satisfied simultaneously by our 3-tuples in all of the groups.

This suggests that by using a *large enough* such family, we could arrive at a group with three elements α, β, γ which satisfy a *smallest* possible set of relations.

How large a family $(G_i, \alpha_i, \beta_i, \gamma_i)$ should we use?

Well, we could be sure of getting the least set of relations if we could use the class of *all* groups and *all* 3-tuples of elements of these. But taking the direct product of such a family would give us set-theoretic indigestion!

We can cut down this surfeit of groups a bit by noting that for any group G_i and three elements $\alpha_i, \beta_i, \gamma_i$, if we let H_i denote the subgroup of G_i generated by these three elements, it will suffice for our product to involve the group H_i, rather than the whole group G_i, since the relations satisfied by α_i, β_i and γ_i in the group G_i and in the subgroup H_i are the same. Now a finitely generated group is countable (meaning finite *or* countably infinite), so we see that it would be enough to let $(G_i, \alpha_i, \beta_i, \gamma_i)$ range over all *countable* groups, and 3-tuples of elements thereof.

However, the class of all countable groups is still not a set. Indeed, even the class of one-element groups is not a set, because we get a different (in the strict set-theoretic sense) group for each choice of that one element. (For those not familiar with such considerations: In set theory, every element of a set is a set. If we had a set of *all* one-element groups, then we could form from this the set of all *members* of their underlying sets, which would be the set of *all* sets; and one knows that this does not exist.) But this is clearly just a quibble – obviously, if we choose any one-element set $\{x\}$, and take the unique group with this underlying set, it will serve as well as any other one-element group so far as honest group-theoretic purposes are concerned. In the same way, I claim we can find a genuine set of countable groups that, *up to isomorphism*, contains *all* the countable groups. Namely, let S be a fixed countably infinite set. Then we can take the set of all groups G whose underlying sets $|G|$ are subsets of S. Or, to hit more precisely what we want, let

$$\{(G_i,\ \alpha_i,\ \beta_i,\ \gamma_i)\ \mid\ i\in I\} \tag{2.3.1}$$

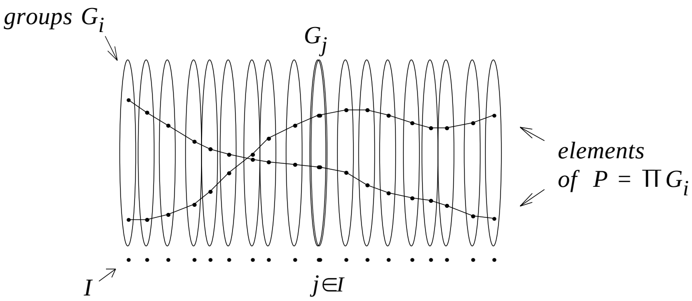
be the set of all 4-tuples such that G_i is a group with $|G_i| \subseteq S$, and α_i, β_i and γ_i are members of $|G_i|$. Now for any countable group H and three elements $\alpha,\ \beta,\ \gamma \in |H|$, we can clearly find an isomorphism θ between one of the G_i's and H, such that $\theta(\alpha_i) = \alpha$, $\theta(\beta_i) = \beta$, $\theta(\gamma_i) = \gamma$.

So, having defined the set (2.3.1), let P be the direct product group $\prod_I G_i$. Let $a,\ b,\ c$ be the I-tuples $(\alpha_i),\ (\beta_i),\ (\gamma_i) \in |P|$, and F the subgroup of P generated by a, b, and c. I claim that F is a free group on a, b, and c.

We could prove this by considering relations satisfied by a, b, and c as suggested above, but let us instead verify directly that F satisfies the *universal property* (2.1.3) characterizing free groups. Let G be any group, and $\alpha,\ \beta,\ \gamma$ three elements of G. We want to prove that there exists a unique homomorphism $h\colon F \to G$ carrying $a,\ b,\ c \in |F|$ to $\alpha,\ \beta,\ \gamma \in |G|$ respectively. Uniqueness will be no problem – by construction F is generated by a, b and c, so if such a homomorphism exists it is unique. To show the existence of h, note that the subgroup H of G generated by $\alpha,\ \beta,\ \gamma$ is countable, hence as we have noted, there exists for some $j \in I$ an isomorphism $\theta\colon G_j \cong H$ carrying $\alpha_j,\ \beta_j,\ \gamma_j \in |G_j|$ to $\alpha,\ \beta,\ \gamma \in |H|$. Now the projection map p_j of the product group $P = \prod G_i$ onto its jth coordinate takes a, b and c to $\alpha_j,\ \beta_j,\ \gamma_j$, hence composing this projection with θ, as shown in the diagram below, we get a homomorphism $h\colon F \to G$ having the desired effect on $a,\ b,\ c$.

$$\begin{tikzcd}
F \arrow[d, hook] & a,\ b,\ c \in |F| \arrow[d, hook] \\
P = \prod G_i \arrow[d, "p_j"'] & \prod |G_i| \arrow[d] \\
G_j \arrow[d, phantom, "\theta\ \wr\|"] & \alpha_j,\ \beta_j,\ \gamma_j \in |G_j| \arrow[d, phantom, "\wr\|"] \\
H \arrow[d, hook] & |H| \arrow[d, hook] \\
G & \alpha,\ \beta,\ \gamma \in |G|
\end{tikzcd}$$

For a useful way to picture this construction, think of P as the group of all functions on the base-space I, taking at each point i a value in G_i. F is the subgroup of functions generated by a, b and c.



Given α, β, γ in any group G, identify the subgroup of G that they generate with an appropriate G_j $(j\in I)$. Then the homomorphism h that we constructed above may be thought of as giving the value of each element of F at the point j, where a, b and c have the values α, β and γ. We have chosen our space I and values for a, b and c sufficiently eclectically so that it is possible to choose points at which a, b and c take on *any* 3-tuple of values in (up to isomorphism!) *any* group. Thus, the functions a, b and c are a ‘‘universal’’ 3-tuple of group-elements.

The same argument works if we replace ‘‘3-tuple’’ by ‘‘X-tuple’’, where X is any countable set. Here we use the observation that a group generated by a countable family of elements is countable. For X of arbitrary cardinality, one can easily show that any group H generated by an X-tuple of elements has cardinality $\leq \max(\mathrm{card}(X), \aleph_0)$. Hence we get:

Proposition 2.3.2. *X be any set. Take a set S of cardinality* $\max(\mathrm{card}(X), \aleph_0)$, *and let* $\{(G_i, u_i) \mid i\in I\}$ *be the set of all pairs such that G_i is a group with $|G_i| \subseteq S$, and u_i is a map $X \to |G_i|$ (i.e., an X-tuple of elements of G_i). Let $P = \prod_I G_i$, and map X into P by defining $u(x)$ $(x\in X)$ to be the element with component $u_i(x)$ at each i. Let F be the subgroup of P generated by* $\{u(x) \mid x\in X\}$.

Then the pair (F, u) is a free group on the set X. □

Digression: Let S_3 be the symmetric group on three letters. Suppose we had begun the above investigation with a less ambitious goal: merely to find a group J with three elements a, b, c such that

(2.3.3) For every choice of three elements $\alpha, \beta, \gamma \in |S_3|$, there exists a unique homomorphism $h: J \to S_3$ taking a, b, c to α, β, γ respectively.

$$\begin{tikzcd} a, b, c\in|J| \arrow[d, "h"] & J \arrow[d] \\ \alpha, \beta, \gamma \in|S_3| & S_3 \end{tikzcd}$$

Then we could have performed the above construction just using 4-tuples $(S_3, \alpha, \beta, \gamma)$ $(\alpha, \beta, \gamma \in|S_3|)$ as our $(G_i, \alpha_i, \beta_i, \gamma_i)$. There are $6^3 = 216$ such 4-tuples, so P would be the direct product of 216 copies of S_3, and a, b, c would be elements of this product which, as one runs over the 216 coordinates, take on all possible combinations of values in S_3. The subgroup J they generate would indeed satisfy (2.3.3). This leads to:

Exercise 2.3:1. Does condition (2.3.3) characterize (J, a, b, c) up to isomorphism? If not, is there some additional condition that (J, a, b, c) satisfies which together with (2.3.3) determines it up to isomorphism?

Exercise 2.3:2. Investigate the structure of the group J, and more generally, of the analogous groups constructed from S_3 using different numbers of generators. To make the problem concrete, try to determine, or estimate as well as possible, the *orders* of these groups, for 1, 2, 3 and generally, for n generators.

The two methods by which we have constructed free groups above go over essentially word-for-word with ''group'' replaced by ''ring'', ''lattice'', or a great many other types of mathematical objects. The determination of just what classes of algebraic structures allow this and related sorts of universal constructions is one of the themes of this course. The next exercise concerns a negative example.

Exercise 2.3:3. State what would be meant by a ''free field on a set X'', and show that no such object exists for any set X. If one attempts to apply the two methods of this and the preceding section to prove the existence of free fields, where does each of them fail?

Exercise 2.3:4. Let $\mathbf{Z}[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over the integers (= the free commutative ring on n generators – cf. §3.12 below). Its field of fractions $\mathbf{Q}(x_1, \ldots, x_n)$, the field of ''rational functions in n indeterminates over the rationals'', looks in some ways like a ''free field on n generators''. E.g., one often speaks of evaluating a rational function at some set of values of the variables. Can some concept of ''free field'' be set up, involving perhaps a modified universal property, or some concept of comparing relations in the *field* operations satisfied by n-tuples of elements in two fields, in terms of which $\mathbf{Q}(x_1, \ldots, x_n)$ would indeed be the free field on n generators?

Exercise 2.3:5. A *division ring* (or *skew field* or *sfield*) is a ring (associative but not necessarily commutative) in which every nonzero element is invertible. If you a find a satisfactory answer to the preceding exercise, you might consider the question of whether there exists in the same sense a *free sfield* on n generators. (This was a longstanding open question, which was finally answered in 1966. I can refer interested students to papers in this area.)

There are many hybrids and variants of the two constructions we have given for free groups. For instance, we might start with the set T of terms in X, and define $p \sim q$ (for $p, q \in T$) to mean that for every map v of X into a group G, one has $p_v = q_v$ in G. Now for each pair $(p, q) \in T \times T$ such that $p \sim q$ *fails* to hold, we can choose a map $u_{p,q}$ of X into a group $G_{p,q}$ such that $p_{u_{p,q}} \neq q_{u_{p,q}}$. We can then form the direct product group $P = \prod G_{p,q}$, take the induced map $u: X \to |P|$, and check that the subgroup F generated by the image of this map will satisfy condition (i) of Corollary 2.1.2. Interestingly, for X countable, this construction uses a product of *fewer* groups $G_{p,q}$ than we used in the version given above.

Finally, consider the following construction, which suffers from severe set-theoretic difficulties, but is still interesting. (I won't try to resolve these difficulties here, but will talk sloppily, as though they did not occur.)

Define a ''generalized group-theoretic operation in three variables'' as *any* function p which associates to every group G and three elements $\alpha, \beta, \gamma \in G$ an element $p(G, \alpha, \beta, \gamma) \in |G|$. We can ''multiply'' two such operations p and q by defining

$$(p \cdot q)(G, \alpha, \beta, \gamma) = p(G, \alpha, \beta, \gamma) \cdot q(G, \alpha, \beta, \gamma) \in |G|.$$

for all groups G and elements $\alpha, \beta, \gamma \in |G|$. We can similarly define the multiplicative inverse of an operation p, and the constant operation, e. We see that the class of generalized group-theoretic operations will satisfy the group axioms under the above three operations. Now consider the three generalized group-theoretic operations a, b and c, defined by

$$a(G,\alpha,\beta,\gamma) = \alpha, \quad b(G,\alpha,\beta,\gamma) = \beta, \quad c(G,\alpha,\beta,\gamma) = \gamma.$$

Let us define a ‘‘derived generalized group-theoretic operation’’ as one obtainable from a, b and c by taking products, inverses, and the neutral element. Then the set of derived generalized group-theoretic operations will form a free group on the generators a, b and c. (This is really just a disguised form of our naive ‘‘direct product of *all* groups’’ idea.)

Exercise 2.3:6. Call a generalized group-theoretic operation p *functorial* if for every homomorphism of groups $f: G \to H$, one has $p(H, f(\alpha), f(\beta), f(\gamma)) = f(p(G, \alpha, \beta, \gamma))$ $(\alpha,\beta,\gamma \in |G|)$. We will see the reason for this term in Chapter 6. Show that all derived group-theoretic operations are functorial. Is the converse true?

Exercise 2.3:7. Same problem for functorial generalized operations on the class of all *finite groups*.

2.4. The classical construction: free groups as groups of words. The constructions discussed above have the disadvantage of not giving very explicit descriptions of free groups. We know that every element of a free group F on the set X arises from a term in the elements of X and the group operations, but we don’t know how to tell whether two such terms – say $(b(a^{-1}b)^{-1})(a^{-1}b)$ and e – yield the same element; in other words, whether $(\beta(\alpha^{-1}\beta)^{-1})(\alpha^{-1}\beta) = e$ is true for all elements α, β of all groups. If it is, then by the results of §2.2 one can obtain this fact *somehow* by the procedures corresponding to conditions (2.2.1-8); if it is not, the ideas of §2.3 suggest we should seek some particular elements for which it fails, in some particular group in which we know how to calculate. But these approaches are hit-and-miss.

In this section, we shall construct the free group on X in a much more explicit way. We will then be able to answer such questions by calculating in the free group itself.

We first recall an important consequence of the associative identity: that ‘‘products can be written without parentheses’’. For example, given elements a, b, c of a group, the elements $(a(c(ab)))$, $(a((ca)b))$, $((ac)(ab))$, $((a(ca))b)$ and $(((ac)a)b)$ are all equal. It is conventional, and usually convenient, to say, ‘‘Let us therefore write their common value as $acab$.’’ However, we will soon want to relate these expressions to group-theoretic *terms*; so instead of dropping parentheses, let us agree to take $(a(c(ab)))$ as the common form to which we shall reduce all of the above five expressions, and generally, let us note that any product of elements can be reduced by the associative law to one with parentheses clustered to the right: $(x_n\,(x_{n-1}\,(\ldots(x_2\,x_1)\ldots)))$.

In particular, given two elements written in this form, we can write down their product and reduce it to this form:

$$\begin{aligned}&((x_n\,(\ldots(x_2\,x_1)\ldots))\cdot(y_m\,(\ldots(y_2\,y_1)\ldots)))\\ &\quad= (x_n\,(\ldots(x_2\,(x_1\,(y_m\,(\ldots(y_2\,y_1)\ldots))))\ldots)).\end{aligned}$$

If we want to find the inverse of an element written in this form, we may use the formula $(xy)^{-1} = y^{-1}x^{-1}$, another consequence of the group laws. By induction this gives $(x_n(\ldots(x_2x_1)\ldots))^{-1} = ((\ldots(x_1^{-1}x_2^{-1})\ldots)\,x_n^{-1})$, which we reduce to $(x_1^{-1}\,(\ldots(x_{n-1}^{-1}\,x_n^{-1})\ldots))$.

More generally, if we started with an expression of the form

$$(x_n^{\pm 1}(\ \ldots\ (x_2^{\pm 1}x_1^{\pm 1})\ldots))$$

(where each factor is either x_i or x_i^{-1}, and the exponents are independent), then the above method together with the fact $(x^{-1})^{-1} = x$ (another consequence of the group axioms) allows us to write the inverse as $(x_1^{\mp 1}(\ldots(x_{n-1}^{\mp 1}\, x_n^{\mp 1})...))$, which is of the same form as the expression we started with; and likewise, the *product* of two expressions of the above form reduces to an expression of the same form.

Note further that if two successive factors $x_i^{\pm 1}$ and $x_{i+1}^{\pm 1}$ are respectively x and x^{-1} for some $x \in X$, or are respectively x^{-1} and x for some $x \in X$, then by the group axioms on inverses and the neutral element (and again, associativity), we can drop this pair of factors – unless they are the only factors in the product, in which case we can rewrite the product as e.

Finally, easy consequences of the group axioms tell us what the inverse of e is (namely e), and how to multiply anything by e. Putting these observations together, we conclude that *given any set* X *of elements of a group* G, the set of elements of G that can be written in one of the forms

(2.4.1)
$$e, \quad \text{or} \quad (x_n^{\pm 1}(\,.\,.\,.(x_2^{\pm 1}\, x_1^{\pm 1})...)),$$
where $n \geq 1$, each $x_i \in X$, and no two successive factors are an element of X and the inverse of the same element, in either order,

is closed under products and inverses. So this set must be the whole subgroup of G generated by X. In other words, any member of the subgroup generated by X can be reduced to an expression (2.4.1).

In the above paragraphs, X has been a subset of a group. Now let X be an arbitrary set, and as in the preceding section, let T be a set of all group-theoretic terms in elements of X (Definition 1.5.1). For convenience, let us assume T chosen so as to contain X, with symb_T being the inclusion map. (If you prefer not to make this assumption, then in the argument to follow, you should insert ‘‘symb_T’’ at appropriate points.) Let $T_{\mathrm{red}} \subseteq T$ denote the set of terms of the form (2.4.1) (‘‘red’’ standing for *reduced*). If $s, t \in T_{\mathrm{red}}$, we can form their product $s\cdot t$ in T, and then, as we have just seen, rearrange parentheses to get an element of T_{red} which is equivalent to $s\cdot t$ so far as evaluation at X-tuples of elements of groups is concerned. Let us call this element $s \odot t$. Thus $s \odot t$ has the properties that for any map $v\colon X \to |G|$, (G a group) one has $(s\cdot t)_v = (s \odot t)_v$, and that $s \odot t \in T_{\mathrm{red}}$. In the same way, given $s \in T_{\mathrm{red}}$, we can obtain from $s^{-1} \in T$ an element we shall call $s^{(-)} \in T_{\mathrm{red}}$, such that for any map $v\colon X \to |G|$, one has $(s_v)^{-1} = (s^{(-)})_v$.

Are any further reductions possible? In a particular group there may be various equalities among the values of such expressions; but we are only interested in reductions that can be done in all groups. No more are obvious; but can we be sure that some sneaky application of the group axioms wouldn’t allow us to prove some two distinct terms of the form (2.4.1) to have the same evaluations at all X-tuples of elements of all groups? (In such a case, we should not lose hope, but should introduce further reductions that would always replace one of these expressions by the other.)

Let us formalize the preceding observations, and indicate the significance of this question:

Lemma 2.4.2. *For each* $s \in T$, *there exists an* $s' \in T_{\mathrm{red}}$ (*i.e., an element of* T *of one of the forms shown in* (2.4.1)) *such that*

(2.4.3) *for every map* v *of* X *into any group* G, $s'_v = s_v$ *in* $|G|$.

Moreover, if one of the following statements is true, all are:

(i) *For each* $s \in T$, *there exists a* unique $s' \in T_{\mathrm{red}}$ *satisfying* (2.4.3).

(ii) *If* $s,\ t$ *are* distinct *elements of* T_{red}, *then* ''$s = t$'' *is* not *an identity for groups; that is, for some* G *and some* $v\colon X \to |G|$, $s_v \neq t_v$.

(iii) *The* 4*-tuple* $F = (T_{\mathrm{red}},\ \odot,\ {}^{(-)},\ e_T)$ *is a group.*

(iv) *The* 4*-tuple* $F = (T_{\mathrm{red}},\ \odot,\ {}^{(-)},\ e_T)$ *is a* free *group on* X.

Proof. We get the first sentence of the lemma by an induction, which I will sketch briefly. The assertion holds for elements $x \in X$: We simply take $x' = x$. Now suppose it true for two terms s, $t \in T$. To establish it for $s \cdot t \in T$, define $(s \cdot t)' = s' \odot t'$. One likewise gets it for s^{-1} using $s^{(-)}$, and it is clear for e. It follows from condition (c) of the definition of ''group-theoretic term'' (Definition 1.5.1) that it is true for all elements of T.

The equivalence of (i) and (ii) is straightforward. Assuming these conditions, let us verify that the 4-tuple F defined in (iii) is a group. Take $p,\ q,\ r \in T_{\mathrm{red}}$. Then $(p \odot (q \odot r))$ and $((p \odot q) \odot r)$ are two elements of T_{red}, call them s and t. For any $v\colon X \to |G|$, $s_v = t_v$ by the associative law for G. Hence by (ii) $s = t$, proving that $\odot$ is associative. The other group laws for F are deduced in the same way.

Conversely, assuming (iii), we claim that for distinct elements $s,\ t \in T_{\mathrm{red}}$, we can prove, as required for (ii), that the equation ''$s = t$'' is not an identity by getting a counterexample in F itself. Indeed if we let v be the inclusion $X \to T_{\mathrm{red}} = |F|$, we can check by induction on n in (2.4.1) that for all $s \in T_{\mathrm{red}}$, $s_v = s$. Hence $s \neq t$ implies $s_v \neq t_v$, as desired.

Since (iv) certainly entails (iii), our proof will be complete if we can show, assuming (iii), that F has the universal property of a free group. Given any map $v\colon X \to |G|$, we map $|F| = T_{\mathrm{red}} \to |G|$ by $s \mapsto s_v \in |G|$. From the properties of $\odot$ and ${}^{(-)}$, we know that this is a homomorphism h such that $h \,|\, X$ (the restriction of h to X) is v; and since X generates F, h is the unique homomorphism with this property, as desired. □

Well – are statements (i)-(iv) true, or not??

The usual way to answer this question is to test (iii) by writing down precisely how the operations $\odot$ and ${}^{(-)}$ are performed, and checking the group axioms on them. Since a term of the form (2.4.1) is uniquely determined by the integer n (which we take to be 0 for the term e) and the n-tuple of elements of X and their inverses, $(x_n^{\pm 1}, \ldots, x_1^{\pm 1})$, one describes $\odot$ and ${}^{(-)}$ as operations on such n-tuples. E.g., one multiplies two tuples $(w, \ldots, x)$ and $(y, \ldots, z)$ (where $w, \ldots, z$ are each an element of X or a symbolic inverse of such an element) by uniting them as $(w, \ldots, x, y, \ldots, z)$, then dropping pairs of factors that may now cancel (e.g., x and y above if y is x^{-1}); and repeating until no such cancelling pairs remain.

But checking the associative law for this recursively defined operation turns out to be very tedious, involving a number of different cases. (E.g., you might try checking associativity for $(v, w, x) \cdot (x^{-1}, w^{-1}, y^{-1}) \cdot (y, w, z)$, and for $(v, w, x) \cdot (x^{-1}, z^{-1}, y^{-1}) \cdot (y, w, z)$, where $w,\ x,\ y$ and z are four distinct elements of X. Both cases work, but they are different computations.)

But there is an elegant trick, not as well known as it ought to be, which rescues us from the toils of this calculation. We construct a certain G which we *know* to be a group, in which we can verify condition (ii) – rather than condition (iii) – of the above lemma.

To see how to construct this G, let us go back to basics and recall where the *group identities* we are trying to verify come from. They are identities which are satisfied by *permutations* of any set A, under composition of permutations, inverses of permutations, and the identity permutation.

So let us try to describe a set A on which the group we want to construct should act by permutations in as ‘‘free’’ a way as possible, by specifying the permutation of A that should represent the image of each $x \in X$.

To start our construction, let a be any symbol not in X. Now define A to be the set of all strings of symbols of the form:

(2.4.4)
$$x_n^{\pm 1}\, x_{n-1}^{\pm 1} \ldots x_1^{\pm 1}\ a$$
where $n \geq 0$, each $x_i \in X$, and no two successive factors $x_i^{\pm 1}$ and $x_{i+1}^{\pm 1}$ are an element of X and the inverse of that same element, in either order.

In particular, taking $n = 0$, note that $a \in A$.

Let G be the group of *all* permutations of A. Define for each $x \in X$ an element $v(x) \in |G|$ as follows. Given $b \in A$,

if b does *not* begin with the symbol x^{-1}, let $v(x)$ take b to the symbol $x\,b$, formed by putting an x at the beginning of the symbol b;

if b does begin with x^{-1}, say $b = x^{-1}c$, let $v(x)$ take b to the symbol c, formed by removing x^{-1} from the beginning of b.

It is immediate from the definition of A that $v(x)(b)$ belongs to A in each case. To check that $v(x)$ is invertible, consider the map which sends a symbol $x\,b$ to b, and a symbol c not beginning with x to the symbol $x^{-1}c$; we find that this is a 2-sided inverse to $v(x)$.

So we now have a map $v\colon X \to |G|$. As in §1.6, this induces an evaluation map $s \mapsto s_v$ taking the set T of terms in X into $|G|$. Now consider any $s = (x_n^{\pm 1}(\ldots(x_2^{\pm 1} x_1^{\pm 1})\ldots)) \in T_{\mathrm{red}}$. It is easy to verify by induction on n that the permutation $s_v \in |G|$ takes $a \in A$ to the symbol $x_n^{\pm 1} \ldots x_1^{\pm 1}\ a$. (In particular, for $s = e$, $s_v(a) = a$.) It follows that if s and t are distinct elements of T_{red}, $s_v(a)$ and $t_v(a)$ are distinct elements of A, so $s_v \neq t_v$ in G, establishing (ii) of Lemma 2.4.2. By that lemma we now have

Proposition 2.4.5. $F = (T_{\mathrm{red}}, \odot, {}^{(-)}, e)$ *is a group; in fact, letting* u *denote the inclusion* $X \to T_{\mathrm{red}}$, *the pair* (F, u) *is a* free group *on* X.

Using parenthesis-free notation for products, and identifying each element of X *with its image in* F, *this says that every element of the free group on* X *can be written uniquely as*

$$e, \qquad \text{or} \qquad x_n^{\pm 1} \ldots x_2^{\pm 1}\, x_1^{\pm 1},$$

where in the latter case, $n \geq 1$, *each* $x_i \in X$, *and no two successive factors* $x_i^{\pm 1}$ *and* $x_{i+1}^{\pm 1}$ *are equal to an element of* X *and the inverse of that same element, in either order.* □

We have obtained what is called a *normal form* for elements of a free group on X – a unique expression for each member of the group, to which we can algorithmically reduce an arbitrary expression. This indeed allows us to calculate explicitly in the free group; e.g., you should find it straightforward to do

Exercise 2.4:1. Determine whether each of the following equations holds for all elements x, y, z of all groups:

(i) $(x^{-1}yx)^{-1}(x^{-1}zx)(x^{-1}yx) = (yx)^{-1}z(yx)$.

(ii) $(x^{-1}y^{-1}xy)^2 = x^{-2}y^{-1}x^2y$.

In the next exercise, we use the group theorists' abbreviations $x^y = y^{-1}xy$ for the *conjugate* of an element x by an element y in a group, and $[x,y] = x^{-1}y^{-1}xy$ for the *commutator* of x and y. Recall also that if H_1, H_2 are subgroups of a group G, then $[H_1, H_2]$ denotes the subgroup of G *generated* by all commutators $[h_1, h_2]$ with $h_1 \in H_1$ and $h_2 \in H_2$.

Exercise 2.4:2. (i) Prove a group identity of the form

$$[[x^{\pm 1}, y^{\pm 1}], z^{\pm 1}]^{x^{\pm 1}} [[z^{\pm 1}, x^{\pm 1}], y^{\pm 1}]^{z^{\pm 1}} [[y^{\pm 1}, z^{\pm 1}], x^{\pm 1}]^{y^{\pm 1}} = e$$

for some choice of the exponents ±1. (There is a certain amount of leeway in these exponents; make your final choice to get maximum symmetry. The result is known as *Phillip Hall's identity*; however its form may vary with the text, depending on whether the above definition of $[x,y]_1$ preferred by most contemporary group theorists is used, or the less common definition $xyx^{-1}y^{-1}$.)

(ii) Deduce that if A, B and C are subgroups of a group G such that two of $[[A, B], C]$, $[[B, C], A]$, $[[C, A], B]$ are trivial, then so is the third. (The ''three subgroups theorem''.)

(iii) Deduce that if A and B are two subgroups of G, and $[A, [A, B]]$ is trivial, then so is $[[A, A], B]$. Is the converse true?

The idea of finding normal forms, or other explicit descriptions, of objects defined by universal properties is a recurring one in algebra. The form we have found is specific to free groups. It might appear at first glance that corresponding forms could be obtained mechanically from any finite system of operations and identities; e.g., those defining rings, lattices, etc.; and thus that the results of this section should generalize painlessly (as those of the two preceding sections indeed do!) to very general classes of structures. But this is not so. An example we shall soon see (§3.5) is that of the Burnside problem, where a sweet and reasonable set of axioms obstinately refuses to yield a normal form. Other nontrivial cases are free Lie algebras [**62**] and free lattices [**4**, §VI.8], for which normal forms are known, but complicated; free modular lattices, for which it has been proved that the word problem is undecidable (no recursive normal form can exist); and groups defined by particular families of generators and relations (§3.3 below), for which the word problem has been proved undecidable in some cases, while nice normal forms exist in others. In general, normal form questions must be tackled case by case, but for certain large families of cases there *are* interesting general methods (cf. [**37**]); I hope eventually to add a chapter on that subject to these notes.

The trick that we used to show that the set of terms T_{red} constitutes a normal form for the elements of the free group is due to van der Waerden, who introduced it in [**102**] to handle the more difficult case of coproducts of groups (§3.6 below). Though the result we proved is, as we have said, specific to groups, the idea behind the proof is a versatile one: If you can reduce all expressions for elements of some universal structure F to members of a set T_{red}, and wish to show that these give a normal form, look for a ''representation'' of F (in whatever sense is appropriate to the structure in question – in the group-theoretic context this was ''an action of the group F on a set A'') which distinguishes the elements of T_{red}. A nice twist which often occurs, as in the above case, is that the object on which we ''represent'' F may be the set T_{red} itself, or some closely related object.

My development of Proposition 2.4.5 was full of motivations, remarks, etc.. You might find it instructive to write out for yourself a concise, direct, self-contained proof that the set of terms indicated in Proposition 2.4.5, under the operations described, forms a group, and that this has the universal property of the free group on X.

Exercise 2.4:3. If X is a set, and $s \neq t$ are two reduced group-theoretic terms in the elements of X (as in Proposition 2.4.5(ii)), will there in general exist a *finite* group G, and a map $v: X \to |G|$, such that $s_v \neq t_v$? (In other words, are the only identities satisfied by all finite groups those holding in all groups?)

If you succeed in answering the above question, you might try the more difficult ones in the next exercise.

Exercise 2.4:4. (i) If X is a set, F the free group on X, H a subgroup of F, and s an element of F such that $s \notin |H|$, will there in general exist a finite group G, and a homomorphism $f: F \to G$, such that $f(s) \notin f(|H|)$?

(ii) Same question, under the assumption that the subgroup H is finitely generated.

Free groups can also be represented by matrices:

Exercise 2.4:5. Let $\mathrm{SL}(2, \mathbf{Z})$ denote the group of all 2×2 matrices of integers with determinant 1, and let H be the subgroup thereof generated by the two elements $x = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$. Show that H is free on $\{x, y\}$. (Hint: Let c be the column vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Examine the form of the column vector obtained by applying an arbitrary reduced group-theoretic word in x and y to c.)

If you do the above, you might like to think further about what pairs of integers, or for that matter, pairs of real or complex numbers, can replace the two ‘‘3’’s in the above matrices. For integers, the answer is known; for the general case there are many partial results (see [**54**]), but one is far from a complete answer.

Chapter 3. A Cook's tour of other universal constructions.

We shall now look at a number of other constructions having many similarities to that of free groups. In each case, the construction can be motivated by a question of the form, ‘‘Suppose we have a structure about which we know only that it satisfies such and such conditions. How much can we say about it on the basis of this information alone?’’ In favorable cases, we shall find that if we collect the ‘‘data’’ we can deduce about such an object, this data itself can be made into an object F, which satisfies the given conditions, and satisfies no relations not implied by these (cf. remark 2.2.10). This F is then a ‘‘universal’’ example of these conditions, and that fact can be translated into a ‘‘universal mapping property’’ for F.

Although the original ‘‘What can we say?’’ question and the ‘‘least set of relations’’ property are useful as motivation and intuition, the universal mapping property gives the characterization of these constructions that is most useful in applications. So, though I will *sometimes* refer explicitly to the motivating ideas, and other times leave them for you to see, we will *always* characterize these objects by universal properties.

The existence of these universal objects may in most cases be proved from scratch by either of the methods of §§2.2 and 2.3: construction from below, as sets of terms modulo necessary relations, and construction from above, as subobjects of big direct products. But often we will, alternatively, be able to combine previously described universal constructions to get the new one.

Where possible, we will get explicit information on the structure of the new object – a normal form or other such description. It is a mark of the skilled algebraist, when working with objects defined by universal properties, to know when to use those properties, and when to turn to an explicit description.

As we move through this chapter, I shall more and more often leave standard details for the reader to fill in: the precise meaning of an object ‘‘universal for’’ a certain property, the verification that such an object exists, etc.. In the later sections, commutative diagrams illustrating universal properties will often be inserted without explanation. These diagrams are not substitutes for assertions, but aids to the reader in visualizing the situation of the assertion he or she needs to formulate.

Constructions of *groups* will receive more than their rightful share of attention here because groups give a wide range of interesting examples, and are more familiar to many students than lattices, noncommutative rings (my own love), Lie algebras, etc..

Let us begin by noting how some familiar elementary group-theoretic constructions can be characterized by universal properties.

3.1. Subgroup and normal subgroup of G generated by $S \subseteq |G|$. Suppose we are explicitly given a group G, and a subset S of $|G|$.

Consider a subgroup A of G about which we are told only that it contains all elements of the set S. How much can we say about A?

Clearly A contains all elements of G that can be obtained from the elements of S by repeated formation of products and inverses, and also contains the neutral element. This is all we can deduce, for it is easy to see that the set of elements which can be so obtained will form the underlying set of a subgroup of G, called *the subgroup* $<S>$ *generated by* the set S. This description builds $<S>$ up ‘‘from below’’. We can also obtain it ‘‘from above’’ as the intersection of all subgroups of G containing S. Whichever way we obtain it, the defining universal property of $<S>$ is that it is a subgroup which contains S, and lies in every subgroup A of G that contains S:

$$\begin{tikzcd} S \arrow[r, phantom, "\subseteq"] \arrow[dr, phantom, "\subseteq" sloped] & {|<S>|} \arrow[d, phantom, "\cap" sloped] & & {<S>} \arrow[d, phantom, "\cap" sloped] \\ & {|A|} & & A \end{tikzcd}$$

(In the second part of the above display, we denote the group homomorphism given by an inclusion map of underlying sets by an inclusion sign between the symbols for the groups; a slight abuse of notation.)

We know a somewhat better description of the elements of $<S>$ than the one I just gave: Each such element is either e or the product of a sequence of elements of S and their inverses. A related observation is that $<S>$ is the image of the map of the free group F on S into G induced by the inclusion-map $S \to |G|$. One may get still better descriptions in particular cases. For instance, if $S = \{a, b, c\}$ and a, b and c commute, then $<S>$ consists of all elements $a^m b^n c^p$; if G is the additive group of integers, then the subgroup generated by $\{1492, 1974\}$ is the subgroup of all even integers; if G is the permutation group S_n and S consists of the two permutations (12) and $(12 \ldots n)$, then $<S>$ is all of G.

There is likewise a least *normal* subgroup of G containing S. This is called ''the normal subgroup of G generated by S'', and has the corresponding universal property, with the word ''normal'' inserted before ''subgroup''.

Exercise 3.1:1. Show that the normal subgroup $N \subseteq G$ generated by S is the subgroup of G generated by $\{gsg^{-1} \mid g \in |G|,\ s \in S\}$.

Can $|N|$ also be described as $\{ghg^{-1} \mid g \in |G|,\ h \in |<S>|\}$?

Exercise 3.1:2. Let G be the free group on two generators x and y, and n a positive integer. Show that the normal subgroup of G generated by x^n and y is generated as a subgroup by x^n and $\{x^i y x^{-i} \mid 0 \leq i < n\}$, and is in fact a *free* group on this set. Also describe the normal subgroup we get if we let $n = 0$.

3.2. Imposing relations on a group; quotient groups. Suppose next that we are given a group G, and are interested in homomorphisms of G into other groups, $f: G \to H$, which make certain specified pairs of elements fall together. That is, let us be given a family of pairs of elements $\{(x_i, y_i) \mid i \in I\} \subseteq |G| \times |G|$ (perhaps only one pair, (x, y)) and consider homomorphisms f from G into other groups, which satisfy

$$(3.2.1) \qquad (\forall i \in I)\ \ f(x_i) = f(y_i).$$

Note that given one homomorphism $f: G \to H$ with this property, we can get more such homomorphisms $G \to K$ by forming composites gf of f with arbitrary homomorphisms $g: H \to K$. It would be nice to know whether there exists one pair (H, f) which satisfies (3.2.1) and is *universal* for this condition, in the sense that given any other pair (K, h) satisfying it, there is a unique homomorphism $g: H \to K$ making the diagram below commute.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "\forall h"] & H \arrow[d, "\exists 1\, g"] \\ & K \end{tikzcd}$$

It is not hard to prove the existence of such a universal pair, either by a ‘‘group-theoretic terms modulo an equivalence relation’’ construction, as in §2.2, or by an ‘‘image in a big direct product’’ construction, as in §2.3. But let us look at the problem another way. Condition (3.2.1) is clearly equivalent to

$$(\forall\, i\in I)\ \ f(x_i y_i^{-1}) \ =\ e. \qquad (3.2.2)$$

So we are looking for a universal homomorphism which annihilates (sends to e) a certain family of elements of $|G|$. We know that the set of elements annihilated by a group homomorphism is always a normal subgroup, so this is equivalent to saying that f should annihilate the normal subgroup N of G generated by $\{x_i y_i^{-1} \mid i\in I\}$, mentioned at the end of the preceding section. And in fact, the pair $(G/N,\, q)$, where N is this normal subgroup, G/N is the quotient group, and $q\colon G \to G/N$ is the quotient map, has precisely the universal property we want:

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[dr, "\forall h"] & G/N \arrow[d, "\exists 1\, g"] \\ & K \end{tikzcd}$$

So this quotient group is the solution to our problem.

If we had never seen the construction of the quotient of a group by a normal subgroup, an approach like the above would lead to a motivation of that construction. We would ask, ‘‘What do we know about a group H, given that it has a homomorphism of G into it satisfying (3.2.1)?’’ We would observe that it contains an image $f(a)$ of each $a\in G$, and that two such images are equal *if* they belong to the same coset of the normal subgroup generated by the $x_i y_i^{-1}$’s. We would discover how the group operations must act on these images-of-cosets, and conclude that this set of cosets, under these operations, was itself a universal example of this situation.

Let us assume even a little more naiveté in

Exercise 3.2:1. Suppose in the above situation that we had not been so astute, and had only noted that $f(a) = f(b)$ when $a b^{-1}$ lies in the *subgroup* generated by $\{x_i y_i^{-1}\}$. Attempt to describe the group operations on the set of equivalence classes under this relation, show where this description fails to be well-defined, and show how this ‘‘failure’’ could lead us to discover the *normality* condition needed.

The construction described above is called *imposing the relations* $x_i = y_i$ on G. We can abbreviate the resulting group $G/(x_i = y_i \mid i\in I)$.

For the next exercise, recall that if G is a group, then a G-*set* means a pair $S = (|S|,\, m)$, where $|S|$ is a set, and $m\colon |G|\times|S| \to |S|$ is a map, which we shall abbreviate by writing $m(g, s) = gs$, satisfying

(3.2.3)
$$(\forall g,\, g'\in|G|,\ s\in|S|)\ \ g(g's)\ =\ (gg')s,$$
$$(\forall s\in|S|)\ \ es\ =\ s,$$

in other words, a set on which G acts by permutations [**28**, §I.5], [**26**, §II.4] [**23**, §1.7]. (We remark that this structure on the set S can be described in two other ways: first, as a homomorphism from G to the group of permutations of $|S|$, and secondly, as a system of unary operations $\bar{g}$ on $|S|$, one for each $g\in|G|$, satisfying *identities* corresponding to all the *relations* holding in G.)

A *homomorphism* $S\to S'$ of G-sets (for a *fixed* group G) means a map $a: |S|\to|S'|$ satisfying

(3.2.4)
$$(\forall g\in|G|,\ s\in|S|)\ \ a(gs)\ =\ g\,a(s).$$

If H is a subgroup of the group G, let $|G/H|$ denote the set of left cosets of H in G. We shall write a typical coset as $[g]=gH$. Then $|G/H|$ can be made the underlying set of a left G-set G/H, by defining $g[g']=[gg']$.

Exercise 3.2:2. Let H be any subgroup of G. Find a universal property characterizing the pair $(G/H,\,[e])$. In particular, what form does this universal mapping property take in the case where $H=\langle x_i^{-1}y_i \mid i\in I\rangle$ for some set $\{(x_i,y_i)\mid i\in I\}\subseteq|G|\times|G|$?

With the concept of imposing relations on a group under our belts, we are now ready to consider

3.3. Groups presented by generators and relations. To start with a concrete example, suppose we are curious about groups G containing two elements a and b satisfying the relation

(3.3.1)
$$ab\ =\ b^2a.$$

One may investigate the consequences of this equation with the help of the group laws. What we would be investigating is, I claim, the structure of the group with a *universal* pair of elements satisfying (3.3.1).

More generally, let X be a set of symbols (in the above example, $X=\{a,b\}$), and let T be the set of all group-theoretic terms in the elements of X. Then formal group-theoretic *relations* in the elements of X mean formulae "$s=t$", where $s,t\in T$. Thus, given any set $R\subseteq T\times T$ of pairs (s,t) of terms, we may consider groups H with X-tuples of elements $v: X\to|H|$ satisfying the corresponding set of relations

(3.3.2)
$$(\forall\,(s,t)\in R)\ \ s_v\ =\ t_v.$$

(For example, when $X=\{a,b\}$ and R is the singleton $\{(ab,\,b^2a)\}$, (3.3.2) becomes (3.3.1).) In this situation, we have

Proposition 3.3.3. *Let X be a set, T the set of group-theoretic terms in X, and R a subset of $T\times T$. Then there exists a universal example of a group with an X-tuple of elements satisfying the relations "$s=t$" ($(s,t)\in R$). I.e., there exists a pair (G,u), where G is a group, and u a map $X\to|G|$ such that*

$$(\forall\,(s,t)\in R)\ \ s_u=t_u,$$

and such that for any group H, and any X-tuple v of elements of H satisfying (3.3.2), *there* *exists a unique homomorphism* $f: G \to H$ *satisfying* $v = fu$ (*in other words, such that the* X*-tuple* v *of elements of* H *is the image under* f *of the* X*-tuple* u *of elements of* G).

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[rd, "\forall v"] & {|G|} \arrow[d] & G \arrow[d, "\exists 1\ f"] \\
 & {|H|} & H
\end{tikzcd}$$

Further, the pair (G, u) *is determined up to canonical isomorphism by these properties, and the group* G *is generated by* $u(X)$.

Methods of Proof. On the one hand, the constructions of §2.2 and §2.3 can be applied essentially word-for-word; all we do is add condition (3.3.2) to the group axioms throughout. (Note that unlike (2.2.1-8), the set of equations (3.3.2) involves no universal quantification over T; it is only assumed to hold for the particular specified X-tuple of elements.)

However, we can get the pair (G, u) now with less work. Let (F, u_F) be the *free* group on X, let N be the normal subgroup of F generated by $\{s_{u_F} t_{u_F}^{-1} \mid (s,t) \in R\}$, i.e., by the set of elements of F that we want to annihilate. Let $G = F/N$, let $q: F \to F/N$ be the canonical map, and let $u = q u_F$. That (G, u) has the desired universal property follows immediately from the universal properties of free groups and quotient groups.

$$\begin{tikzcd}
X \arrow[r, "u_F"] \arrow[rrd, "\forall v"] & {|F|} \arrow[r, "q"] \arrow[rd] & {|G|} \arrow[d] & F \arrow[r, "q"] \arrow[rd] & G \arrow[d, "\exists 1\ f"] \\
 & & {|H|} & & H
\end{tikzcd}$$

If (G', u') is another pair with the same universal property, then by the universal property of G there exists a homomorphism $i: G \to G'$ such that $iu = u'$, and by the universal property of G', an $i': G' \to G$ such that $i'u' = u$. These are inverses of one another; indeed, note that $i'iu = u$, hence by the uniqueness condition in the universal property of G, $i'i$ equals the identity map of G; by a like argument, ii' is the identity of G', so i is invertible, and gives the asserted isomorphism.

That G is generated by $u(X)$ can be seen from our construction, but let us also show from the universal property that this *must* be so. Consider the subgroup $<u(X)>$ of G generated by $u(X)$. The universal property of G gives a homomorphism $j: G \to <u(X)>$ which is the identity on elements of $u(X)$. Following it by the inclusion of $<u(X)>$ in G yields an endomorphism of G which agrees with the identity map on $u(X)$, and so, by the universal property, *is* the identity. So the inclusion of $<u(X)>$ in G is surjective, as desired. □

Though in the above proof, we spoke of getting our construction by combining two known constructions as being ‘‘less work’’ than constructing it from scratch, a more important advantage in seeing that a new construction can be obtained from known constructions is that results proved about the known constructions yield results about the new one.

The group G of the preceding proposition is called *the group presented by the generators* X *and relations* R. A common notation for this is

$$G \;=\; < X \mid R >.$$

For example, the universal group corresponding to (3.3.1) is written

$$< a,\, b \mid ab = b^2 a >.$$

In a group presented by generators and relations, one often uses the same symbols for the elements of X and their images in $|G|$, even if the map u is not one-to-one. For instance, from the well-known lemma saying that if an element η of a group (or monoid) has both a right inverse ζ and a left inverse ξ, then $\xi = \zeta$, it follows that in the group $< x, y, z \mid xy = e = yz >$, one has $u(x) = u(z)$. Unless there is a special need to be more precise, we may express this by saying ‘‘in $< x, y, z \mid xy = e = yz >$, one has $x = z$’’.

Recall that the concept of group-theoretic term was introduced both for the consideration of what relations hold among families of elements of *all* groups, and to write specific relations that hold among particular families of elements of particular groups. For the purpose of discussing identities holding in all groups, it was necessary to distinguish between expressions such as $(xy)^{-1}$ and $y^{-1}x^{-1}$, between $(xy)z$ and $x(yz)$, etc.. But in considering relations in particular groups we can generally take for granted the group identities, i.e., not distinguish pairs of expressions that have the same evaluations in all groups. For example, in (3.3.1), the right hand side could be replaced by $b(ba)$ without changing the effect of the condition. Hence in considering groups presented by generators and relations, one often considers the relations to be given by pairs, not of *terms*, but of their equivalence classes under the relation of having equal values in all groups – in other words pairs $(s, t) \in |F| \times |F|$, where F is the free group on X. For such (s, t), an X-tuple v of elements of a group G is considered to ‘‘satisfy $s = t$’’ if $h(s) = h(t)$, for h the homomorphism $F \to G$ induced by v, as in Definition 2.1.3.

Whether s and t are group-theoretic terms as in Proposition 3.3.3, or elements of a free group as in the above paragraph, we should note that there is a certain abuse of language in saying that a family v of elements of a group G ‘‘satisfies the relation $s = t$’’, and in writing equations ‘‘$s = t$’’ in presentations of groups. What we mean in such cases is that a certain equation *obtained from* the pair (s, t) and the X-tuple v holds in G; but the equation $s = t$ among our terms or free group elements is itself generally false! As with other convenient but imprecise usages, once we are conscious of its impreciseness, we may use it, but should be ready to frame more precise statements when imprecision could lead to confusion (for instance, if we also want to discuss which of certain terms or elements of a free group are really equal).

We have noted that a relation (s, t) is satisfied by an X-tuple v of elements of a group G if and only if $(s t^{-1})_v = e$ in G; in other words, if and only if v satisfies the relation $(s t^{-1}, e)$. Thus, every presentation of a group can be reduced to one in which the relations all have the form (r, e) for terms (or free-group elements) r. The elements r are then called the *relators* in the resulting presentation, and one may express the group in question by listing the relators, rather than the relations. E.g., in this notation, the group we wrote earlier as $< a, b \mid ab = b^2 a >$ would be written $< a, b \mid aba^{-1} b^{-2} >$. However, I will stick to our original notation in these notes.

Exercise 3.3:1. Show that the three groups described below are isomorphic (i.e., isomorphic as groups, ignoring the maps ‘‘$X \to |G|$’’ coming from the presentations of the first two.)

(i) $G \;=\; < a, b \mid a^2 = e,\; ab = b^{-1} a >$.

(ii) $H \;=\; < s, t \mid s^2 = t^2 = e >$.

(iii) The group of all distance-preserving permutations of $\mathbf{Z}$, i.e., all translation-maps $n \mapsto n+c \ (c \in \mathbf{Z})$ and all reflection-maps $n \mapsto -n+d \ (d \in \mathbf{Z})$.

The universal property of a group presented by generators and relations is extremely useful in considerations such as that of

Exercise 3.3:2. Find all endomorphisms of the group of the preceding exercise. Describe the structure of the monoid of these endomorphisms.

Returning to the example with which we started this section –

Exercise 3.3:3. Find a normal form or other convenient description for the group presented by two generators a, b and the one relation (3.3.1): $ab = b^2 a$.

The following question, suggested by a member of the class some years ago, is harder, but has a nice solution:

Exercise 3.3:4. (D. Hickerson.) Do the same for $< a, b \mid ab = b^2 a^2 >$.

Any group G can be presented by some system of generators and relations. E.g., take $|G|$ itself for generating set, and the multiplication table of G as a set of relations. But it is often of interest to find concise presentations for given groups. Note that the *free* group on a set X may be presented by the generating set X and the empty set of relations!

Exercise 3.3:5. Suppose $f(x,y)$ and $g(y)$ are group-theoretic terms in two and one variables respectively. What can you prove about the group with presentation

$$< w, x, y \mid w = f(x,y), \ \ x = g(y) > ?$$

Generalize if you can.

Exercise 3.3:6. Consider the set $\mathbf{Z} \times \mathbf{Z}$ of ‘‘lattice points’’ in the plane. Let G be the group of ‘‘symmetries’’ of this set, i.e., maps $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ which preserve distances between points.

(i) Find a simple *description* of G. (Cf. the description of the group of symmetries of $\mathbf{Z}$ in terms of translations and reflections in Exercise 3.3:1(iii).)

(ii) Find a simple *presentation* for G.

(iii) Find a *normal form* for elements of G, in terms of the generators used in your presentation.

Exercise 3.3:7. Suppose G is a group of n elements. Then the observation made above, on how to present any group by generators and relations, yields bounds on the minimum numbers of generators and relations needed to present G. Write down these bounds; then see to what extent you can improve on them.

The above observations show that every *finite* group is *finitely presented*, i.e., has a presentation in terms of finitely many generators and finitely many relations. Of course, there are also finitely presented groups which are infinite. The next two exercises, of which the first should not be difficult, while the second requires some ingenuity or experience with infinite groups, concern this property of finite presentability.

Exercise 3.3:8. Show that if G is a finitely presented group, and $< x_1, \ldots, x_n \mid R >$ is any presentation of G using finitely many generators, then there is a finite subset $R_0 \subseteq R$ such that $< x_1, \ldots, x_n \mid R_0 >$ is also a presentation of G.

Exercise 3.3:9. Find a finitely generated group that is not finitely presented.

Another kind of question one can ask is typified by

Exercise 3.3:10. Is the group

$$< x,\ y \mid xyx^{-1} = y^2,\ \ yxy^{-1} = x^2 >$$

trivial (= $\{e\}$)? What about

$$< x,\ y \mid xyx^{-1} = y^2,\ \ yxy^{-1} = x^3 > ?$$

(If you prove either or both of these groups trivial, you should present your calculation in a way that makes it clear at each stage which defining relation you are applying, and to what part of what expression.)

For the group-theory buff, here are two harder, but still tractable examples.

Exercise 3.3:11. (J. Simon [**83**].) (i) Is either of the groups

$$< a,\ b \mid (b^{-1}a)^4 a^{-3} = e = b^{10}(b^{-1}a)^{-3} > \qquad \text{or} \qquad < a,\ b \mid (ba^{-1})^{-3} a^{-2} = e = b^9 (ba^{-1})^4 >$$

trivial?

(ii) In the group $< a,\ b \mid ba^{-4}bab^{-1}a = e >$, is the subgroup generated by $ba(b^{-1}a)^2$ and a^3b^{-1} free abelian?

An interesting text on groups presented by generators and relations, which assumes only an undergraduate background, but goes deep into the techniques of the subject, is [**27**].

Let us observe a consequence of the universal property of Proposition 3.3.3, characterizing the group G with presentation $< X \mid R >$: For any group H, the set of homomorphisms $\mathrm{Hom}(G, H)$ is in natural one-to-one correspondence with the set of X-tuples of elements of H satisfying the relations R.

For instance, if n is a positive integer, we observe that $< x \mid x^n = e >$ is $\mathbf{Z}_n$, the cyclic group of order n; hence for any group H, we get a natural bijection between $\mathrm{Hom}(\mathbf{Z}_n, H)$ and $\{a \in |H| \mid a^n = e\}$, each such $a \in |H|$ corresponding to the unique homomorphism $\mathbf{Z}_n \to H$ carrying x to a. (Terminological note: A group element $a \in |H|$ which satisfies $a^n = e$ is said to have *exponent n*. This is equivalent to its having order *dividing* n.)

Similarly, one finds that $< x, y \mid xy = yx >$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$, hence $\mathrm{Hom}(\mathbf{Z} \times \mathbf{Z}, H)$ corresponds to the set of all ordered pairs of commuting elements of H.

Thus, presentations of groups by generators and relations provide a bridge between the internal structure of groups, and their ''external'' behavior under homomorphisms. This will be of particular importance when we turn to category theory, which treats mathematical objects in terms of the homomorphisms among them.

The last exercise of this section describes one of my favorite groups, though most of its interesting properties cannot be given here.

Exercise 3.3:12. Let $G = < x, y \mid y^{-1}x^2y = x^{-2},\ x^{-1}y^2x = y^{-2} >$.

(i) Find a normal form or other convenient description for elements of G. Verify from this description that G has no nonidentity elements of finite order.

(ii) Calling the group characterized in several ways in Exercise 3.3:1 ''D'', show that G has exactly three normal subgroups N such that $G/N \cong D$, and that the intersection of these three subgroups is $\{e\}$.

(iii) It follows from (ii) above that G can be identified with a subgroup of $D \times D \times D$. Give a criterion for an element of $D \times D \times D$ to lie in this subgroup, and prove directly from this criterion that no element of this subgroup has finite order.

Though group presentations often yield groups for which a normal form can be found, it has been proved by Novikov, Boone and Britton that there exist finitely presented groups G such that no algorithm can decide whether an arbitrary pair of terms of G represent the same element. A proof of this result is given in the last chapter of [**29**].

3.4. Abelian groups, free abelian groups, and abelianizations. An *abelian group* is a group A satisfying the further identity

$$(\forall x, y \in |A|) \quad xy = yx.$$

The discussion of §2.1 carries over without essential change and gives us the concept of a *free abelian group* (F, u) on a set X; the method of §2.2 establishes the existence of such groups by constructing them as quotients of sets T of terms by appropriate equivalence relations, and the method of §2.3 yields an alternative construction as subgroups of direct products of large enough families of abelian groups. We may clearly also obtain the free abelian group on a set X as the group presented by the generating set X and the relations $st = ts$, as s and t range over all elements of T. This big set of relations is easily shown to be equivalent, for any X-tuple of elements of any group, to the smaller family $xy = yx\ (x, y \in X)$, so the free abelian group on X may be presented as

$$< X \mid xy = yx \ \ (x, y \in X) >.$$

To investigate the *structure* of free abelian groups, let us consider, say, three elements a, b, c of an arbitrary abelian group A, and look at elements of A that can be obtained from these by group-theoretic operations. We know from §2.4 that any such element g may be written either as e, or as a product of the elements $a, a^{-1}, b, b^{-1}, c, c^{-1}$. We can now use the commutativity of A to rearrange this product so that it begins with all factors a (if any), followed by all factors a^{-1} (if any), then all factors b (if any), etc.. Now performing cancellations if both a and a^{-1} occur, or both b and b^{-1} occur, or both c and c^{-1} occur, we reduce g to an expression $a^i b^j c^k$, where i, j and k are integers (positive, negative, or 0; exponentiation by negative integers and 0 being defined by the usual conventions). It is easy to describe the group operations on the set $T_{\text{ab-red}}$ of such expressions, and to check that under these operations, $T_{\text{ab-red}}$ forms an abelian group F. It follows as in §2.4 that this F is the *free* abelian group on $\{a, b, c\}$, and thus that the set $T_{\text{ab-red}}$ of terms $a^i b^j c^k$ is a normal form for elements of the free abelian group on three generators. In this verification, we do not need any analog of ‘‘van der Waerden’s trick’’ (§2.4). Rather, the result that $T_{\text{ab-red}}$ is an abelian group under the induced operations,

$$\begin{aligned}(a^i b^j c^k) \odot (a^{i'} b^{j'} c^{k'}) &=_{\text{def}} a^{i+i'} b^{j+j'} c^{k+k'},\\ (a^i b^j c^k)^{(-)} &=_{\text{def}} a^{-i} b^{-j} c^{-k}\\ e &=_{\text{def}} a^0 b^0 c^0\end{aligned}$$

follows from the known fact that the integers form an abelian group under $+$, $-$, and 0. (Note that the symbols $\odot$, ${}^{(-)}$ represent different operations from those represented by the same symbols in §2.4, though the idea is the same as that of that section.)

This normal form is certainly simpler than that of the free *group* on $\{a, b, c\}$. Yet there is a curious way in which it is more complicated: It is based on our choice of ‘‘alphabetic order’’ for the generating set $\{a, b, c\}$. Using different orderings, we get different normal forms, e.g., $b^j c^k a^i$, etc.. If we want to generalize our normal form to the free abelian group on a finite set X without any particular structure, we must begin by ordering X, say writing $X = \{x_1, x_2, \ldots, x_n\}$. Only then can we speak of ‘‘the set of all expressions $x_1^{i_1} \ldots x_n^{i_n}$’’. If we want a normal form in the free abelian group on an *infinite* set X, we must again choose a total ordering of X, and then either talk about ''formally infinite products with all but finitely many factors equal to e'', or modify the normal form, say to ''e or $x^{i(x)} y^{i(y)} \ldots z^{i(z)}$ where $x < y < \ldots < z \in X$, and all exponents shown are nonzero'' (the last two conditions to ensure uniqueness!).

We may be satisfied with one of these approaches, or we may prefer to go to a slightly different kind of representation for F, which we discover as follows: Note that if g is a member of the free abelian group F on X, then for each $x \in X$, the exponent $i(x)$ to which x appears in our normal forms for g is the same for these various forms; only the position in which $x^{i(x)}$ is written (and if $i(x) = 0$, whether it is written) changes from one normal form to another. Clearly, any of our normal forms for g, and hence the element g itself, is determined by the X-tuple of exponents $(i(x))_{x \in X}$. So let us ''represent'' g by this X-tuple; that is, identify F with a certain set of integer-valued functions on X. It is easy to see that the group operations of F correspond to componentwise addition of such X-tuples, componentwise additive inverse, and the constant X-tuple 0; and that the X-tuple corresponding to each generator $x \in X$ is the function δ_x having value 1 at x and 0 at all other elements $y \in X$. The X-tuples that actually correspond to members of F are those which are nonzero at only finitely many components. Thus we get the familiar description of the free abelian group on X as the subgroup of $\mathbf{Z}^X$ consisting of all functions having finite support in X. (The *support* of a function f means $\{x \mid f(x) \neq 0\}$.)

Exercise 3.4:1. If X is infinite, it is clear that the whole group $\mathbf{Z}^X$ is *not* a free abelian group on X under the map $x \mapsto \delta_x$, since it is not generated by the δ_x. Show that $\mathbf{Z}^X$ is not a free abelian group on *any* set of generators.

(For further results on $\mathbf{Z}^X$ and its subgroups when X is countably infinite, see Specker [**97**]. Among other things, it is shown there that the uncountable group $\mathbf{Z}^X$ has only countably many homomorphisms into $\mathbf{Z}$, though its countable subgroup F clearly has uncountably many! It is also shown that the subgroup of *bounded* functions on X is free abelian, on uncountably many generators. This fact was generalized to not necessarily countable X by Nöbeling [**87**]. For a simpler proof of this result, using ring theory, see [**36**, §1].)

The concept of an *abelian group* presented by a system of *generators and relations* may be formulated exactly like that of a group presented by generators and relations. It may also be constructed analogously: as the quotient of the free abelian group on the given generators by the subgroup generated by the relators st^{-1} (we don't have to say ''normal subgroup'' because normality is automatic for subgroups of abelian groups); or alternatively, as the *group* presented by the given generators and relations, together with the additional relations saying that all the generators commute with one another.

Suppose now that we start with an arbitrary group G, and impose relations saying that for all $x, y \in |G|$, x and y should commute: ''$xy = yx$''. That is, we form the quotient of G by the normal subgroup generated by the elements $(yx)^{-1}(xy) = x^{-1}y^{-1}xy$. These elements are known as *commutators*, and often abbreviated

$$x^{-1}y^{-1}xy \ = \ [x, y].$$

(Another notation is (x, y), but we will not use this to avoid confusion with ordered pairs. Incidentally, it is to conform to group-theorists' usage that I am using $(yx)^{-1}(xy) = x^{-1}y^{-1}xy$, rather than the choice $(xy)(yx)^{-1} = xyx^{-1}y^{-1}$ that our habit of converting a relation $s = t$ to a relator st^{-1} would have led to.) The normal subgroup that they generate is called the *commutator subgroup*, or *derived subgroup* of G, written $[G, G]$, and often abbreviated by group theorists to G'. The quotient group, $G^{\mathrm{ab}} =_{\mathrm{def}} G/[G, G]$, is an abelian group with a homomorphism q of the given group G into it, which is *universal* among homomorphisms of G into abelian groups A, the diagram for the universal property being

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[dr, "\forall v"'] & G^{\mathrm{ab}} \arrow[d, "\exists 1\ f"] \\ & A. \end{tikzcd}$$

This group G^{ab} (or more precisely, the pair (G^{ab}, q), or any isomorphic pair) is called the *abelianization* or *commutator factor group* of G.

Suppose now that we write down any system of generators and relations for a group, and compare the *group* G and the *abelian group* H, that these same generators and relations define. By the universal property of G, there will exist a unique homomorphism $r: G \to H$ taking the generators to corresponding generators. It is easy to check that (H, r) has the universal property characterizing the abelianization of G. So this gives another way of describing abelianization. Note, as a consequence, that given an arbitrary system of generators and group-theoretic relations, the group these present will determine, up to natural isomorphism, the abelian group that they present (but not vice versa).

Exercise 3.4:2. Find the structures of the abelianizations of the groups presented in Exercises 3.3:1, 3.3:3, 3.3:4, 3.3:10 and 3.3:11.

Exercise 3.4:3. Show that any group homomorphism $f: G \to H$ induces a homomorphism of abelian groups $f^{\mathrm{ab}}: G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. State precisely the condition relating f and f^{ab}. Show that for a composite of group homomorphisms, one has $(fg)^{\mathrm{ab}} = f^{\mathrm{ab}} g^{\mathrm{ab}}$. Conclude that for any group G, there is a natural homomorphism of monoids, $\mathrm{End}(G) \to \mathrm{End}(G^{\mathrm{ab}})$, and a natural homomorphism of groups $\mathrm{Aut}(G) \to \mathrm{Aut}(G^{\mathrm{ab}})$.

Exercise 3.4:4. For G as in Exercises 3.3:1 and 3.3:2, is the natural homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(G^{\mathrm{ab}})$ of the above exercise one-to-one?

Exercise 3.4:5. If H is a subgroup of G, what can be said about the relation between H^{ab} and G^{ab}? Same question if H is a homomorphic image of G.

Exercise 3.4:6. Let K be a field, n a positive integer, and $\mathrm{GL}(n, K)$ the group of invertible $n \times n$ matrices over K. Determine as much as you can about the structure of $\mathrm{GL}(n, K)^{\mathrm{ab}}$.

Exercise 3.4:7. If G is a group, will there exist a universal homomorphism of G into a *solvable* group, $G \to G^{\mathrm{solv}}$? What if G is assumed finite?

Does there exist a ‘‘free solvable group’’ on a set X, or some similar construction?

Exercise 3.4:8. Show that the free abelian group on n generators cannot be presented *as a group* by fewer than n generators and $n(n-1)/2$ relations.

3.5. The Burnside problem.

In 1902, W. Burnside [**46**] asked whether a finitely generated group, all of whose elements had finite order, must be finite. This problem was hard to approach because, with nothing assumed about the *values* of the finite orders of the elements, one had no place to begin a calculation. So Burnside also posed this question under the stronger hypothesis that there be a *common* finite bound on the orders of all elements of G.

The original question with no bound on the orders was suddenly answered negatively in 1964, with a counterexample arising from the Golod-Shafarevich construction [**60**]; there is a short and fairly self-contained presentation of this material in the last chapter of [**25**]. In the opposite direction, Burnside himself proved that if G is a finitely generated group of *matrices* over a field and all elements of G have finite order, then G is finite [**47**].

Turning to the question of a general group G with a common bound on the orders of its elements, note that if m is such a bound, then $m!$ is a common *exponent* for these elements; while if n is a common exponent, it is also a bound on their orders. So ‘‘all elements are of bounded order’’ is equivalent to ‘‘all elements have a common exponent’’. The latter condition is more convenient to study, since the statement that x has exponent n has the form of a single identity. So for any positive integer n, one defines the *Burnside problem for exponent* n to be the question of whether every finitely generated group satisfying

$$(\forall x)\ x^n = e \tag{3.5.1}$$

is finite.

For $n = 1$, the answer is obviously yes, for $n = 2$ the same result is an easy exercise, for $n = 3$ it is not very hard to show, and it has also been proved for $n = 4, 6$. On the other hand, it has been shown in recent years that the answer is negative for all odd $n \geq 665$ [**31**], and for all $n > 8000$ [**78**]. This still leaves a large but finite set of open cases: all odd values strictly between 3 and 665, and all even values strictly between 6 and 8000. We won’t get involved in these hard group-theoretic problems here. But the concept of universal constructions does allow us to understand the nature of the question better. Call a group G an *n-Burnside group* if it satisfies the identity (3.5.1). One may define the *free* n-Burnside group on any set X by the obvious universal property, and it will exist for the usual reasons. In particular, it can be presented, as a group, by the generating set X, and the infinite family of relations equating the nth powers of *all* terms in the elements of X to e. I leave it to you to think through the following relationships:

Exercise 3.5:1. Let n and r be positive integers.

(i) What implications hold among the following statements?

(a) Every n-Burnside group which can be generated by r elements is finite.

(b) The free n-Burnside group on r generators is finite.

(c) The group $<x_1, \ldots, x_r \mid x_1^n = \ldots = x_r^n = e>$ is finite.

(d) There exists a finite r-generator group having a finite presentation in which all relators are nth powers, $<x_1, \ldots, x_r \mid w_1^n = \ldots = w_s^n = e>$ (where each w_i is a term in $x_1, \ldots, x_r$. Cf. Exercises 3.3:7 and 3.3:8.)

(e) There exists an integer N such that all n-Burnside groups generated by r elements have order $\leq N$.

(f) There exists an integer N such that all *finite* n-Burnside groups generated by r elements have order $\leq N$ (‘‘the restricted Burnside problem’’).

(ii) What implications hold among cases of statement (a) involving the same value of n but different values of r? involving the same value of r but different values of n?

Note that if for a given n and r we could find a *normal form* for the free n-Burnside group on r generators, we would know whether (b) was true! But except when n or r is very small, such normal forms are not known. For further discussion of these questions, see [**24**, Chapter 18]. For recent results, including a partial solution to the restricted Burnside problem ((f) above), and some negative results on the *word problem* for free Burnside groups, see [**71**], [**84**], [**101**], and references given in those works.

A group G is called *residually finite* if for any two elements $x \neq y \in |G|$, there exists a homomorphism f of G into a *finite* group such that $f(x) \neq f(y)$.

Exercise 3.5:2. Investigate implications involving conditions (a)-(f) of the preceding exercise, together with

(g) The free n-Burnside group on r generators is residually finite.

Exercise 3.5:3. (i) Restate Exercise 2.4:3 as a question about residual finiteness (showing, of course, that your restatement is equivalent to the original question).

(ii) If G is a group, does there exist a universal homomorphism $G \to G^{\mathrm{rf}}$, of G into a residually finite group?

3.6. Products and coproducts. Let G and H be groups. Consider the following two situations:

(a) a group P given with a homomorphism $p_G : P \to G$ and a homomorphism $p_H : P \to H$, and

(b) a group Q given with homomorphisms $q_G : G \to Q$, and $q_H : H \to Q$ (diagrams below).

Note that if in situation (a) we have any homomorphism a of any other group P' into P, then P' also acquires homomorphisms into G and H, namely $p_G\, a$ and $p_H\, a$; and similarly, if in situation (b) we have any homomorphism b of Q into a group Q', then Q' acquires homomorphisms $b\, q_G$ and $b\, q_H$ of G and H into it:

$$\begin{tikzcd} & & G & G \arrow[dr, "q_G"] & & \\ P' \arrow[r, "a"] & P \arrow[ur, "p_G"] \arrow[dr, "p_H"'] & & & Q \arrow[r, "b"] & Q'. \\ & & H & H \arrow[ur, "q_H"'] & & \end{tikzcd}$$

So we may ask whether there exists a *universal* example of a P with maps into G and H, that is, a 3-tuple (P, p_G, p_H) such that for any group P', every pair of maps $p'_G : P' \to G$ and $p'_H : P' \to H$ arises by composition of p_G and p_H with a unique homomorphism $a: P' \to P$; and, dually, whether there exists a universal example of a group Q with maps of G and H into it.

In both cases, the answer is yes. The universal P is simply the direct product group $G \times H$, with its projection maps p_G and p_H onto the two factors; the universal property is easy to verify. The universal Q, on the other hand, can be constructed by generators and relations. It has to have for each $g \in |G|$ an element $q_G(g)$ – let us abbreviate this to $\bar{g}$ – and for each $h \in |H|$ an element $q_H(h)$ – call this $\tilde{h}$. So let us take for generators a set of symbols

(3.6.1) $$\{\bar{g}, \tilde{h} \mid g \in |G|,\ h \in |H|\}.$$

The relations these must satisfy are those saying that q_G and q_H are homomorphisms:

(3.6.2) $$\bar{g}\,\bar{g}' = \overline{gg'} \quad (g, g' \in |G|), \qquad \tilde{h}\tilde{h}' = \widetilde{hh'} \quad (h, h' \in |H|).$$

It is immediate that the group so presented has the desired universal mapping property. (We might have supplemented (3.6.2) with the further relations $\overline{e_G} = e$, $\widetilde{e_H} = e$, $\overline{g^{-1}} = \bar{g}^{-1}$, $\widetilde{h^{-1}} = \tilde{h}^{-1}$. But these are implied by the relations listed, since as is well known, any set map between groups which preserves products also preserves neutral elements and inverses.) More generally, if G is a group which can be presented as $< X \mid R >$, and if similarly $H = < Y \mid S >$, then we may take for generators of Q a disjoint union $X \sqcup Y$, and for relations the union of R and S. For instance, if

$$G = \mathbf{Z}_3 =$$

$< x \mid x^3 = e >$ and $H = \mathbf{Z}_2 = < x \mid x^2 = e >$, then Q may be presented as

$$< x,\, x' \mid x^3 = e,\ x'^{\,2} = e >,$$

with q_G and q_H given by $x \mapsto x$ and $x \mapsto x'$, respectively. You should be able to verify the universal property of Q from this presentation.

(If you were not familiar with the concept of a ''disjoint union'' $X \sqcup Y$ of two sets X and Y, I hope that the above discussion suggests the meaning. Explicitly, it means the union of a bijective copy of X and a bijective copy of Y, chosen to be disjoint. So, if $X = \{a, b, c\}$, $Y = \{b, c, d, e\}$, where a, b, c, d, e are distinct, then their ordinary set-theoretic union is the 5-element set $X \cup Y = \{a, b, c, d, e\}$, but an example of a ''disjoint union'' would be any set of the form $X \sqcup Y = \{a, b, c, b', c', d', e'\}$, where a, b, c, b', c', d', e' are distinct, given with the obvious maps taking X to the 3-element subset $\{a, b, c\}$ of this set, and Y to the disjoint 4-element subset $\{b', c', d', e'\}$. Though there is not a unique way of choosing a disjoint union of two sets, the construction is unique in the ways we care about; e.g., note that in the above example, any disjoint union of X and Y will have $|X| + |Y| = 7$ elements. Hence one often speaks of ''the'' disjoint union. We will see, a few sections from now, that disjoint union of sets is itself a universal construction – of set theory.)

To see for general G and H what the group determined by this universal property ''looks like'', let us again think about an arbitrary group Q with homomorphisms of G and H into it, which we abbreviate $g \mapsto \bar{g}$ and $h \mapsto \tilde{h}$. The elements of Q which we can name in this situation are, of course, the products

$$x_n^{\pm 1}\, x_{n-1}^{\pm 1} \ldots x_1^{\pm 1} \quad \text{with} \quad x_i \in \{\bar{g}, \tilde{h} \mid g \in |G|,\ h \in |H|\} \ \text{and} \ n \geq 0.$$

(Notational remark: In §2.4, I generally kept $n \geq 1$, and introduced ''e'' as a separate kind of expression. Here I shall adopt the convenient convention that the product of the empty (length 0) sequence of factors is e, so that the case ''e'' may be absorbed in the general case.)

But for any $g \in |G|$ or $h \in |H|$ we have noted that $\bar{g}^{-1} = \overline{g^{-1}}$ and $\tilde{h}^{-1} = \widetilde{h^{-1}}$ in Q; so the inverse of any member of the generating set $\{\bar{g} \mid g \in |G|\} \cup \{\tilde{h} \mid h \in |H|\}$ is another member of that set; hence we may simplify the above product so that no exponents $^{-1}$ occur. We also know that $\bar{e} = \tilde{e} = e$, so wherever instances of $\bar{e}$ or $\tilde{e}$ occur in our product, we may drop them. Finally, the relations (3.6.2) allow us to replace any occurrence of two successive factors from $\{\bar{g} \mid g \in |G|\}$ by a single such factor, and to do the same if two factors from $\{\tilde{h} \mid h \in |H|\}$ occur together. So the elements of Q that we can construct can all be reduced to the form

(3.6.3)
$$x_1 \ldots x_n$$
where $n \geq 0$, $x_i \in \{\bar{g} \mid g \in |G| - \{e\}\} \cup \{\tilde{h} \mid h \in |H| - \{e\}\}$, and no two successive x's come from the same set, $\{\bar{g} \mid g \in |G| - \{e\}\}$ or $\{\tilde{h} \mid h \in |H| - \{e\}\}$.

We can express the *product* of two elements (3.6.3) as another such expression, by putting the sequences of factors together, and reducing the resulting expression to the above form in the obvious way; likewise it is clear how to find expressions of that form for inverses of elements (3.6.3), and for the element e. In any particular group Q with homomorphisms of G and H into it, there may be other elements than those expressed by (3.6.3), and there may be some equalities among such products. But as far as we can see, there don't *seem* to be any cases left of two expressions (3.6.3) that must represent the same element in *every* such group Q. *If* in fact there are none, then, as in §2.4, the expressions (3.6.3) will correspond to the distinct elements of the *universal* Q we are trying to describe, and thus will give a normal form for the elements of this group.

We can use the same stratagem as in §2.4 to show that there are in fact no undiscovered necessary equalities – it was for this situation that van der Waerden devised it!

Proposition 3.6.4 (van der Waerden [**102**]). *Let* G, H *be groups, and* Q *the group with a universal pair of homomorphisms* $G \to Q$, $H \to Q$, *written* $g \mapsto \bar{g}$, $h \mapsto \tilde{h}$. *Then every element of* Q *can be written uniquely in the form* (3.6.3).

Proof. Let us introduce an additional symbol a, and denote by A the set of all symbols

$$(3.6.5) \qquad x_n \,...\, x_1\, a, \quad \text{where } x_1, \ldots, x_n \text{ are as in (3.6.3).}$$

We would like to describe actions of G and H on this set. It is clear what these actions *should* be, but an explicit description is a bit messy, because of the need to describe separately the cases where the element of A on which we are acting does or does not begin with an element of the group we are acting by, and if it does, the cases where this beginning element is or is not the inverse of the element by which we are acting. This makes still more messy the formal verification that the ‘‘actions’’ give homomorphisms of G and of H into the permutation group of A.

But we shall get around these annoyances (which are in any case minor compared with the difficulties of doing things without van der Waerden’s method) by another trick. Let us describe a set A_G which is in bijective correspondence with A: For those elements $b \in A$ which already begin with a symbol $\bar{g}$ $(g \in |G| - \{e\})$, we let A_G contain the same element b. For elements b which do not, let the corresponding element of A_G be the expression $\bar{e}\,b$. Thus *every* element of A_G begins with a symbol $\bar{g}$ $(g \in |G|)$, and we can now describe the action of $g' \in |G|$ on A_G as simply taking an element $\bar{g}\,c$ to $\overline{g'g}\,c$. It is trivial to verify that *this* is a homomorphism of G into the permutation group of A_G. This action on A_G now clearly induces an action on the bijectively related set A. Likewise, an action of H on A can be defined, via an action on an exactly analogous set A_H.

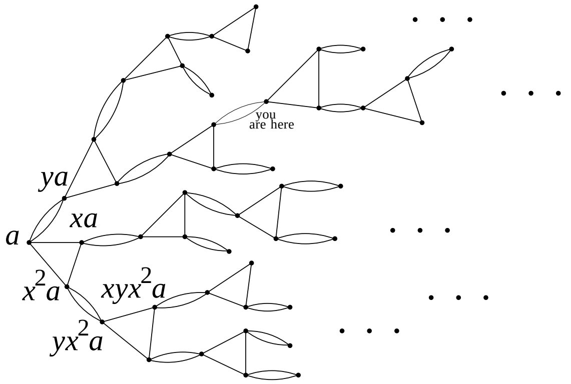
Thus we have homomorphisms of both G and H into the permutation group of A; this is equivalent to giving a homomorphism of the group Q we are interested in into this permutation group. Further, given any element (3.6.3) of Q, it is easy to see by induction on n that its image in the permutation group of A sends the ‘‘starting point’’ element a to precisely $x_n \ldots x_1\, a$. Hence two distinct expressions (3.6.3) correspond to elements of Q having distinct actions on a, hence these elements of Q are themselves distinct. So not only can every element of Q be written in the form (3.6.3), but distinct such expressions correspond to distinct elements of Q, proving the proposition. □

For a concrete example, again let $G = \mathbf{Z}_3 = \langle x \mid x^3 = e \rangle$ and let $H = \mathbf{Z}_2 = \langle y \mid y^2 = e \rangle$. Then A will consist of strings like a, ya, xyx^2a, etc.. (We can drop ‘‘$^{-}$’’ and ‘‘$^{\sim}$’’ here because $|G| - \{e\}$ and $|H| - \{e\}$ use no symbols in common.) $G = \mathbf{Z}_3$ will act on this set by

$$\begin{tikzcd} & xb \arrow[d] \\ b \arrow[ur] & \\ & x^2b \arrow[ul] \end{tikzcd}$$

(for strings b not beginning with x), while $H = \mathbf{Z}_2$ acts by permuting pairs of symbols $b \rightleftarrows yb$ (b not beginning with y). If we want to see that say, $yxyx^2$ and x^2yxy have distinct actions on A, we simply note that the first sends the symbol a to the symbol

$yxyx^2a$, while the second takes it to x^2yxya. A picture of the Q-set A, for this G and H, looks like some kind of seaweed:



(G acts by rotating the triangles, H by transposing pairs of points marked ⟷.)

We recall that the universal group ''P'' considered at the beginning of this section turned out to be the direct product of G and H. Since Q is characterized by the dual universal property, we shall call it the *coproduct* of G and H.

Because of the similarity of the normal form of this construction to that of *free groups*, group-theorists have long called it the *free product* of the given groups in these notes. However, the constructions for *sets, commutative rings, abelian groups, topological spaces*, etc. characterized by this same universal property show a great diversity of forms, and have been known under different names in the respective disciplines. The general name ''coproduct'' introduced by category theory (Chapter 6 below) unifies the terminology, and we shall follow it in these notes. On the other hand, the ''P'' constructions look very similar in all these cases, and have generally all had the name ''direct product'', which is retained (shortened to ''product'') by category theorists.

In both our product and coproduct constructions, the pair of groups G and H may be replaced by an arbitrary family $(G_i)_{i\in I}$. The direct product $\prod_I G_i$ with its I-tuple of projection maps again gives the universal example of a group P given with an I-tuple of maps $p_i: P \to G_i$. The coproduct $Q = \coprod_I G_i$, generated by the images of a universal family of maps of the G_i's *into* Q, can be constructed, just as above, using strings of nonidentity elements from a disjoint union of the underlying sets of these groups, such that two factors from the same group G_i never occur consecutively. (Note that the coproduct symbol $\coprod$ is the direct product symbol $\prod$ turned upside-down.)

Exercise 3.6:1. If X is a set, then a coproduct of copies of the infinite cyclic group $\mathbf{Z}$, indexed by X, $\coprod_X \mathbf{Z}$, will be a free group on X. Show this by universal properties, and describe the correspondence of normal forms. Can you find any other families of groups whose coproduct is a free group?

Exercise 3.6:2. Let us (following group-theorists' notation) write coproducts of finite families of groups as $Q = G*H$, $Q = F*G*H$, etc.. Prove that for any three groups F, G and H, one has $(F*G)*H \cong F*G*H \cong F*(G*H)$, using (a) universal properties, and (b) normal forms.

Exercise 3.6:3. For any two groups G and H, show how to define natural isomorphisms $i_{G,H}: G\times H \cong H\times G$, and $j_{G,H}: G*H \cong H*G$. What form do these isomorphisms take when $G = H$? (Describe them on elements.)

It is sometimes said that ''We may identify $G\times H$ with $H\times G$, and $G*H$ with $H*G$, by treating the isomorphisms $i_{G,H}$ and $j_{G,H}$ as the identity, and identifying the corresponding group elements.'' Is this reasonable when $G = H$?

Exercise 3.6:4. Show that in a coproduct group $G*H$, the only elements of finite order are the conjugates of the images of elements of finite order of G and H. (First step: Find how to determine, from the normal form of an element of $G*H$, whether it is a conjugate of an element of G or H.)

Can you similarly describe all *finite subgroups* of $G*H$?

There is a fact about the direct product group which one would not at first expect from its universal property: It also has two natural maps *into* it: $f_G: G \to G\times H$ and $f_H: H \to G\times H$, given by $g \mapsto (g, e)$ and $h \mapsto (e, h)$. (Note that there is no analogous construction on direct products of *sets*.) To examine this phenomenon, we recall that the universal property of $G\times H$ says that to map a group A into $G\times H$ is equivalent to giving a map $A \to G$ and a map $A \to H$. Looking at f_G, we see that the two maps it corresponds to are the identity map $\mathrm{id}_G: G \to G$, defined by $\mathrm{id}_G(g) = g$, and the trivial map $e: G \to H$, defined by $e(g) = e$. The map f_H is characterized similarly, with the roles of G and H reversed.

The group $G\times H$ has, in fact, a second universal property, in terms of this pair of maps. The 3-tuple $(G\times H, f_G, f_H)$ is universal among 3-tuples (K, a, b) such that K is a group, $a: G \to K$ and $b: H \to K$ are homomorphisms, and the images in K of these homomorphisms *centralize* one another:

$$(\forall g\in|G|,\ h\in|H|)\quad a(g)\,b(h) \ = \ b(h)\,a(g),$$

equivalently:

$$[a(G),\ b(H)] \ = \ \{e\}.$$

If $P = \prod_I G_i$ is a direct product of arbitrarily many groups, one similarly has maps $f_i: G_i \to P$, but if the index set I is infinite, the images of the f_i will not in general generate P, and it follows from this that P cannot have the same universal property. But one finds that the subgroup P_0 of P generated by the images $f_i(G_i)$ is again a universal group with maps of the G_i into it having images that centralize one another. This subgroup consists of those elements of P having only finitely many coordinates $\neq e$.

Exercise 3.6:5. Prove the above new universal property of $G\times H$.

Describe the map

$$m:\ G*H \ \to \ G\times H$$

which the universal property of $G\times H$ associates to the pair of maps f_G, f_H and deduce that this map is surjective, and that its kernel is the normal subgroup of $G*H$ generated by the commutators $[\bar g, \tilde h]$ $(g\in|G|,\ h\in|H|)$.

Generalize this construction to products and coproducts of arbitrary families $(G_i)_{i\in I}$.

One may wonder why commutativity suddenly came up like this, since the original universal property by which we characterized $G\times H$ had nothing to do with it. The following observation throws a little light on this. The set of relations that will be satisfied in $G\times H$ by the images of elements of G and H under the two maps f_G and f_H defined above will be the intersection of the sets of relations satisfied by their images in K under $a: G \to K$, $b: H \to K$, in the two cases

(i) $K = G;\ a = \mathrm{id}_G,\ b = e,$

(ii) $K = H;\ a = e,\ b = \mathrm{id}_H.$ (Why?)

And what are such relations? Clearly $a(g)b(h) = b(h)a(g)$ holds in each case. The above second universal property of $G\times H$ is equivalent to saying that no relations hold in both cases *except* this family of relations and their consequences.

A coproduct group $G{*}H$ similarly has natural maps $u_G\colon G{*}H \to G$ and $u_H\colon G{*}H \to H$, constructed from the identity maps of G and H and the trivial maps between them, but u_G and u_H have no unexpected properties that I know of.

Exercise 3.6:6. If G is a group, construct maps $G \to G \times G$ and $G{*}G \to G$ using universal properties, and the identity map, but *not* using the trivial map of G. Describe how these maps behave on elements.

Exercise 3.6:7. Suppose $(G_i)_{i\in I}$ is a family of groups, and we wish to consider groups G given with homomorphisms $G_i \to G$ such that the images of *certain* pairs G_i, $G_{i'}$ commute, while no condition is imposed on the remaining pairs. To formalize this, let $J \subseteq I \times I$ be a symmetric antireflexive relation on our index set I (antireflexive means $(\forall\, i\in I)\ (i,i)\notin J$); and let H be the universal group with maps $r_i\colon G_i \to H$ $(i\in I)$ such that for $(i,i')\in J$, $[r_i(G_i),\ r_{i'}(G_{i'})] = \{e\}$.

Study the structure of this H, and obtain a normal form if possible. You may assume the index set I finite if this helps.

3.7. Products and coproducts of abelian groups. Let A and B be abelian groups. Following the model of the preceding section, we may look for abelian groups P and Q having universal pairs of maps:

$$\begin{tikzcd}
 & A & & A \arrow[dr, "q_A"] & \\
P \arrow[ur, "p_A"] \arrow[dr, "p_B"'] & & & & Q. \\
 & B & & B \arrow[ur, "q_B"'] &
\end{tikzcd}$$

Again abelian groups with both these properties exist – but this time, they turn out to be the same group, namely $A \times B$! (The reader should verify both universal properties.) To look at this another way, if we construct abelian groups P and Q with the universal properties of the direct product and coproduct of A and B respectively, and then form the homomorphism $m\colon P \to Q$ analogous to that of Exercise 3.6:5, this turns out to be an isomorphism.

Note that though $A \times B$ is the universal *abelian* group with homomorphisms of A and B into it, this is not the same as the universal *group* with homomorphisms of A and B into it – that group, $A{*}B$, constructed in the preceding section, will generally not be abelian when A and B are. Thus, the coproduct of two abelian groups A and B as *abelian groups* is generally not the same as their coproduct as *groups*. Rather, we can see by comparing universal properties that the coproduct as abelian groups is the abelianization of the coproduct as groups: $A \times B = (A{*}B)^{\mathrm{ab}}$.

Hence, in using the coproduct symbol ''$\amalg$'', we have to specify what kind of coproduct we are talking about, $\amalg_{\mathrm{gp}} A_i$ or $\amalg_{\mathrm{ab\,gp}} A_i$, unless this is clear from context. On the other hand, *direct products* of abelian groups as abelian groups are the same as their direct products as groups.

For a not necessarily finite family $(A_i)_{i\in I}$ of abelian groups, the coproduct still *embeds in* the direct product under the map ''m''. It can in fact be described as the subgroup of that direct product group consisting of those elements almost all of whose coordinates are e. When abelian groups are written additively, this coproduct is generally called the ''direct sum'' of the groups, and denoted $\bigoplus_I A_i$; in the case of two groups we write this $A \oplus B$.

Notes on confused terminology: Some people extend the term ''direct sum'' to mean ''coproduct'' in all contexts – groups, rings, etc.. Other writers, because of the form that ''direct sum'' has for finite families of abelian groups, use the phrase ''direct sum'' as a synonym of ''direct product'', even in the case of infinite families of groups! The coproduct of an infinite family of abelian groups is sometimes called their ‘‘restricted direct product’’ or ‘‘restricted direct sum’’, the direct product then being called the ‘‘complete direct product’’ or ‘‘complete direct sum’’. In these notes, we shall stick with the terms ‘‘product’’ and ‘‘coproduct’’, as defined above (except that we shall often expand ‘‘product’’ to ‘‘direct product’’, to avoid possible confusion with meanings such as a product of elements under a multiplication).

What is special about abelian groups, that makes finite products and coproducts come out the same; and why only *finite* products and coproducts? One may consider the key property to be the fact that homomorphisms of abelian groups can be ‘‘added’’; i.e., that given two homomorphisms $f, g: A \to B$, the map $f+g: A \to B$ defined by $(f+g)(a) = f(a) + g(a)$ is again a homomorphism. (This is not true for nonabelian groups.) Temporarily writing $*_{\text{ab gp}}$ for the coproduct of two abelian groups, one finds, in fact, that the inverse of the map $m: G *_{\text{ab gp}} H \to G \times H$ is given by the sum

$$q_G\, p_G + q_H\, p_H: \ G \times H \to G *_{\text{ab gp}} H.$$

For coproducts of noncommutative groups, the corresponding map is not a group homomorphism, while for coproducts of infinite families of abelian groups, no analog of the above map can be constructed because one cannot make sense of an infinite sum. So only when the coproduct is taken in the class of abelian groups, and the given family of groups is finite, do we get our inverse to m.

3.8. Right and left universal properties. The universal property of direct products differs in a basic way from the other universal properties we have looked at so far. In all other cases, we constructed an object (e.g., a group) F with specified ‘‘additional structure’’ or conditions (e.g., a map of a given set X into $|F|$), such that each structure of the same sort on any other object U could be obtained by a unique homomorphism *from* the universal object F *to* the object U. A direct product $P = G \times H$ is an object with the opposite sort of universal property: all groups with the specified additional structure (a map into G, and a map into H) are obtained by mapping arbitrary groups U *into* the universal example P. Thus, while the free group on a set X, the abelianization of a group G, the coproduct of two groups G and H, etc., can be thought of as ‘‘first’’ or diagrammatically ‘‘leftmost’’ groups with given kinds of structure, the direct product $G \times H$ is the ‘‘last’’ or ‘‘rightmost’’ group with maps into G and H. We shall refer to these two types of conditions as ‘‘left’’ and ‘‘right’’ universal properties respectively. (This terminology is based on thinking of arrows as going from left to right, though it happens that in most of the diagrams in preceding sections, the arrow from the universal object was drawn downward.)

The philosophy of how to construct objects with properties of either kind is in broad outline the same: Figure out as much information as possible about an arbitrary object (*not* assumed universal) with the given ‘‘additional structure’’, and see whether that information can itself be considered as a description of an object. If it can, this object will in general turn out to be universal for the given structure! In the case of ‘‘left’’ universal constructions (free groups, coproducts, ...), this ‘‘information’’ means answers to the question, ‘‘What elements do we know exist, and what equalities must hold among them?’’ (Cf. remark 2.2.10.) In the right universal case, on the other hand, the question is, ‘‘*Given* an element of our object, what *data* can we describe about it in terms of the additional structure?’’

Let us illustrate this with the case of the direct product of groups. Given groups G and H, consider any group P with specified homomorphisms p_G, p_H into G and H. What data can we find about an element x of P using these maps? Obviously, we can get from x a pair of elements $(g, h) \in |G| \times |H|$, namely

$$g = p_G(x) \in |G|, \qquad h = p_H(x) \in |H|.$$

Can we get any more data? We can also obtain elements $p_G(x^2)$, $p_H(x^{-1})$, etc.; but these can be found by group operations from the elements $g = p_G(x)$ and $h = p_H(x)$, so they give no new information about x. All right then, let us agree to classify elements of P according to the pairs $(g, h) \in |G| \times |H|$ which they determine.

Now suppose $x \in |P|$ gives the pair (g, h), and y gives the pair (g', h'). Can we find from these the pair given by $xy \in |P|$? the pair given by x^{-1}? Clearly so: these will be (gg', hh'), and (g^{-1}, h^{-1}) respectively. And we can likewise write down the pair that $e \in |P|$ yields: (e_G, e_H).

Very well, let us take the ''data'' that classifies elements of our arbitrary P, namely the set of pairs (g, h) $(g \in |G|,\ h \in |H|)$ – together with the law of composition we have found for these pairs, $(g, h) \cdot (g', h') = (gg', hh')$, the inverse operation $(g, h) \mapsto (g^{-1}, h^{-1})$, and the neutral element pair (e_G, e_H) – and ask whether this forms a group. It does! And, because of the way this group was constructed, it will have homomorphisms into G and H, and be universal for this property. It is, of course, the product group $G \times H$.

Here is a pair of examples we have not yet discussed. Suppose we are given a homomorphism of groups

$$f\colon\ G \to H.$$

Now consider

(a) homomorphisms $a\colon A \to G$, from arbitrary groups A into G, whose composites with f are the trivial homomorphism, i.e., which satisfy $fa = e$; and

(b) homomorphisms $b\colon H \to B$, from H into arbitrary groups B, whose composites with f are the trivial homomorphism, i.e., which satisfy $bf = e$.

Given a homomorphism of the first sort, one can get further homomorphisms with the same property by composing with homomorphisms $A' \to A$, for arbitrary groups A'; so one may look for a pair (A, a) with the *right* universal property that every such pair (A', a') arises from (A, a) via a unique homomorphism $A' \to A$. For (ii), one would want a corresponding *left* universal B.

To try to find the right-universal A, we ask: Given an arbitrary homomorphism $A \to G$ with $fa = e$ as in (i), what data can we attach to any element $x \in |A|$? Its image $g = a(x)$, certainly. This must be an element which f carries to the neutral element, since $fa = e$; thus the set of possibilities is $\{g \in |G| \mid f(g) = e\}$. We find that this set forms a group (with a map into G, namely the inclusion) having the desired universal property. This is the *kernel* of f.

We get the left universal example of (ii) by familiar methods: Given arbitrary $b\colon H \to B$ with $bf = e$ as in (ii), B must contain an image $\bar{h} = b(h)$ of each element $h \in |H|$. The fact that $bf = e$ tells us that the images in B of all elements of $f(G)$ must be the neutral element, and we quickly discover that the universal example is the quotient group $B = H/N$, where N is the normal subgroup of H generated by $f(G)$. This group is called the *cokernel* of the map f.

Right universal constructions are not as conspicuous in algebra as left universal constructions. When they occur, they are often fairly elementary and familiar constructions (e.g., the direct product of two groups; the kernel of a homomorphism). However, we shall see less trivial cases in later chapters; some of the exercises below also give interesting examples.

Exercise 3.8:1. Let G be a group, and X a set. Show that there exist

(i) a G-set S with a universal map $f: |S| \to X$, and

(ii) a G-set T with a universal map $g: X \to |T|$.

First state the universal properties explicitly.

(Hint to (i): Given any G-set S with a map $f: |S| \to X$, an element $s \in |S|$ will determine not only an element $x = f(s) \in X$, but for every $g \in |G|$ an element $x_g = f(gs) \in X$. From the *family* of elements, $(x_g)_{g \in |G|}$ determined by an $s \in S$, can one describe the family associated with hs for any $h \in |G|$?)

One can carry the idea of the above exercise further in several directions:

(a) Given a group homomorphism $f: G_1 \to G_2$, note that from any G_2-set S one can get a G_1-set S_f, by taking the same underlying set, and defining for $g \in |G_1|$, $s \in |S|$

$$gs =_{\mathrm{def}} f(g)\, s.$$

Now given a G_1-set X, one can look for a G_2-set S with a universal homomorphism $S_f \to X$, or for a G_2-set T with a universal homomorphism $X \to T_f$. The above exercise corresponds to the case $G_1 = \{e\}$. (An $\{e\}$-set is essentially a set with no additional structure. You should verify that for $G = \{e\}$, the above universal questions reduce to those of the exercise.)

(b) Instead of looking at *sets* S on which a group G acts by permutations, one can consider abelian groups, vector spaces, etc., on which G acts by automorphisms. In this case, the analogous universal constructions are still possible, and give the important concept of ‘‘induced representations’’ of a group.

(c) The preceding point introduced extra structure on the *sets* on which our groups act. One can also consider the case where one's *groups* G have additional structure, say topological or measure-theoretic, and restrict attention to continuous, measurable, etc., G-actions on appropriately structured spaces S. The versions of ‘‘induced representation’’ that one then obtains are at the heart of the modern representation theory of topological groups.

Exercise 3.8:2. Formulate right universal properties analogous to the left universal property defining free groups and the abelianization of a group, and show that no constructions exist having these properties. What goes wrong when we attempt to apply the general approach of this section?

Exercise 3.8:3. If X is a set and S a subset of X, then given any set map $f: Y \to X$, one gets a subset of Y, $T = f^{-1}(S)$. Does there exist a *universal* pair (X, S), such that for any set Y, every subset $T \subseteq Y$ is induced in this way via a unique set map $f: Y \to X$?

Exercise 3.8:4. Let A, B be fixed sets. Suppose X is another set, and $f: A \times X \to B$ is a set map. Then for any set Y, and map $m: Y \to X$, a set map $A \times Y \to B$ is induced. (How?) Does there exist, for each A and B, a universal set X and map f as above, i.e., an X and an f such that for any Y, all maps $A \times Y \to B$ are induced by unique maps $Y \to X$?

Exercise 3.8:5. Let R be a ring with 1. (Commutative if you like. If you consider general R, then for ‘‘module’’ understand ‘‘left module’’ below.) Before attempting each of the following questions, formulate precisely the universal property desired.

(i) Given a set X, does there exist an R-module M with a universal set map $|M| \to X$?

(ii) If M is an R-module, let M_{add} denote the underlying additive group of M. Given an abelian group A, does there exist an R-module M with a universal homomorphism of abelian groups $M_{\mathrm{add}} \to A$?

(iii) and (iv): What about the left universal analogs of the above right universal questions?

3.9. Tensor products. Let A, B and C be abelian groups, which we shall write additively. Then by a *bilinear map* $\beta: (A, B) \to C$ we shall mean a set map $\beta: |A| \times |B| \to |C|$ such that

(i) for each $a \in |A|$, the map $\beta(a, -): |B| \to |C|$ (that is, the map taking each element $b \in |B|$ to $\beta(a, b) \in |C|$) is a *linear* map (homomorphism of abelian groups) from B to C, and

(ii) for each $b \in |B|$, the map $\beta(-, b): |A| \to |C|$ is a linear map from A to C.

This is usually called a bilinear map ''from $A \times B$ to C'' (I usually call it that myself). However, that terminology misleads many students into thinking that it has something to do with the *group* $A \times B$. In fact, although the definition of bilinear map involves the group structures of A and B, and involves the set $|A| \times |B|$, it has nothing to do with the structure of direct product group that one can put on this set. This is illustrated by:

Exercise 3.9:1. Show that for any abelian groups A, B, C, the only map $|A| \times |B| \to |C|$ which is both a linear map $A \times B \to C$, and a bilinear map $(A, B) \to C$, is the zero map.

As an example to keep in mind, take any ring $R = (|R|, +, \cdot, -, 0, 1)$, and let R^+ denote the additive group $(|R|, +, -, 0)$. Then the maps $(x, y) \mapsto x + y$ and $(x, y) \mapsto x - y$ are *group homomorphisms* $R^+ \times R^+ \to R^+$ (but not bilinear maps), while the multiplication map $(x, y) \mapsto x \cdot y$ is a *bilinear map* $(R^+, R^+) \to R^+$ (but not a group homomorphism $R^+ \times R^+ \to R^+$).

I am speaking about abelian groups to keep the widest possible audience, but everything I have said and will say about bilinear maps among such groups applies to bilinear maps of modules over a commutative ring, and in particular, to bilinear maps of vector spaces over a field, with the adjustment that ''linear map'' in (i) and (ii) above is interpreted to mean module homomorphism. (There are also extensions of all these concepts to left modules, right modules, and bimodules over noncommutative rings, which we will look at with the help of a more sophisticated perspective in §9.7; but these are not obvious if you haven't seen them before.)

Given two abelian groups A and B, let us construct an abelian group $A \otimes B$ (called the *tensor product* of A and B) as follows: We present it using a set of generators which we write $a \otimes b$, one for each $a \in |A|$, $b \in |B|$, and defining relations which are precisely the conditions required to make the map $(a, b) \mapsto a \otimes b$ bilinear; namely

$$\begin{aligned} (a + a') \otimes b &= a \otimes b + a' \otimes b, \\ a \otimes (b + b') &= a \otimes b + a \otimes b' \end{aligned} \qquad (a, a' \in |A|,\ b, b' \in |B|).$$

(If we are working with R-modules, we also need the R-module relations

$$(ra) \otimes b = r(a \otimes b) = a \otimes (rb) \qquad (a \in |A|,\ b \in |B|,\ r \in |R|).$$

To indicate that one is referring to the module so constructed, rather than the tensor product as abelian groups, one often writes this object $A \otimes_R B$.)

Then by construction, $A \otimes B$ will be an abelian group with a bilinear map $\otimes: (A, B) \to A \otimes B$, and the universal property of the abelian group presented by these generators and relations translates to say that this map will be universal among bilinear maps on (A, B).

$$\begin{tikzcd}
(A,\, B) \arrow[r, "{(a,b) \mapsto a \otimes b}"] \arrow[dr, "\forall \beta"'] & A \otimes B \arrow[d, "\exists 1\, h"] \\
 & C
\end{tikzcd}$$

We can get simpler presentations of this group if we are given presentations of A and B. A presentation of A can be looked at as a representation $A = F(X)/{<}S{>}$, where $F(X)$ is the free abelian group on the given set of generators, and ${<}S{>}$ is the subgroup of $F(X)$ generated by the family S of relators (elements that are required to go to 0). If A is so presented, and likewise B is written as $F(Y)/{<}T{>}$, then it is not hard to show (and you may do so as the next exercise) that

$$A \otimes B \ \cong\ F(X) \otimes F(Y) / {<}S \otimes Y \cup X \otimes T{>} \tag{3.9.1}$$

where $S \otimes Y$ means $\{s \otimes y \mid s \in S,\, y \in Y\} \subseteq |F(X) \otimes F(Y)|$, and $X \otimes T$ is defined analogously. One finds that $F(X) \otimes F(Y)$ is a free abelian group on the family $X \otimes Y$ (precisely: a free abelian group on $X \times Y$ via the mapping $(x,y) \mapsto u(x) \otimes v(y)$, where $u \colon X \to |F(X)|$ and $v\colon Y \to |F(Y)|$ are the universal maps associated with the free groups $F(X)$ and $F(Y)$). Hence (3.9.1) is equivalent to a presentation of $A \otimes B$ by the generating set $X \times Y$ and a certain set of relations.

In the following exercises, unless the contrary is stated, you may substitute ‘‘R-module’’ for ‘‘abelian group’’ and get the results for this more general case.

Exercise 3.9:2. Prove (3.9.1), and the assertion that $F(X) \otimes F(Y)$ is free abelian on $X \otimes Y$. Can the ‘‘denominator’’ of (3.9.1) be replaced simply by ${<}S \otimes T{>}$?

Exercise 3.9:3. (i) Given abelian groups A and C, is there a universal pair (B, β), of an abelian group B and a bilinear map $\beta\colon (A, B) \to C$?

(ii) Given an abelian group C, is there a universal 3-tuple (A, B, β), such that A and B are abelian groups and β a bilinear map $(A, B) \to C$?

Before answering each part, say what the universal property would be and whether it would be a right or left universal property. Try the approach suggested in the preceding section for finding such objects.

Why have we defined bilinear maps only for *abelian* groups? This is answered by

Exercise 3.9:4. Let F, G and H be not necessarily abelian groups (so this exercise has no generalization to R-modules), and suppose $\beta\colon |F| \times |G| \to |H|$ is a map such that

$$\begin{aligned} &(\forall f \in |F|) \text{ the map } g \mapsto \beta(f,g) \text{ is a group homomorphism: } G \to H; \\ &(\forall g \in |G|) \text{ the map } f \mapsto \beta(f,g) \text{ is a group homomorphism: } F \to H. \end{aligned} \tag{3.9.2}$$

(i) Show that the subgroup H_0 of H generated by the image of β is abelian.

(ii) Deduce that the map β has a natural factorization

$$|F| \times |G| \ \to\ |F^{\mathrm{ab}}| \times |G^{\mathrm{ab}}| \ \xrightarrow{\beta'}\ |H_0| \ \hookrightarrow\ |H|,$$

where β' is bilinear. Thus, the study of maps satisfying (3.9.2) is reduced to the study of bilinear maps of *abelian* groups. This makes it easy to do

(iii) For general groups F and G, deduce a description of the group H with a universal map β satisfying (3.9.2) in terms of tensor products of abelian groups.

Remark: the above exercises, together with the observation that the multiplication map of a ring is a *bilinear* operation with respect to the ring's additive group structure, show why, though one often deals with rings having noncommutative multiplication, one does not have a natural concept of ''ring with noncommutative addition''.

(Nonetheless, there are sometimes ways of generalizing a concept other than the obvious ones, and some group-theorists have introduced a version of the concept of bilinear map which does not collapse in the manner described above in the noncommutative case. The student interested in this can look at [**45**] and other papers referred to there.)

Although the image of a group homomorphism is a subgroup of the codomain group, this is not generally true of the image of a bilinear map:

Exercise 3.9:5. (i) Let U, V, W be finite-dimensional vector spaces over a field, and consider composition of linear maps as a set map $\mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W)$. Note that if we regard these hom-sets as additive groups, this map is bilinear. Suppose V is one-dimensional; then describe the range of this composition map. Is it a subgroup of $\mathrm{Hom}(U, W)$?

(ii) If A and B are abelian groups, does every element of $A \otimes B$ have the form $a \otimes b$ for some $a \in |A|$, $b \in |B|$? (Prove your answer, of course.)

Another important property of tensor products is noted in

Exercise 3.9:6. If A, B and C are abelian groups, show that there is a natural isomorphism $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$.

State an analogous result holding for sets A, B, C and set maps.

To motivate the next exercise, let n be a positive integer, and $\mathbf{Z}_n$ the cyclic group of order n, which can be presented by one generator y and one relation $ny = 0$. (Since we are now using additive notation, ny means $y + ... + y$, with n summands.) If A is any abelian group, you should not find it hard to verify that $A \otimes \mathbf{Z}_n$ is isomorphic to A / nA, where nA is the subgroup of A consisting of all elements na $(a \in |A|)$; i.e., that the homomorphism $x \mapsto x \otimes y$ is surjective, and has kernel nA.

To generalize this observation, let us replace $\mathbf{Z}_n$ by an arbitrary abelian group B with a presentation

$$B \;=\; F(Y) \,/\, {<}\, T \,{>}, \tag{3.9.3}$$

where $F(Y)$ is the free abelian group on Y. Given any abelian group A, one finds that $A \otimes F(Y)$ is a direct sum of copies of A, indexed by Y, $\bigoplus_Y A$. I claim now that we can get $A \otimes B$ from this group by dividing out by another sum of homomorphic images of A, indexed by T. To describe this sum, we need a way of specifying certain maps of A into $\bigoplus_Y A$. Because the latter group is a coproduct, it has associated with it a universal Y-tuple of maps,

$$q_y \colon\; A \;\to\; \bigoplus\nolimits_Y A \qquad (y \in Y).$$

Since $\mathrm{Hom}(A, \bigoplus_Y A)$ is an abelian group, this Y-tuple of elements determines a homomorphism $\psi \colon F(Y) \to \mathrm{Hom}(A, \bigoplus_Y A)$, such that for each $y \in Y$, $\psi(y) = q_y$. Having defined this homomorphism ψ, we can now apply it to other elements of $F(Y)$; in particular, we can define

$$C \;=\; (\bigoplus\nolimits_Y A\,) \,/\, \Sigma_{t \in T}\, \psi(t)(A), \tag{3.9.4}$$

where the denominator means the subgroup of $\bigoplus_Y A$ generated by the images of A under all the homomorphisms $\psi(t) \colon A \to \bigoplus_Y A$, as t ranges over the relator-set T of (3.9.3). Now for any $f \in |F(Y)|$, let $[f]$ denote its image in B (cf. (3.9.3)), and for any $x \in |\bigoplus_Y A|$, let $[x]$ denote its image in C (cf. (3.9.4)). The rest I leave to you:

Exercise 3.9:7. Show that the formula $\tau(a, [f]) = [\psi(f)(a)]$ gives a well-defined map $\tau\colon |A| \times |B| \to |C|$, that this map is bilinear, and that the pair (C, τ) has the universal property of $(A \otimes B, \otimes)$. Conclude that the right hand side of (3.9.4) is isomorphic to $A \otimes B$.

Apply this to the case $B = \mathbf{Z}_n$, and recover the description of $A \otimes \mathbf{Z}_n$ given in the motivating remarks above.

Another interesting task is

Exercise 3.9:8. Investigate conditions on abelian groups (or R-modules) A and B under which $A \otimes B = \{0\}$.

Note: In subsequent sections, we shall occasionally refer again to bilinear maps. In these situations, we may use either the notation ''$(A, B) \to C$'' introduced here, or the more standard notation ''$A \times B \to C$''. (Of course, if all we have to say is something like ''this map $|A| \times |B| \to |C|$ is bilinear'', we will not need to use either notation.)

3.10. Monoids. So far, we have been moving within the realm of groups. It is time to broaden our horizons. We begin with semigroups and monoids, objects which are very much like groups in some ways, yet quite different in others.

We recall that a *semigroup* means an ordered pair $S = (|S|, \cdot)$ such that $|S|$ is a set and $\cdot$ a map $|S| \times |S| \to |S|$ satisfying the associative identity, while a *monoid* is a 3-tuple $S = (|S|, \cdot, e)$ where $|S|$ and $\cdot$ are as above, and the third component, e, is a *neutral element* for the operation $\cdot$. As with groups, the multiplication of semigroups and monoids is most often written without the ''$\cdot$'' when there is no need to be explicit. A *homomorphism* of semigroups $f\colon S \to T$ means a set map $f\colon |S| \to |T|$ which respects ''$\cdot$''; a *monoid homomorphism* is required to respect neutral elements as well: $f(e_S) = e_T$.

(Incidentally, I have long considered the use of two unrelated terms, ''semigroup'' and ''monoid'', for these very closely related types of objects to be an unnecessary proliferation of terminology. In most areas of mathematics, distinctions between related concepts are made by modifying phrases, e.g., ''commutative group'' versus ''not necessarily commutative group'', ''ring with 1'' versus ''ring without 1'', ''manifold with boundary'' versus ''manifold without boundary''. The author of a paper considering one of these concepts will generally begin by setting conventions, such as ''In this note, unless the contrary is stated, rings will have unit element, and ring homomorphisms will be understood to respect this element''. In papers of mine where monoids have come up, I have followed the same principle, and called them ''semigroups with neutral element'' or, after saying what this would mean, simply ''semigroups''. I did the same in these notes through the 1995 edition. However, it looks as though the term ''monoid'' is here to stay; so starting with the 1998 edition of these notes, I am following standard usage, given above.)

The concept of monoid seems somewhat more basic than that of semigroup. If X is any set, then the set of all maps $X \to X$ has a natural monoid structure, with functional composition as the multiplication and the identity map as the neutral element, and more generally, this is true of the set of endomorphisms of any mathematical object. Sets whose natural structure is one of semigroup and not of monoid tend to arise as subsidiary constructions, when one considers those elements of a naturally occurring monoid that satisfy some restrictions that exclude the neutral element; e.g., the set of maps $X \to X$ having finite range, or the set of even integers under multiplication. However, ''semigroup'' is the older of the two terms, so the study of semigroups and monoids is called ''semigroup theory''.

If $(|S|, \cdot, e)$ is a monoid, one can, of course, look at the semigroup $(|S|, \cdot)$, while if $(|S|, \cdot)$ is a semigroup, one can ''adjoin a unit'' and get a monoid $(|S| \sqcup \{e\}, \cdot, e)$. Thus, results on monoids yield results on semigroups and vice versa. To avoid repetitiveness, we will focus here on monoids, and mention semigroups only when there is a contrast to be made. Most of our observations on monoids have obvious analogs for semigroups, the exceptions being those relating to invertible elements.

The concept of a free monoid (F, u) on a set X is defined by the expected universal property (diagram below).

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, "\forall v"] & {|F|} \arrow[d] & F \arrow[d, "\exists 1\ f"] \\
& {|S|} & S
\end{tikzcd}$$

Free monoids on all sets exist, by the general arguments of §2.2 and §2.3. One also has a normal form in the free monoid on X, analogous to that of §2.4, but without any negative exponents. That is, every element can be written uniquely as a product,

$$x_n \dots x_1,$$

where $x_1, \dots, x_n \in |X|$, and $n \geq 0$ (the product of 0 factors being understood to mean the neutral element). Multiplication is performed by juxtaposing such products. ''Van der Waerden's trick'' is not needed to establish this normal form, since there is no cancellation to complicate a direct verification of associativity. Note that the free monoid on X is isomorphic to the submonoid generated by X within the *free group* on X.

If X is a set, and R a set of pairs of *monoid terms* in the elements of X, there will likewise exist a monoid determined by ''generators X and relations R'', i.e., a monoid S with a map $u: X \to |S|$ such that for each of the pairs of terms $(s, t) \in R$, one has $s_u = t_u$ in S, and which is universal for this property. As in the group case, this S can be obtained by a direct construction, using terms modulo identifications deducible from the monoid laws and the system of relations R, or as a submonoid of a large direct product, or by taking the free monoid F on the set X, and imposing the given relations.

But how does one ''impose relations'' on a monoid? In a group, we noted that any relation $x = y$ was equivalent to $xy^{-1} = e$, hence to study relations satisfied in a homomorphic image of a given group G, it sufficed to consider which elements of G go to e; so the construction of imposing relations on G reduced to that of dividing out by a normal subgroup. But for monoids, the question of which elements fall together does not come down to that of which elements go to e. For instance, let S be the free monoid on $\{x, y\}$, and map S homomorphically to the free monoid on $\{x\}$ by sending both x and y to x. Note that any product of m x's and n y's goes to x^{m+n} under this map. Thus the only element going to e is e itself, though the homomorphism is far from one-to-one.

So to study relations satisfied in the image of a monoid homomorphism $f: S \to T$, one should look at the whole set

$$K_f \;=\; \{(s,t) \mid f(s)=f(t)\} \;\subseteq\; |S|\times|S|.$$

We note the following properties of K_f:

(3.10.1) $(\forall\, s\in S)\;\; (s,s)\in K_f.$

(3.10.2) $(\forall\, s,\, t\in S)\;\; (s,t)\in K_f \;\Rightarrow\; (t,s)\in K_f.$

(3.10.3) $(\forall\, s,\, t,\, u\in S)\;\; (s,t)\in K_f,\; (t,u)\in K_f \;\Rightarrow\; (s,u)\in K_f.$

(3.10.4) $(\forall\, s,\, t,\, s',\, t'\in S)\;\; (s,t)\in K_f,\; (s',t')\in K_f \;\Rightarrow\; (ss',tt')\in K_f.$

Here (3.10.1-3) say that K_f is an equivalence relation, and (3.10.4) says that it ‘‘respects’’ the monoid operation.

I claim, conversely, that if S is a monoid, and $K\subseteq|S|\times|S|$ is any subset satisfying (3.10.1-4), then there exists a homomorphism f of S into a monoid T such that $K_f = K$. Indeed, since K is an equivalence relation on $|S|$, we may define $|T| = |S|/K$ and let $f: |S| \to |T|$ be the quotient map $x \mapsto [x]$. It is now easy to see from (3.10.4) that the formula $[s]\cdot[t] = [st]$ defines an operation on equivalence classes; and it is straightforward that this makes $T = (|T|,\, \cdot,\, [e])$ a monoid such that f is a homomorphism, and $K_f = K$.

Exercise 3.10:1. (i) Compare this construction with that of §2.2. Why did we need the conditions (2.2.1-3) in that construction, but not the corresponding conditions here?

(ii) Given two monoid homomorphisms $f: S \to T$ and $f': S \to T'$, show that there exist isomorphisms between their images making the diagram below commute if and only if $K_f = K_{f'}$.

$$\begin{tikzcd}[row sep=tiny] & f(S) \subseteq T \\ S \arrow[ur] \arrow[dr] & \|\wr \\ & f'(S) \subseteq T' \end{tikzcd}$$

Definition 3.10.5. *For any monoid S, a binary relation K on $|S|$ satisfying* (3.10.1-4) *above is called a* congruence *on S. The equivalence class of an element is called its* congruence class *under K; the monoid T constructed above is called the* quotient *or* factor *monoid of S by K, written S/K.*

Given a set R of pairs of elements of a monoid S, it is clear that one can construct the *least* congruence K containing R by closing R under four operations corresponding to conditions (3.10.1-4). The quotient S/K has the correct universal property to be called the monoid obtained by imposing the relations R on the monoid S. We shall sometimes denote this S/R, or $S/(s=t \mid (s,t)\in R)$, or, if the elements of R are listed as $(s_i,\, t_i)$ $(i\in I)$, as $S/(s_i = t_i \mid i\in I)$.

In particular, by imposing relations on a free monoid, we can get a monoid presented by any families of generators X and relations R. As for groups, this is written $<X \mid R>$. If there is danger of ambiguity, the group- and monoid-constructions can be distinguished as $<X \mid R>_{\mathrm{gp}}$ and $<X \mid R>_{\mathrm{md}}$.

Exercise 3.10:2. Given congruences K and K' on a monoid S, will there exist a least congruence containing both K and K'? A greatest congruence contained in both? Will set-theoretic union and intersection give such congruences? If not, what useful descriptions can you find for them? Is there a least congruence on S? A greatest?

If K is a congruence on S, characterize congruences on $T = S/K$ in terms of congruences on S.

Exercise 3.10:3. If X is a subset of $|S| \times |S|$, will there be a largest congruence contained in X? If not, will this become true under additional assumptions, such as that X is an equivalence relation on $|S|$, or that X is the underlying set of a submonoid of $S \times S$?

One can speak similarly of congruences on *groups*, *rings*, *lattices*, etc.. They are defined in each case by conditions (3.10.1-3), plus a family of conditions analogous to (3.10.4), one for each operation of positive arity on our algebras.

The fact that is special about *groups* can now be reformulated: ''A congruence K on a group G is uniquely determined by the congruence class of the neutral element $e \in |G|$, which can be any *normal subgroup* N of G. The congruence classes of K are then the *cosets* of N in G.'' Hence in group theory, rather than considering congruences, one almost always talks about normal subgroups.

Since a ring R has an additive group structure, a congruence on a ring will in particular be a congruence on its additive group, and hence will be determined by the congruence class J of the additive neutral element 0. The possibilities for J turn out to be precisely the *ideals* of R, so in ring theory, one works with ideals rather than congruences. However, historically, the congruence relation ''$a \equiv b \ (\mathrm{mod}\ n)$'' on the ring of integers $\mathbf{Z}$ was talked about before one had the concept of the ideal $n\mathbf{Z}$. Ring theorists still occasionally find it suggestive to write $a \equiv b \ (\mathrm{mod}\ J)$ rather than $a - b \in J$.

For objects such as monoids and lattices, congruences cannot be reduced to anything simpler, and are studied as such.

As usual, questions of the *structure* of monoids presented by generators and relations must be tackled case by case. For example:

Exercise 3.10:4. Find a normal form or other description for the monoid presented by two generators a and b and the one relation $ab = e$.

(Note that in the above and the next few exercises, letters a through d denote general monoid elements, but e is always the neutral element. If you prefer to write 1 instead of e in your solutions, you may do so.)

Exercise 3.10:5. (i) Same problem for generators a, b, c, d and relations

$$ab = ac = dc = e.$$

(ii) Same problem for generators a, b, c, d and relations

$$ab = ac = cd = e.$$

Exercise 3.10:6. Same problem for generators a, b, c and relations

$$ab = ac, \quad ba = bc, \quad ca = cb.$$

Exercise 3.10:7. Same problem for generators a, b and the relation $ab = b^2 a$.

One may define the *product* and the *coproduct* of two or of an arbitrary family of monoids, by the same universal properties as for groups,

$$\begin{tikzcd}[row sep=small] & S & & S \arrow[dr] & \\ S \times T \arrow[ur] \arrow[dr] & & & & S * T. \\ & T & & T \arrow[ur] & \end{tikzcd}$$

These also turn out to have the same descriptions as for groups: The direct product of an I-tuple of monoids consists of all I-tuples of elements of the given monoids, with componentwise operations;

the coproduct consists of formal products of strings of elements other than the neutral element taken from the given monoids, such that no two successive factors come from the same monoid. Van der Waerden's method *is* used in establishing this normal form, since multiplication of two such products can involve ''cancellation'' if some of the given monoids have elements satisfying $ab = e$.

On monoids, as on groups, one has the construction of *abelianization*, gotten by imposing the relations $ab = ba$ for all $a, b \in |S|$.

One may also define the *kernel* and *cokernel* of a monoid homomorphism $f: S \to S'$ as for groups:

$$\text{(3.10.6)} \qquad \operatorname{Ker} f \;=\; \text{submonoid of } S \text{ with underlying set } \{s \in |S| \mid f(s) = e\},$$

$$\text{(3.10.7)} \qquad \operatorname{Cok} f \;=\; S' / (f(s) = e \mid s \in |S|).$$

But since, as we have seen, the structure of the image of a monoid homomorphism f is not determined by the kernel of f, and, likewise, not every homomorphic image T of a monoid S' can be written as the cokernel (3.10.7) of a homomorphism of another monoid S into S' (e.g., the image of S under a non-one-to-one homomorphism with trivial kernel cannot), these are not such important concepts in the theory of monoids as in group theory.

We have seen that for f a homomorphism of monoids, a better analog of the group-theoretic concept of kernel is the congruence

$$\text{(3.10.8)} \qquad K_f \;=\; \{(s,t) \mid f(s) = f(t)\} \;\subseteq\; |S| \times |S|.$$

Note that this set K_f is the underlying set of a submonoid of $S \times S$, which we may call $\operatorname{Cong} f$. Likewise, since to impose relations on a monoid we specify, not that some elements should go to e, but that some pairs of elements should fall together, it seems reasonable that a good generalization of the cokernel concept should be, not an image $q(S)$ universal for the condition $qf = e$, where f is a given monoid homomorphism into S, but an image $q(S)$ universal for the condition $qf = qg$, for some *pair* of homomorphisms

$$\text{(3.10.9)} \qquad f, g\colon\; T \;\to\; S.$$

Such a homomorphic image is called a *difference cokernel* of the pair of maps f and g (because in the case of abelian groups, it can be described as the *cokernel* of the *difference-map* $f - g$).

This in turn suggests a dual construction: Given f, g as in (3.10.9), one can get a universal map p into T such that $fp = gp$; its domain monoid is called the *difference kernel* of f and g.

Explicitly, the difference *cokernel* of f and g is the quotient of the monoid S by the congruence generated by all pairs $(f(t), g(t))$ $(t \in |T|)$, and the difference *kernel* is the submonoid of T whose underlying set is $\{t \mid f(t) = g(t)\}$.

Exercise 3.10:8. Let $f: S \to T$ be a monoid homomorphism.

(i) Note that there is a natural pair of monoid homomorphisms from $\operatorname{Cong} f$ to S. Characterize $\operatorname{Cong} f$ and these two maps by a universal property.

(ii) What can be said of the difference kernel and difference cokernel of this pair of maps?

(iii) Can you construct from f a monoid $\operatorname{CoCong} f$ with a pair of maps into it, having a dual universal property? If so, again, look at the difference kernel and difference cokernel of this pair.

Exercise 3.10:9. The definition of difference kernel can be applied to groups as well as monoids. If G is a group, investigate which subgroups of G can occur as difference kernels of pairs of homomorphisms on G.

3.11. Groups to monoids and back again. If S is a monoid, we can get a group S^{gp} from S by ‘‘adjoining inverses’’ to all its elements in a universal manner. Thus, S^{gp} is a *group* G having a map $q\colon |S| \to |G|$ which respects products and neutral elements, and is universal among all such maps from S to groups.

But what kind of a map, exactly, is q? Since $S = (|S|, \cdot, e)$ is a monoid and $S^{\mathrm{gp}} = G = (G, \cdot, {}^{-1}, e)$ is a group, we cannot call it a group homomorphism or a monoid homomorphism from S to G. But it is more than just a set map, since it respects $\cdot$ and e. The answer is that q is a monoid homomorphism from S to the *monoid* $(|G|, \cdot, e)$ (i.e., $(|G|, \mu_G, e_G)$). So for an arbitrary *group* H, let us write $H_{\mathrm{md}} = (|H|, \mu_H, e_H)$, that is, ‘‘$H$ considered as a monoid’’. We can now state the universal property of S^{gp} and q neatly: S^{gp} is a group G, and q is a monoid homomorphism from S to G_{md}, such that for any group H and any monoid homomorphism $a\colon S \to H_{\mathrm{md}}$, there exists a unique group homomorphism $f\colon G \to H$ such that $a = fq\colon S \to H_{\mathrm{md}}$.

$$\begin{tikzcd}
S \arrow[r, "q"] \arrow[dr, "\forall a"] & G_{\mathrm{md}} \arrow[d] & G \arrow[d, "\exists 1\, f"] \\
 & H_{\mathrm{md}} & H
\end{tikzcd}$$

We shall call S^{gp} the *universal enveloping group* of the monoid S. It may be presented *as a group* by taking a generator for each element of S, and taking for defining relations the full multiplication table of S. More generally, if we are given some presentation of S by generators and relations as a monoid, G will be a group presented by the same generators and relations.

Exercise 3.11:1. Show that a monoid S is ‘‘embeddable in a group’’ (meaning embeddable in the monoid H_{md} for some group H) if and only if the universal map $q\colon S \to S^{\mathrm{gp}}$ is one-to-one.

Exercise 3.11:2. Describe the universal enveloping groups of the monoids of Exercises 3.10:4-7, and also of the monoid presented by generators a, b, c and the one relation $ab = ac$.

The last part of the above exercise reveals one necessary condition for the one-one-ness referred to in the preceding exercise to hold: The monoid S must have the ‘‘cancellation’’ property $xy = xy' \Rightarrow y = y'$. An interesting way of obtaining a full set of necessary and sufficient conditions for the universal map of a given monoid into a group to be one-to-one was found by A. I. Mal’cev [**80**], [**81**] (see also [**5**, §VII.3]).

Exercise 3.11:3. Let G be a group and S a *submonoid* of G_{md}, which generates G as a group. Observe that the inclusion of S in G_{md} induces a homomorphism $S^{\mathrm{gp}} \to G$. Will this in general be one-to-one? Onto?

If you have done Exercise 3.3:3, consider the case where G is the group of that exercise, and S the submonoid generated by a and ba. Describe the structure of S and of S^{gp}.

Suppose S is an *abelian* monoid. In this situation, important applications of the universal enveloping group construction have been made by A. Grothendieck; the group S^{gp} for S an abelian monoid is therefore often called ‘‘the Grothendieck group $K(S)$’’. This group is also abelian, and has a simple description: Using additive notation, and writing $\bar{a}$ for $q(a)$, one finds that every element of $K(S)$ can be written $\bar{a}-\bar{b}$ $(a,\,b\in|S|)$, and that one has equality $\bar{a}-\bar{b}=\overline{a'}-\overline{b'}$ between two such elements if and only if there exists $c\in|S|$ such that $a+b'+c=a'+b+c$ [**28**, p.40]. (If you have seen the construction of the *localization* RS^{-1} of a commutative ring at a multiplicative subset S, you will see that these constructions are closely related. In particular, the multiplicative group of nonzero elements of the field of fractions F of a commutative integral domain R is the Grothendieck group of the multiplicative monoid of nonzero elements of R.) The application of this construction to the abelian monoid of isomorphism classes of finite-dimensional vector bundles on a topological space X, with monoid operation corresponding to the operation ‘‘$\oplus$’’ on vector bundles, is the starting point of ‘‘K-theory’’. But perhaps this idea has been pushed too much – it is annoyingly predictable that when I mention a monoid of isomorphism classes of modules under ‘‘$\oplus$’’, people will say, ‘‘Oh, and then you go to its Grothendieck group!’’, when in fact I wanted to talk about the monoid itself.

Given a monoid S, there is also a *right-universal* way of obtaining a group: The set of *invertible elements* (‘‘units’’) of S can be made a group $U(S)$ in an obvious way, and the inclusion $U(S)\to S$ is universal among ‘‘homomorphisms of groups into S’’, in the sense indicated in the diagram below.

$$\begin{tikzcd}
G_{\mathrm{md}} \arrow[d] \arrow[rrd, "\forall\, h"] & & & G \arrow[d, "\exists 1\ f"] \\
U(S)_{\mathrm{md}} \arrow[rr, "i"] & & S & U(S)
\end{tikzcd}$$

Exercise 3.11:4. Let S be the monoid defined by generators $x,\,y,\,z$ and relations $xyz=e$, $zxy=e$. Investigate the structures of S and its abelianization S^{ab}. Describe the groups $U(S)$, $U(S)^{\mathrm{ab}}$, and $U(S^{\mathrm{ab}})$.

The constructions that relate semigroups and monoids, mentioned near the beginning of the preceding section, are related in a way paralleling $(\ )^{\mathrm{gp}}$ and $(\ )_{\mathrm{md}}$:

Exercise 3.11:5. (i) If $S=(|S|,\,\cdot)$ is a semigroup, describe how to extend the multiplication ‘‘$\cdot$’’ to $|S|\sqcup\{e\}$ so that $(|S|\sqcup\{e\},\,\cdot,\,e)$ becomes a monoid.

Let us call the monoid resulting from the above construction S^{md}, while if $S'=(|S'|,\,\cdot,\,e)$ is a monoid, let us write S'_{sg} for the semigroup $(|S'|,\,\cdot)$.

(ii) Show that given a semigroup S, the monoid S^{md} is universal among monoids T given with semigroup homomorphisms $S\to T_{\mathrm{sg}}$.

(iii) Given a monoid $S=(|S|,\,\cdot,\,e)$, what is the relation between the monoids S and $(S_{\mathrm{sg}})^{\mathrm{md}}$? Is there a natural homomorphism in either direction between them?

3.12. Associative and commutative rings. An *associative ring* R means a 6-tuple

$$R\ =\ (|R|,\ +,\ \cdot,\ -,\ 0,\ 1)$$

such that $(|R|,\,+,\,-,\,0)$ is an abelian group, $(|R|,\,\cdot,\,1)$ is a monoid, and the monoid operation $\cdot:|R|\times|R|\to|R|$ is *bilinear* with respect to the additive group structure. (Dropping the ‘‘1’’ from this definition, one gets a concept of ‘‘ring without 1’’, but we shall not consider these in this section, except in one exercise.) A *ring homomorphism* is a map of underlying sets respecting all the operations including 1. (Some writers, although requiring their rings to have 1, perversely allow ‘‘homomorphisms’’ that may not preserve 1; but we shall stick to the above definition.) An associative ring is called *commutative* if the multiplication $\cdot$ is so.

‘‘Commutative associative ring’’ is usually abbreviated to ‘‘commutative ring’’. Depending on the focus of a given work, *either* the term ‘‘associative ring’’ *or* the term ‘‘commutative ring’’ is usually shortened further to ‘‘ring’’; an author should always make clear what his or her usage will be. Here, I shall generally shorten ‘‘associative ring’’ to ‘‘ring’’; though I will sometimes retain the word ‘‘associative’’ when I want to emphasize that commutativity is *not* being assumed. When one deals with *nonassociative* rings – which we shall not do here – it is the associativity condition on the *multiplication* that is removed; frequently one then considers other identities (for instance, the identities of Lie or Jordan rings, which involve both the addition and the multiplication) in its place. The assumption that $(|R|, +, -, 0)$ is an abelian group, and that multiplication is a bilinear map with respect to this group structure, is made in all versions of ring theory: commutative, associative and nonassociative.

If k is a fixed commutative ring, then k-modules form a natural generalization of abelian groups, on which a concept of bilinear map is also defined, as noted parenthetically in §3.9 above. Hence one can generalize the definition of associative ring by replacing the abelian group structure by a k-module structure, and the bilinear map of abelian groups by a bilinear map of k-modules. The result is the definition of an *associative algebra* over k. The reader familiar with these concepts may note that everything I shall say below for rings remains valid, mutatis mutandis, for k-algebras. (An associative k-algebra is sometimes defined differently, as ring R given with a homomorphism of k into its center; but the two formulations are equivalent: Given a k-algebra R in the present sense of a ring with appropriate k-module structure, the map $c \mapsto c1_R$ $(c\in k)$ is a homomorphism of k into the center of that ring; while given a homomorphism g of k into the center of a ring, the definition $c\cdot r = g(c)r$ gives an appropriate module structure; and these constructions are inverse to one another. For algebras without 1, and for nonassociative algebras, this equivalence does not hold, and the ‘‘ring with k-module structure’’ definition is the useful one.)

The subject of universal constructions in ring theory is a vast one. In this section and the next, we will mainly look at the analogs of some of the constructions we have considered for groups and monoids.

First, free rings. Let us begin with the commutative case, since that is the more familiar one. Suppose R is a commutative ring, and x, y, z are three elements of R. Given any ring-theoretic combination of x, y and z, we can use the distributive law of multiplication (i.e., bilinearity of $\cdot$) to expand this as a sum of products of x, y and z (monomials) and additive inverses of such products. Using the commutativity and associativity of multiplication, we can write each monomial so that all factors x come first, followed by all y’s, followed by all z’s. We can then use commutativity of addition to bring together all occurrences of each monomial (arranging the distinct monomials in some specified order), and finally use distributivity again to combine occurrences of the same monomial using integer coefficients. If we now consider all *ring-theoretic terms* in symbols x, y and z, of the forms to which we have just shown we can bring any combination of *elements* x, y and z in any ring, we see, by the same argument as in §2.4, that the set of these ‘‘reduced terms’’ should give a normal form for the free commutative ring on three generators x, y and z – *if* they form a commutative ring under the obvious operations. It is, of course, well known that the set of such expressions *does* form a commutative ring, called the *polynomial ring* in three indeterminates, and written $\mathbf{Z}[x, y, z]$.

So polynomial rings over $\mathbf{Z}$ are free commutative rings. (More generally, the free commutative k-algebra on a set X is the polynomial algebra $k[X]$.) The universal mapping property corresponds to the familiar operation of *substituting values* for the indeterminates in a polynomial.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, "\forall v"'] & {|\mathbf{Z}[X]|} \arrow[d] & \mathbf{Z}[X] \arrow[d, "\exists 1\, f"] \\
 & {|R|} & R
\end{tikzcd}$$

When we drop the commutativity assumption, and look at general associative rings, the situation is similar, except that we cannot rewrite each monomial so that ‘‘all x’s come first’’ etc.. Thus we end up with linear combinations, with coefficients in $\mathbf{Z}$, of arbitrary products of our generators. We claim that formal linear combinations of such products give a normal form for elements of the free associative ring on the set X. This ring is written $\mathbf{Z}{<}X{>}$, and sometimes called the ring of *noncommuting polynomials* in X.

We were sketchy in talking about $\mathbf{Z}[X]$ because it is a well-known construction, but let us stop and sort out just what we mean by the above description of $\mathbf{Z}{<}X{>}$, before looking for a way to prove it.

We could choose a particular way of arranging the parentheses in every monomial term (say, nested to the right), a particular way of arranging the different monomials, and of arranging the parentheses in every sum or difference, and so obtain a set of ring-theoretic terms to which every term could be reduced, which we would prove constituted a normal form for the free ring. But observe that the question of putting parentheses into monomial terms is really just one of how to write elements in a *free monoid*, while the question of expressing sums and differences of such monomials is that of describing an element of the free abelian group on a set of generators. Let us therefore assume that we have chosen one or another way of calculating in free abelian groups – whether using a normal form, or a representation by integer-valued functions with only finitely many nonzero values, or whatever – and likewise that we have chosen a way of calculating in free monoids. Then we can calculate in free rings! Indeed, formalizing the above ideas, we get

Lemma 3.12.1. *Let $\mathbf{Z}{<}X{>}$ denote the free ring on the set X. Then the additive group of $\mathbf{Z}{<}X{>}$ is a free abelian group on the set of products in $\mathbf{Z}{<}X{>}$ of elements of X, and this set of products forms a free monoid on X.*

Proof. Let S denote the free monoid on X, and $F(|S|)$ the free abelian group on the underlying set of this monoid. We shall begin by describing a map $F(|S|) \to |\mathbf{Z}{<}X{>}|$.

If we write u for the universal map $X \to |\mathbf{Z}{<}X{>}|$, then by the universal property of free monoids, it induces a homomorphism u' from the free monoid S into the multiplicative monoid of $\mathbf{Z}{<}X{>}$. Hence by the universal property of free abelian groups, there exists a unique abelian group homomorphism u'' from the free abelian group $F(|S|)$ into the additive group of $\mathbf{Z}{<}X{>}$ which acts by u' on elements of $|S|$. (Note that our considerations so far are valid for any ring R given with a set map $X \to |R|$.) Clearly the image of the monoid S in $\mathbf{Z}{<}X{>}$ is closed under multiplication and contains the multiplicative neutral element; it is easy to deduce from this and the distributive law that the image of the abelian group $F(|S|)$ is closed under all the ring operations.

Since this image contains X, and $\mathbf{Z}<X>$ is generated as a ring by X, the image is all of $\mathbf{Z}<X>$, i.e., u'' is surjective. (The above argument formalizes our observation that every element of the subring generated by an X-tuple of elements of an arbitrary ring R can be expressed as a linear combination of products of elements of the given X-tuple.)

We now wish to show that u'' is one-to-one. To do this it will suffice to show that there is *some* ring R with an X-tuple v of elements, such that under the induced homomorphism $\mathbf{Z}<X> \to R$, elements of $\mathbf{Z}<X>$ which are images of distinct elements of $F(|S|)$ are mapped to distinct elements of R.

How do we find such an R? Van der Waerden's trick for groups suggests that we should obtain it from some natural *representation* of the desired free ring. We noted in §2.4 that the *group* operations and identities arise as the operations and identities of the permutations of a set, so for ''representations'' of groups, we used actions on sets. The operations and identities for associative rings arise as the natural structure on the set of all endomorphisms of an abelian group A – one can compose such endomorphisms, and add and subtract them, and under these operations they form a ring $\mathrm{End}(A)$. So we should look for an appropriate family of endomorphisms of some abelian group to represent $\mathbf{Z}<X>$.

Let us, as in (2.4.4), introduce a symbol a; let Sa denote the set of symbols $x_n \ldots x_1 a$ ($x_i \in X$, $n \geq 0$); and this time let us further write $F(Sa)$ for the free abelian group on this set Sa. For every $x \in X$, let $\bar{x}: Sa \to Sa$ denote the map carrying each symbol $b \in Sa$ to the symbol xb. This extends uniquely (by the universal property of free abelian groups) to an additive group homomorphism $\bar{\bar{x}}: F(Sa) \to F(Sa)$. Thus $(\bar{\bar{x}})_{x\in X}$ is an X-tuple of elements of the associative ring $\mathrm{End}(F(Sa))$.

Taking $R = \mathrm{End}(F(Sa))$, the above X-tuple induces a homomorphism

$$f:\ \mathbf{Z}<X> \ \to\ R.$$

Now given any element of $F(|S|)$, which we may write

(3.12.2) $$r \;=\; \Sigma_{s\in|S|}\, n_s\, s \qquad (n_s \in \mathbf{Z},\ \text{almost all } n_s = 0),$$

we verify easily that the element $fu''(r) \in \mathrm{End}(F(Sa))$ carries a to $\Sigma\, n_s sa$. Hence distinct expressions (3.12.2) give distinct elements $u''(r) \in \mathbf{Z}<X>$, which proves the one-one-ness of u'' and establishes the lemma. □

For many fascinating results and open problems on free algebras, see [**50**]. For a smaller dose, you could try my paper [**35**], which answers the question, ''When do two elements of a free algebra commute?'' The problem is not of great importance itself, but it leads to the development of a number of beautiful and useful ring-theoretic tools.

Exercise 3.12:1. Let α denote the automorphism of the polynomial ring $\mathbf{Z}[x, y]$ which interchanges x and y. It is a standard result that the fixed ring of α, i.e., $\{a \in \mathbf{Z}[x,y] \mid \alpha(a) = a\}$, can be described as the polynomial ring in the two elements $x+y$ and xy.

(i) Consider analogously the automorphism β of the free associative ring $\mathbf{Z}<x,y>$ interchanging x and y. Show that the fixed ring of β is generated by the elements $x+y$, x^2+y^2, x^3+y^3, ... and is a free ring on this infinite set.

(ii) Observe that the homomorphism $\mathbf{Z}<x,y> \to \mathbf{Z}[x, y]$ taking x to x and y to y must take the fixed ring of β into the fixed ring of α. Will it take it *onto* the fixed ring of α?

(iii) If G is the free group on generators x and y, and if γ is the automorphism interchanging x and y in this group, describe the fixed subgroup of γ. Same question for the free *abelian* group on x and y.

Our description of the free ring on a set X involved the free monoid on X, and the description of the free commutative ring (the polynomial ring) can be seen to have an analogous relationship to the free commutative monoid. These connections between rings and monoids can be explained in terms of another universal construction:

If $R = (|R|, +, \cdot, -, 0, 1)$ is an associative ring, let R_{mult} denote its multiplicative monoid, $(|R|, \cdot, 1)$. Then for any monoid S, there will exist, by the usual arguments, a ring R with a *universal* monoid homomorphism $u\colon S \to R_{\text{mult}}$.

$$\begin{tikzcd}
S \arrow[r, "u"] \arrow[dr, "\forall v"'] & R_{\text{mult}} \arrow[d] & & R \arrow[d, "\exists 1\ f"] \\
& R'_{\text{mult}} & & R'
\end{tikzcd}$$

To study this object, let us fix S, and consider any ring R' with a homomorphism $S \to R'_{\text{mult}}$. The elements of R' that we can capture using this map are the linear combinations of images of elements of S, with integer coefficients. (Why is there no need to go on to products of such elements?) One finds that the *universal* such ring R will have as additive structure the free abelian group on $|S|$, with multiplicative structure determined by the bilinearity condition, and the condition that the given map $|S| \to |R|$ respect multiplication. The result is called the *monoid ring* on S, denoted $\mathbf{Z}S$.

Given a presentation of S by generators and relations (written multiplicatively), a presentation of $\mathbf{Z}S$ as a ring will be given by the same generators and relations. In particular, if we take for S the *free* monoid on a set X, presented by generators X and no relations, then $\mathbf{Z}S$ will be presented as a *ring* by generators X and no relations, and so will be the free ring on X, which is just what we saw in Lemma 3.12.1. If we take for S a free *abelian* monoid, then S may be presented as a monoid by generators X and relations $xy = yx$ $(x, y \in X)$, hence this is also a presentation of $\mathbf{Z}S$ as a ring. Since commutativity of a set of generators of a ring is equivalent to commutativity of the whole ring, the above presentation makes $\mathbf{Z}S$ the free commutative ring on X.

If S is a monoid and A an abelian group, then a ‘‘linear action’’ or ‘‘representation’’ of S on A means a homomorphism of S into the multiplicative monoid of the endomorphism ring $\mathrm{End}(A)$ of A. By the universal property of $\mathbf{Z}S$, this is equivalent to a ring homomorphism of $\mathbf{Z}S$ into $\mathrm{End}(A)$, which is in turn equivalent to a structure of left $\mathbf{Z}S$-module on the abelian group A. In particular, to give an action of a *group* G by automorphisms on an abelian group A corresponds to making A a left module over the *group ring* $\mathbf{Z}G$. Much of modern group theory revolves around linear actions, and hence is closely connected with the properties of $\mathbf{Z}G$ (and more generally, with group *algebras* kG where k is a commutative ring, so that left kG-modules correspond to actions of G on *k-modules*). For some of the elementary theory, see [**28**, Chapter XVIII]. A major work on group algebras is [**88**].

Above, we ‘‘factored’’ the construction of the free associative or commutative ring on a set X into two constructions: the free (respectively, free abelian) monoid construction, which universally closes X under a multiplication with a neutral element, and the monoid-ring construction, which brings in an additive structure in a universal way. In fact, these free ring constructions can also be factored the other way around! Given a set X, we can first map it into an abelian group in a universal way, getting the free abelian group A on X, then form a ring (respectively a commutative ring) R with a universal additive group homomorphism $A \to R_{\mathrm{add}}$. For any abelian group A, this universal associative ring is called the *tensor ring* on A, because its additive group structure turns out to have the form

$$\mathbf{Z} \oplus A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \ldots ,$$

though we shall not show this here. The corresponding universal *commutative* ring is called the *symmetric ring* on A; its structure for general A is more difficult to describe. For more details see [**28**, §§XVI.7, 8] or [**41**]. Thus, a free associative ring can be described as the tensor ring on a free abelian group, and a polynomial ring as the symmetric ring on a free abelian group.

On to other constructions. Suppose R is a commutative ring, and (f_i, g_i) $(i \in I)$ a family of pairs of elements of R. To impose the relations $f_i = g_i$ on R, one forms the factor-ring R/J, where J is the ideal generated by the elements $f_i - g_i$. This ideal is often written $(f_i - g_i)_{i \in I}$. Another common notation, preferable because it is more explicit, is $\Sigma_{i \in I} R(f_i - g_i)$, or, if we set $U = \{f_i - g_i \mid i \in I\}$, simply RU. It consists of all sums

(3.12.3) $\quad \Sigma\, r_i(f_i - g_i) \qquad (r_i \in |R|,$ nonzero for only finitely many $i \in I)$.

The construction of imposing relations on a *noncommutative* ring R is of the same form, but with ''ideal'' taken to mean ''two-sided ideal'' – an additive subgroup of R closed under both left and right multiplication by members of R. The two-sided ideal generated by $\{f_i - g_i \mid i \in I\}$ is also often written $(f_i - g_i)_{i \in I}$, but again there is a more expressive notation, $\Sigma_{i \in I} R(f_i - g_i)R$, or simply RUR. This ideal consists of all sums of products of the form $r(f_i - g_i)r'$ $(i \in I, r, r' \in R)$, but in the noncommutative case, it is not in general enough to have, as in (3.12.3), *one* such summand for each $i \in I$. For instance, in $\mathbf{Z}\langle x, y\rangle$, the ideal generated by the one element x contains the element $yxy^2 + y^2xy$, which cannot be simplified to a single product rxr'.

Exercise 3.12:2. Let R be a commutative ring. Will there, in general, exist a universal homomorphism of R into an *integral domain* R'? If not, can you find conditions on R for such a homomorphism to exist? Consider in particular the cases $R = \mathbf{Z}$, $\mathbf{Z}_6$, $\mathbf{Z}_4$.

Exercise 3.12:3. (i) Obtain a normal form for elements of the associative ring A presented by two generators x, y, and one relation $yx - xy = 1$.

(ii) Let $\mathbf{Z}[x]_{\mathrm{add}}$ be the *additive group* of polynomials in one indeterminate x. Show that there exists a homomorphism f of the ring A of part (i) into the endomorphism ring of this abelian group, such that $f(x)$ is the operation of multiplying polynomials by x in $\mathbf{Z}[x]$, and $f(y)$ the operation of differentiating with respect to x. Is this homomorphism one-to-one?

The ring of the above example, or rather the corresponding algebra over a field k, is called the *Weyl algebra*. It is of importance in quantum mechanics, where multiplication by the coordinate function x corresponds to determining the x-coordinate of a particle, and differentiating with respect to x corresponds to determining its momentum in the x-direction. The fact that these operators do not commute leads, via the mysterious reasoning of quantum mechanics, to the impossibility of measuring those two quantities simultaneously, the simplest case of the ''Heisenberg uncertainty principle''.

Direct products $\prod_I R_i$ of associative rings and of commutative rings turn out, as expected, to be gotten by taking direct products of underlying sets, with componentwise operations.

Exercise 3.12:4. (Andreas Dress) (i) Find all subrings of $\mathbf{Z}\times\mathbf{Z}$. (Remember: a subring must have the same multiplicative neutral element 1. Try to formulate your description of each such subring R as a necessary and sufficient condition for an arbitrary $(a, b)\in\mathbf{Z}\times\mathbf{Z}$ to lie in $|R|$.)

A much harder problem is:

(ii) Is there a similar characterization of all subrings of $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$?

Exercise 3.12:5. Show that the commutative ring presented by one generator x, and one relation $x^2 = x$, is isomorphic (as a ring) to the product ring $\mathbf{Z}\times\mathbf{Z}$.

Exercise 3.12:6. Given generators and relations for two rings, R and S, show how to obtain generators and relations for $R\times S$.

Exercise 3.12:7. Describe

(i) the commutative ring A presented by one generator x, and one relation $2x = 1$, and

(ii) the commutative ring B presented by one generator x and two relations $4x = 2$, $2x^2 = x$. (Note that both of these relations are implied by the relation of (i).)

Are these isomorphic? Show that each of them has the property that for any ring R (commutative if you wish) there is *at most* one homomorphism of the presented ring (A, respectively B) into R.

Exercise 3.12:8. Suppose R is a ring whose underlying abelian group is finitely generated. Show that as a ring, R is finitely presented. (You may use the fact that every finitely generated abelian group is finitely presented.)

If you are comfortable with algebras over a commutative ring k, try to generalize this result to that context.

In discussing universal properties, I have neglected to mention some trivial cases. Let me give these in the next two exercises. Even if you do not write them up, think through the ‘‘ring’’ cases of parts (i) and (iii) of the next exercise, since some later exercises use them.

Exercise 3.12:9. (i) Consider the free group, monoid, associative ring, and commutative ring on the *empty* set of generators. Reformulate the universal properties of these objects in as simple a form as possible. Display the group, monoid, ring, and commutative ring characterized by these properties, if they exist.

(ii) State, similarly, the universal properties that would characterize the product and coproduct of an empty family of groups, monoids, rings, or commutative rings, and determine these objects, if any.

(iii) Give as simple as possible a system of defining generators and relations for the rings $\mathbf{Z}$ and $\mathbf{Z}_n$.

The next exercise concerns semigroups and rings without neutral elements. Note that when we say ‘‘without 1’’ etc., this does not *forbid* the existence of an element 1 satisfying $(\forall x)\ 1x = x = x1$. It just means that we don’t require the existence of such elements, and that when they exist, we don’t give them a special place in the definition, or require homomorphisms to respect them.

Exercise 3.12:10. Same as parts (i) and (ii) of the preceding exercise, but for semigroups and for rings without 1. Same for sets. Same for G-sets for a fixed group G.

Now back to rings with 1.

3.13. Coproducts and tensor products of rings.

We have noted that the descriptions of *coproducts* vary from one sort of algebraic object to another. We shall see below that they are different for commutative and for noncommutative rings. Let us again start with the commutative case.

Suppose S and T are fixed commutative rings, and we are given homomorphisms $s \mapsto \bar{s}$ and $t \mapsto \tilde{t}$ of these into a third commutative ring R. What elements of R can we capture? Obviously, elements $\bar{s}$ $(s \in |S|)$ and $\tilde{t}$ $(t \in |T|)$; from these we can form products $\bar{s}\tilde{t}$, and we can then form sums of elements of all these sorts:

(3.13.1) $$\bar{s} + \tilde{t} + \bar{s}_1\tilde{t}_1 + \ldots + \bar{s}_n\tilde{t}_n .$$

We don’t get more elements by multiplying such sums together, because a product $(\bar{s}\,\tilde{t}\,)(\bar{s}'\,\widetilde{t'}\,)$ reduces to $\overline{ss'}\,\widetilde{tt'}$. Let us note that the lone summands $\bar{s}$ and $\tilde{t}$ in (3.13.1) can actually be written in the same form as the others, for since $\overline{1_S} = \widetilde{1_T} = 1_R$, we have $\bar{s} = \bar{s}\,\widetilde{1_T}$ and $\tilde{t} = \overline{1_S}\,\tilde{t}$. So the subring of R that we get is generated as an additive group by the image of the map

(3.13.2) $$(s,\, t) \ \mapsto \ \bar{s}\,\tilde{t}$$

of $|S| \times |T|$ into $|R|$. If we look for equalities among sums of elements of this form, we find

$$\overline{(s+s')}\,\tilde{t} \ = \ \bar{s}\,\tilde{t} + \bar{s}'\,\tilde{t}, \quad \text{and} \quad \bar{s}\,\widetilde{(t+t')} \ = \ \bar{s}\,\tilde{t} + \bar{s}\,\tilde{t}',$$

in other words, relations saying that (3.13.2) is bilinear. These relations and their consequences turn out to be *all* we can find, and one can show that the *universal* R with ring homomorphisms of S and T into it, that is, the coproduct of S and T as commutative rings, has the additive structure of the *tensor product* of the additive groups of S and T. Its multiplication is determined by the formula

(3.13.3) $$(s \otimes t)(s' \otimes t') \ = \ ss' \otimes tt'$$

describing how to multiply the additive generators of this tensor product group. For a proof that this extends to a bilinear operation on all of $S \otimes T$, and that this operation makes the additive group $S \otimes T$ into a ring, see Lang [**28**, §XVI.6]. (Note: Lang works in the context of algebras over a commutative ring k, and he defines such an algebra as a homomorphism f of k into the center of a ring R – what I prefer to call, for intuitive comprehensibility, a ring R given with a homomorphism of k into its center; cf. parenthetical remark near the beginning of §3.12 above. Thus, when he defines the coproduct to be a certain *map*, look at the codomain of the map to see the *ring* that he means.) Of course, you can try writing down such a proof yourself, using the universal property of the tensor product, and perhaps some version of van der Waerden’s trick.

This coproduct construction is called ‘‘tensor product of commutative rings’’. The universal maps of S and T into $S \otimes T$ which make it their coproduct are given by

$$s \mapsto s \otimes 1, \qquad t \mapsto 1 \otimes t.$$

Exercise 3.13:1. If m and n are integers, find the structure of the tensor product ring $\mathbf{Z}_m \otimes \mathbf{Z}_n$ by two methods:

(i) By constructing the tensor product of abelian groups, and describing multiplication defined above.

(ii) By using the fact that a presentation of a coproduct can be obtained by ‘‘putting together’’ presentations for the two given objects. (Cf. Exercise 3.12:9.)

Exercise 3.13:2. Let $\mathbf{Z}[i]$ denote the ring of *Gaussian integers* (complex numbers $a+bi$ such that a and b are integers). This may be presented as a commutative ring by one generator i, and one relation $i^2 = -1$. Examine the structures of the rings $\mathbf{Z}[i] \otimes \mathbf{Z}_p$ (p a prime). E.g., will they be integral domains for all p? For some p?

The next two exercises concern tensor products of algebras over a field k, for students familiar with these concepts. Such tensor products are actually simpler to work with than the tensor products of rings described above, because every algebra over a field k is free as a k-module (since every k-vector-space has a basis), and tensor products of free modules are easily described (cf. lines following (3.9.1) above).

Exercise 3.13:3. Let K and L be extensions of a field k. A *compositum* of K and L means a 3-tuple (E, f, g) where E is a field extension of k, and $f: K \to E$, $g: L \to E$ are k-algebra homomorphisms such that E is generated by $f(|K|) \cup g(|L|)$ as a field (i.e., under the ring operations, and the partial operation of multiplicative inverse).

(i) Suppose K and L are *finite-dimensional* over k, and we form their tensor product algebra $K \otimes_k L$, which is a commutative k-algebra, but not necessarily a field. Show that up to isomorphism, all the composita of K and L over k are given by the factor rings $K \otimes L / P$, for prime ideals $P \subseteq K \otimes L$. (First write down what should be meant by an isomorphism between composita of K and L.)

(ii) What if K and L are not assumed finite-dimensional?

Exercise 3.13:4. (i) Determine the structure of the tensor product $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers and $\mathbf{R}$ the field of real numbers. In particular, can it be described as a nontrivial direct product of $\mathbf{R}$-algebras?

(ii) Do the same for $\mathbf{Q}(2^{1/3}) \otimes_{\mathbf{Q}} \mathbf{Q}(2^{1/3})$.

(iii) Relate the above results to the preceding exercise.

You can carry this exercise much farther if you like – find a general description of a tensor product of a finite Galois extension with itself; then of two arbitrary finite separable extensions (by taking them to lie in a common Galois extension, and considering the subgroups of the Galois group they correspond to); then try some examples with inseparable extensions... . In fact, one modern approach to the whole subject of Galois theory is via properties of such tensor products.

If S and T are arbitrary (not necessarily commutative) associative rings, one can still make the tensor product of the additive groups of S and T into a ring with a multiplication satisfying (3.13.3). It is not hard to verify that this will be universal among rings R given with homomorphisms $f: S \to R$, $g: T \to R$ such that all elements of $f(S)$ *commute* with all elements of $g(T)$ (cf. the ‘‘second universal property’’ of the direct product of two groups, end of §3.6 above. In fact, some early ring-theorists wrote $S \times T$ where we now write $S \otimes T$, considering this construction as the ring-theoretic analog of the direct product construction on groups.)

Exercise 3.13:5. Show that if S and T are monoids, then $\mathbf{Z}\,S \otimes \mathbf{Z}\,T \cong \mathbf{Z}\,(S \times T)$.

Exercise 3.13:6. Suppose S and T are associative rings, and we form the additive group $R_{\mathrm{add}} = S_{\mathrm{add}} \otimes T_{\mathrm{add}}$. Is the multiplication described above in general the unique multiplication on R_{add} which makes it into a ring R such that the maps $s \mapsto s \otimes 1_T$ and $t \mapsto 1_S \otimes s$ are ring homomorphisms? You might look, in particular, at the case $S = \mathbf{Z}[x]$, $T = \mathbf{Z}[y]$.

We shall encounter tensor products again in §9.7.

Let us now look at coproducts of not necessarily commutative rings. These exist, by the usual general nonsense, and again, a presentation of $S * T$ can be gotten by putting together presentations of S and T. But the explicit description of these coproducts is more complicated than for the constructions we have considered so far. For S and T arbitrary associative rings, there *is* no neat explicit description of $S * T$. Suppose, however, that as abelian groups, S is free on a basis

$\{1_S\} \cup B_S$, and T is free on a basis $\{1_T\} \cup B_T$. (For example, the rings $\mathbf{Z}[x]$ and $\mathbf{Z}[i]$ have such bases, with the B parts being $\{x, x^2, \ldots\}$ and $\{i\}$ respectively.) Then we see that given a ring R and homomorphisms $S \to R$, $T \to R$, written $s \mapsto \bar{s}$ and $t \mapsto \tilde{t}$, the elements of R that we get by ring operations from the images of S and T can be written as linear combinations, with integer coefficients, of products $x_n \ldots x_1$ where $x_i \in \overline{B_S} \cup \widetilde{B_T}$ (i.e., $\{\bar{b} \mid b \in B_S\} \cup \{\tilde{b} \mid b \in B_T\}$), such that no two elements from the same basis-set occur successively. (In thinking this through, note that a product of two elements from $\overline{B_S}$ can be rewritten as a linear combination of single elements from $\overline{B_S} \cup \{\overline{1_S}\}$, and that occurrences of $\overline{1_S}$ can be eliminated because in R, $\overline{1_S} = 1_R$; and the same considerations apply to elements from $\widetilde{B_T}$. In this description we are again considering 1_R as the ‘‘empty’’ or ‘‘length 0’’ product.) In fact, the coproduct of S and T as associative rings turns out to have precisely the set of such products as an additive basis.

Exercise 3.13:7. Verify the above assertion, using an appropriate modification of van der Waerden’s trick.

Exercise 3.13:8. (i) Examine the coproduct ring $\mathbf{Z}[i] * \mathbf{Z}[i]$ (where $\mathbf{Z}[i]$ denotes the Gaussian integers, as in Exercise 3.13:2). In particular, try to determine its center, and whether it has any zero-divisors.

(ii) In general, if S and T are rings free as abelian groups on *two-element* bases, of the forms $\{1, s\}$ and $\{1, t\}$, what can be said about the structure and center of $S * T$?

The next part shows that the above situation is exceptional.

(iii) Suppose as in (ii) that S and T each have additive bases containing 1, and that neither of these bases consists of 1 alone; but now suppose that at least one of them has more than two elements. Show that in this situation, the center of $S * T$ is just $\mathbf{Z}$.

Some surprising results on the module theory of ring coproducts are obtained in [**38**]. (That paper presumes familiarity with basic properties of semisimple artin rings and their modules. The reader who is familiar with such rings and modules, but not with homological algebra, should not be deterred by the discussion of homological properties of coproducts in the first section; these are applications of the main result of the paper, but that result and its proof do not require homological methods.)

3.14. Boolean algebras and Boolean rings. Let S be a set, and let $\mathbf{P}(S)$ denote the power set of S, that is, $\{T \mid T \subseteq S\}$. There are various natural operations on $\mathbf{P}(S)$: union, intersection, complement (i.e., ${}^{c}T = \{s \in S \mid s \notin T\}$), and the two zeroary operations, $\varnothing \in \mathbf{P}(S)$ and $S = {}^{c}\varnothing \in \mathbf{P}(S)$. Thus we can regard $\mathbf{P}(S)$ as the underlying set of an algebraic structure

$$(\mathbf{P}(S),\ \cup,\ \cap,\ {}^{c},\ \varnothing,\ S). \tag{3.14.1}$$

This structure, and more generally, any 6-tuple consisting of a set given with five operations of arities $2, 2, 1, 0, 0$ satisfying all the identities satisfied by structures of the form (3.14.1) for S a set, is called a *Boolean algebra*.

Such 6-tuples do not quite fit any of the pigeonholes we have considered so far. For instance, neither of the operations $\cup$, $\cap$ is the composition operation of an abelian group, hence a ‘‘Boolean algebra’’ is not a ring.

However, there is a way of looking at $\mathbf{P}(S)$ which reduces us to ring theory. There is a standard one-to-one correspondence between the power set $\mathbf{P}(S)$ of a set S and the set of functions 2^S, where 2 means the 2-element set $\{0, 1\}$; namely, the correspondence associating to each $T \in \mathbf{P}(S)$ its characteristic function (1 on elements of T, and 0 on elements of ${}^{\mathrm{c}}T$). If we try to do arithmetic with these functions, we run into the difficulty that the sum of two $\{0,1\}$-valued functions is not generally $\{0,1\}$-valued. But if we think of $\{0,1\}$ as the underlying set of the ring $\mathbf{Z}_2$ rather than as a subset of $\mathbf{Z}$, this problem is circumvented: 2^S becomes the ring $\mathbf{Z}_2^S$ – the direct product of an S-tuple of copies of $\mathbf{Z}_2$. Moreover, it is possible to describe union, intersection, etc., of subsets of S in terms of the ring operations of $\mathbf{Z}_2^S$. Namely, writing $\bar{a}$ for the characteristic function of $a \subseteq S$, we have

$$\text{(3.14.2)} \qquad \overline{a \cap b} \ = \ \bar{a}\,\bar{b}, \qquad \overline{a \cup b} \ = \ \bar{a} + \bar{b} + \bar{a}\,\bar{b}, \qquad \overline{{}^{\mathrm{c}}a} \ = \ 1 + \bar{a}, \qquad \overline{\varnothing} = 0, \qquad \bar{S} = 1.$$

Conversely, each *ring* operation of $\mathbf{Z}_2^S$, translated into an operation on subsets of S, can be expressed in terms of our set-theoretic Boolean algebra operations. The expressions for multiplication, for 0, and for 1 are clear from (3.14.2); additive inverse is the identity operation, and $+$ is described by

$$\text{(3.14.3)} \qquad \bar{a} + \bar{b} \ = \ \overline{(a \cap {}^{\mathrm{c}}b) \cup ({}^{\mathrm{c}}a \cap b)}\,.$$

(The above set $(a \cap {}^{\mathrm{c}}b) \cup ({}^{\mathrm{c}}a \cap b)$ is called the ‘‘symmetric difference’’ of the sets a and b.)

Now the ring $B = \mathbf{Z}_2^S = (2^S, +, \cdot, -, 0, 1)$ will clearly, like $\mathbf{Z}_2$, satisfy

$$\text{(3.14.4)} \qquad (\forall x \in |B|) \quad x^2 \ = \ x,$$

from which one easily deduces the further identities,

$$\text{(3.14.5)} \qquad \begin{array}{ll} (\forall x, y \in |B|) & xy \ = \ yx, \\ (\forall x \in |B|) & x + x \ = \ 0 \quad (\text{equivalently: } 1 + 1 = 0 \text{ in } B). \end{array}$$

An associative ring satisfying (3.14.4) (and so also (3.14.5)) is called a *Boolean ring*. We shall see below (Exercise 3.14:2) that the identities defining a Boolean ring, i.e., the identities of associative rings together with (3.14.4), imply *all* identities satisfied by rings $\mathbf{Z}_2^S$. Hence Boolean rings and Boolean algebras are essentially equivalent – one can turn one into the other using (3.14.2) and (3.14.3).

Exercise 3.14:1. The *free Boolean ring* $F(X)$ on any set X exists by the usual general arguments. Find a normal form for the elements of $F(X)$ when X is finite. To prove that distinct expressions in normal form represent distinct elements, you will need some kind of representation of $F(X)$; use a representation by subsets of a set S.

Exercise 3.14:2. Assume here the result implicit in the last sentence of the preceding exercise, that the free Boolean ring on any finite set X can be embedded in the Boolean ring of subsets of some set S.

(i) Deduce that all identities satisfied by the rings $\mathbf{Z}_2^S$ (S a set) follow from the identities by which we defined Boolean rings.

(ii) Conclude that the free Boolean ring on an *arbitrary* set X can be embedded in the Boolean ring of $\{0,1\}$-valued functions on some set (if you did not already prove this as part of your proof of (i)).

(iii) Deduce that there exists a finite list of identities for Boolean *algebras* which implies all identities holding for such structures (i.e., all identities holding in sets $\mathbf{P}(S)$ under $\cup$, $\cap$, ${}^{\mathrm{c}}$, 0 and 1).

Exercise 3.14:3. An element a of a ring (or semigroup or monoid) is called *idempotent* if $a^2 = a$. If R is a commutative ring, let us define

$$\mathrm{Idpt}(R) \;=\; (\{a\in R \mid a^2 = a\},\ \dot{+},\ \cdot,\ \dot{-},\ 0,\ 1),$$

where $a \dot{+} b = a + b - 2ab$ and $\dot{-}a = a$.

(i) Verify that each of the above operations carries the set $|\mathrm{Idpt}(R)|$ into itself.

(ii) Show that if $a\in|\mathrm{Idpt}(R)|$, then R can (up to isomorphism) be written $R_1 \times R_2$, in such a way that the element a has the form $(0,1)$ in this direct product. Deduce that if $a_1, \dots, a_i \in|\mathrm{Idpt}(R)|$, then R can be written as a finite direct product in such a way that each a_i has each coordinate 0 or 1. This result can be used to get a proof of the next point that is conceptual, rather than purely computational:

(iii) Show that for any commutative ring R, $\mathrm{Idpt}(R)$ is a Boolean ring.

(iv) Given any Boolean ring B, show that there is a universal pair (R, f) where R is a commutative ring, and $f: B \to \mathrm{Idpt}(R)$ a homomorphism.

(v) Investigate the structure of the R of the above construction in some simple cases, e.g., $B = \mathbf{Z}_2$, $B = \mathbf{Z}_2^2$, $B = \mathbf{Z}_2^X$.

(Students familiar with algebraic geometry will recognize that the idempotent elements of a commutative ring R correspond to the continuous $\{0,1\}$-valued functions on $\mathrm{Spec}(R)$, showing that the Boolean rings $\mathrm{Idpt}(R)$ of the above exercise are analogous to Boolean rings of $\{0,1\}$-valued functions on sets.)

Exercise 3.14:4. (i) If $f: U \to V$ is a set map, what sort of homomorphism does it induce between the Boolean rings $\mathbf{Z}_2^U$ and $\mathbf{Z}_2^V$?

(ii) Let B be a Boolean ring. Formulate universal properties for a ''universal representation of B by subsets of a set'', in each of the following senses:

(a) A universal pair (S, f), where S is a set, and f a Boolean ring homomorphism $B \to \mathbf{Z}_2^S$.

(b) A universal pair (T, g), where T is a set, and g a Boolean ring homomorphism $\mathbf{Z}_2^T \to B$.

(iii) Investigate whether such universal representations exist. If such representations are obtained, investigate whether the maps f, g will in general be one-to-one, or onto.

Exercise 3.14:5. (i) Show that every finite Boolean ring is isomorphic to one of the form 2^S for some finite set S.

(ii) For what finite sets S is the Boolean ring 2^S free? What will be the number of free generators?

Exercise 3.14:6. A subset T of a set S is said to be *cofinite* in S if ${}^{c}T$ (taken relative to S, i.e., $S - T$) is finite. Show that $\{T \subseteq \mathbf{Z} \mid T$ is finite or cofinite$\}$ is the underlying set of a Boolean subring of $2^{\mathbf{Z}}$, which is neither free, nor isomorphic to a Boolean ring 2^U for any set U.

Above, I have for purposes of exposition distinguished between the power set $\mathbf{P}(S)$ of a set S and the function-set 2^S. But these notations are often used interchangeably, and I may use them that way myself elsewhere in these notes.

3.15. Sets. The objects we have been studying have been sets with additional operations. Let us briefly note the forms that some of the constructions we have discovered take for plain sets.

Given a family of sets $(S_i)_{i\in I}$, the object with the universal property characterizing products is the usual direct product, $\prod_I S_i$, which may be described as the set of functions on I whose value at each element i belongs to the set S_i. The ith projection map takes each such function to its value at i. Note that the product of the vacuous family of sets (indexed by the empty set!) is a one-element set.

The coproduct of a family $(S_i)_{i\in I}$ is their *disjoint union* $\bigsqcup_I S_i$, to which we referred in passing in §3.6. If the S_i are themselves disjoint, one can take for this set their ordinary union; the inclusions of the S_i in this union give the universal family of maps $q_j: S_j \to \bigsqcup_I S_i$ $(j\in I)$. A construction that will work without any disjointness assumption is to take

$$\bigsqcup_I S_i \;=\; \{(i,s) \mid i\in I,\ s\in S_i\} \tag{3.15.1}$$

with universal maps given by

$$q_i(s) \;=\; (i,s) \qquad (i\in I,\ s\in S_i). \tag{3.15.2}$$

A frequent practice in mathematical writing is to assume (‘‘without loss of generality’’) that a family of sets is disjoint, if this would be notationally convenient, and if there is nothing logically forcing them to have elements in common. In such cases one can, as noted, take the universal maps involved in the definition of a coproduct of sets to be inclusions. But in other cases, for instance if we want to consider a coproduct of a set with itself, or of a set and a subset, a construction like (3.15.1) is needed. Note that when a construction is described ‘‘in general’’ under such a disjointness assumption, and is later applied in a situation where one cannot make that assumption, one must be careful to insert q_i’s where appropriate.

Exercise 3.15:1. Investigate laws such as ‘‘associativity’’, ‘‘distributivity’’, etc.. which are satisfied ‘‘up to natural isomorphism’’ by the constructions of pairwise product and coproduct of sets.

Show that some of these laws are also satisfied by products and coproducts of groups, while others are not.

Sets can also be constructed by ‘‘generators and relations’’. If X is a set, then relations are specified by a set R of ordered pairs of elements of X, which we want to make fall together. The universal image of X under a map making the components of each of these pairs fall together is easily seen to be the quotient of X by the least equivalence relation containing R.

The constructions examined in this section – direct product of sets, disjoint union, and quotient by the equivalence relation generated by a given binary relation – were, of course, already used in earlier sections. So the point of the above observations was not to introduce those constructions, but to show their relation to our general concepts.

3.16. Some structures we have not looked at. ... Lattices, modular lattices, distributive lattices; cylindric algebras; partially ordered sets; heaps, loops; Lie algebras, Jordan algebras, general nonassociative algebras; rings with polynomial identity, rings with involution, fields, division rings, Hopf algebras; modules, bimodules; filtered groups, filtered rings, filtered modules; ordered groups, lattice-ordered groups, ...

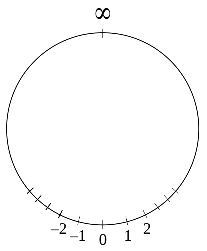
We’ll look at some of these in later chapters.

On the objects we *have* considered here, we have only looked at basic and familiar universal constructions. Once we develop a general theory of universal constructions, we shall see that they come in many more varied forms.

For diversity, I will end this chapter with two examples for those who have had a little topology.

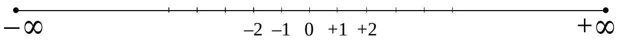
3.17. The Stone-Čech compactification of a topological space. As is well known, the real line $\mathbf{R}$ is not compact. It is frequently convenient, when studying the limit-behavior of $\mathbf{R}$-valued functions or sequences, to adjoin to $\mathbf{R}$ an additional point, ‘‘∞’’. The resulting compact space, $\mathbf{R}\cup\{\infty\}$, is shown below.

$\mathbf{R}\cup\{\infty\}$:



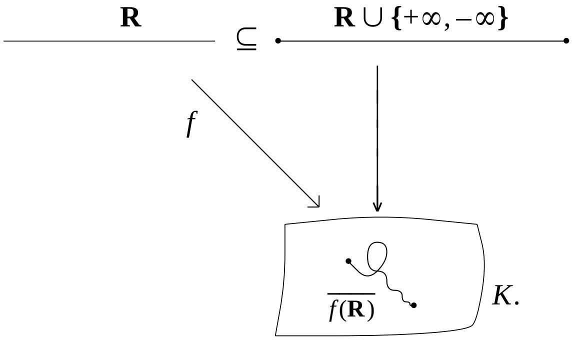
At other times, one adjoins to $\mathbf{R}$ two points, $+\infty$ and $-\infty$, getting a space

$\mathbf{R}\cup\{+\infty,-\infty\}$:

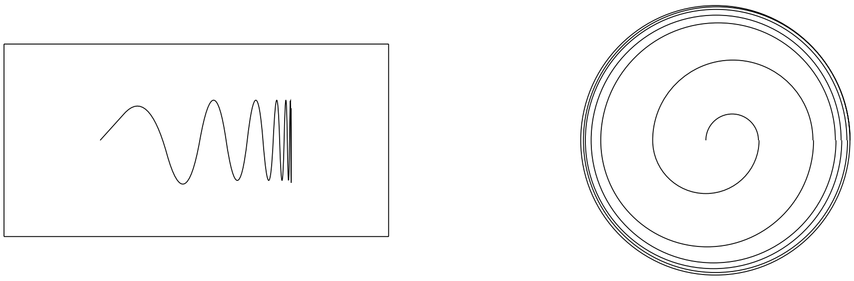


Note that $\mathbf{R}\cup\{\infty\}$ may be obtained from $\mathbf{R}\cup\{+\infty,-\infty\}$ by an *identification*. Hence $\mathbf{R}\cup\{+\infty,-\infty\}$ can be thought of as making ‘‘finer distinctions’’ in limiting behavior than $\mathbf{R}\cup\{\infty\}$.

One might imagine that $\mathbf{R}\cup\{+\infty,-\infty\}$ makes ‘‘the finest possible distinctions’’. A precise formulation of this would be a statement that for any continuous map f of $\mathbf{R}$ into a compact Hausdorff space K, the closure of the image of $\mathbf{R}$ should be an image of $\mathbf{R}\cup\{+\infty,-\infty\}$; i.e., that the map f should factor through the inclusion $\mathbf{R}\subseteq\mathbf{R}\cup\{+\infty,-\infty\}$. Here is a picture of an example where this is true:



But from the following pictures we can see that this will not hold in general:



We can nevertheless ask whether there is *some* compactification of $\mathbf{R}$ which makes ‘‘the most possible distinctions''. Let us raise the same question with $\mathbf{R}$ replaced by a general topological space X, and give the desired compactification a name.

Definition 3.17.1. *Let* X *be a topological space. A* Stone-Čech compactification *of* X *will mean a pair* (C, u), *where* C *is a compact Hausdorff space and* u *a continuous map* $X \to C$, *universal among all continuous maps of* X *into compact Hausdorff spaces* K (*diagram at right*).

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "\forall v"'] & C \arrow[d, "\exists 1\, g"] \\ & K \end{tikzcd}$$

Exercise 3.17:1. Show that if a pair (C, u) as in the above definition exists, then $u(X)$ is dense in C. In fact, show that if (C, u) has the stated universal property but without the condition of uniqueness of factoring maps g (as in the above diagram), then

(i) uniqueness of such maps holds if and only if $u(X)$ is dense in C; and

(ii) if C' is the closure of $u(X)$ in C, the pair (C', u) has the full universal property.

We want to know whether such compactifications always exist.

The analog of our construction of free groups from terms as in §2.2 would be to adjoin to X some kinds of ''formal limit points''. But limit points of what? Not every sequence in a compact Hausdorff space K converges, nor need every point of the closure of a subset $J \subseteq K$ be the limit of a sequence of points of J (unless K is first countable); so adjoining limits of *sequences* would not do. The approach of adjoining limit points can in fact be made to work, but it requires considerable study of how such points may be described; the end result is a construction of the Stone-Čech compactification of X in terms of *ultrafilters*. We shall not pursue that approach here; it is used in [**100**, Theorem 17.17 et seq.]. (NB: The compactification constructed there may not be Hausdorff when X is ''bad'', so in such cases it will not satisfy our definition.)

The ''big direct product'' approach is more easily adapted. If $v_1 : X \to K_1$ and $v_2 : X \to K_2$ are two continuous maps of X into compact Hausdorff spaces, then the induced map $(v_1, v_2) : X \to K_1 \times K_2$ will ''make all the distinctions among limit points made by either v_1 or v_2'', since the maps v_1 and v_2 can each be factored through it; further, if we let K' denote the *closure* of the image of X in $K_1 \times K_2$, and $v' : X \to K'$ the induced map, then all these distinctions are still made in K', and the image of X is dense in this space. We can do the same with an arbitrary family of maps $v_i : X \to K_i$ $(i \in I)$, since Tychonoff's Theorem tells us that the product space $\prod_I K_i$ is again compact.

As in the construction of free groups, we now have to find some *set* of pairs (K_i, v_i) which are ''as good as'' the class of *all* maps of X into *all* compact Hausdorff spaces K. For this purpose, we want a bound on the cardinalities of the *closures* of all images of X under maps into compact Hausdorff spaces K. To get this, we would like to say that every point of the closure of the image of X ''depends'' in some way on images of elements of X, in such a fashion that different points ''depend'' on these differently; and then bound the number of kinds of ''dependence'' there can be in terms of the cardinality of X. The next lemma establishes the ''different points depend on X in different ways'' idea, and the corollary that follows gives the desired bound.

Lemma 3.17.2. *Let* K *be a Hausdorff topological space, and for any* $k \in |K|$, *let* $N(k)$ *denote the set of all open neighborhoods of* k (*open sets in* K *containing* k). *Then for any map* v *from a set* X *into* K, *and any two points* $k_1 \neq k_2$ *of the closure of* $v(X)$ *in* K, *one has*

$v^{-1}(N(k_1)) \neq v^{-1}(N(k_2))$ *(where by* $v^{-1}(N(k))$ *we mean* $\{v^{-1}(U) \mid U \in N(k)\}$, *a subset of* $\mathbf{P}(X)$).

Proof. Since k_i $(i = 1, 2)$ is in the closure of $v(X)$, every neighborhood of k_i in K has nonempty intersection with $v(X)$, i.e., every member of $v^{-1}(N(k_i))$ is nonempty. Since $N(k_i)$ is closed under pairwise *intersections*, so is $v^{-1}(N(k_i))$. But since K is Hausdorff and $k_1 \neq k_2$, these two points possess disjoint neighborhoods, whose inverse images in X will have empty intersection. If the sets $v^{-1}(N(k_1))$ and $v^{-1}(N(k_2))$ were the same, this would give a contradiction. □

Thus, we can associate to distinct points of the closure of $v(X)$ distinct sets of subsets of X. Hence,

Corollary 3.17.3. *In the situation of the above lemma, the cardinality of the closure of* $v(X)$ *in* K *is* $\leq 2^{2^{\mathrm{card}\,X}}$. □

So now, given any topological space X, let us choose a set S of cardinality $2^{2^{\mathrm{card}\,|X|}}$, and let A denote the set of all pairs $a = (K_a, u_a)$ such that K_a is a compact Hausdorff topological space with underlying set $|K_a| \subseteq S$, and u_a is a continuous map $X \to K_a$. (We no longer need to keep track of cardinalities, but if we want to, $\mathrm{card}\,A \leq 2^{2^{2^{2^{\mathrm{card}\,|X|}}}}$, assuming X infinite. The two additional exponentials come in when we estimate the number of topologies on a set of $\leq 2^{2^{\mathrm{card}\,|X|}}$ elements.) Thus, if v is any continuous map of X into a compact Hausdorff space K, and we write K' for the closure of $v(X)$ in K, then the pair (K', v) will be ''isomorphic'' to some pair $(K_a, u_a) \in A$, in the sense that there exists a homeomorphism making the diagram below commute.

$$\begin{tikzcd} & K' \subseteq K \arrow[dd, "\wr"', "\text{homeomorphism}"] \\ X \arrow[ur, "v"] \arrow[dr, "u_a"'] & \\ & K_a \end{tikzcd}$$

We now form the compact Hausdorff space $P = \prod_{a \in A} K_a$, and the map $u: X \to P$ induced by the u_a's, and let $C \subseteq P$ be the closure of $u(X)$. It is easy to show, as we did for groups in §2.3 that the pair (C, u) satisfies the universal property 3.17.1. Thus:

Theorem 3.17.4. *Every topological space* X *has a Stone-Čech compactification* (C, u) *in the sense of Definition 3.17.1.* □

Exercise 3.17:2. Show that in the above construction, $u(X)$ will be homeomorphic to X under u if and only if X can be *embedded* in a compact Hausdorff space (where an ''embedding'' means a continuous map f inducing a homeomorphism between X and $f(X)$, the latter set being given the subspace topology). Examine conditions on X under which these equivalent statements will hold. Show that for any topological space X, there exists a universal map into a space Y embeddable in a compact Hausdorff space; that this map is always onto, but that it may not be one-to-one. Can it be one-to-one and onto but not a homeomorphism?

Note: Most authors use the term ''compactification'' to mean a dense *embedding* in a compact space. Hence, they only consider a space X to have a Stone-Čech compactification if the map u that we have constructed *is* an embedding.

Exercise 3.17:3. Suppose we leave off the condition ''Hausdorff'' – does a space X always have a universal map into a *compact* space C? A compact T_1 space C? ...

Exercise 3.17:4. Let C be the Stone-Čech compactification of the real line $\mathbf{R}$, and regard $\mathbf{R}$ as a subspace of C.

(i) Show that $C - \mathbf{R}$ has exactly two connected components.

(The above shows that there was a grain of truth in the naive idea that $\mathbf{R} \cup \{+\infty, -\infty\}$ was the universal compactification of $\mathbf{R}$. Exercise 3.17:5 will also be relevant to this idea.)

(ii) What can you say about path-connected components of $C - \mathbf{R}$?

(iii) Show that no *sequence* in $\mathbf{R}$ converges to a point of $C - \mathbf{R}$.

A continuous map of $\mathbf{R}$ into a topological space K may be thought of as an open *curve* in K. If K is a *metric space* one can define the *length* (possibly infinite) of this curve.

Exercise 3.17:5. Show that if $v: \mathbf{R} \to K$ is a curve of *finite length* in a compact (or more generally, a complete) metric space K, then v factors through the inclusion of $\mathbf{R}$ in $\mathbf{R} \cup \{+\infty, -\infty\}$.

Is the converse true? I.e., must every map $\mathbf{R} \to K$ which factors through the inclusion of $\mathbf{R}$ in $\mathbf{R} \cup \{+\infty, -\infty\}$ have finite length?

Exercise 3.17:6. (Exploring possible variants of Exercises 3.17:4-3.17:5.) It would be nice to get a result like the first assertion of the above exercise with a purely topological hypothesis on the map v, rather than a condition involving a metric on K. Consider, for instance, the following condition on a map v of the real line into a compact Hausdorff space K:

(3.17.5) For every closed set $V \subseteq K$, and open set $U \supseteq V$, the set $v^{-1}(U) \subseteq \mathbf{R}$ has only finitely many connected components that contain points of $v^{-1}(V)$. (Suggestion: draw a picture.)

(i) Can we replace the assumptions in Exercise 3.17:5 that K is a metric space and v has finite length by (3.17.5) or some similar condition?

(ii) Let X be the open unit disc, C the closed unit disc, and $u: X \to C$ the inclusion map. Does the pair (C, u) have any universal property with respect to X, like that indicated for $\mathbf{R} \cup \{+\infty, -\infty\}$ with respect to $\mathbf{R}$ in the preceding exercise?

(iii) Does the open disc have a universal path-connected compactification?

(iv) In general, if C is the Stone-Čech compactification of a ''nice'' space X, what can be said about connected components, path components, homotopy, cohomotopy, etc. of $C - X$?

One can also consider universal constructions for objects which mix topological and algebraic structure:

Exercise 3.17:7. Let G be any Hausdorff topological *group* (a group given with a Hausdorff topology on its underlying set, such that the group operations are continuous). Show that there exists a universal pair (C, h), where C is a *compact* Hausdorff topological group, and $h: G \to C$ a continuous group homomorphism. This is called the *Bohr compactification* of G. Show that $h(G)$ is dense in C. Is h generally one-to-one? A topological embedding? What will be the relation between C and the Stone-Čech compactification of the underlying topological space of G?

If it helps, you might consider some of these questions in the particular case where G is the additive group of the real line.

In §2.4 we saw that we could improve on the construction of the free group on X from ''terms'' by noting that a certain subset of the terms would make do for all of them. For the Stone-Čech compactification, the ''big direct product'' construction is subject to a similar simplification. In that construction, we made use of all maps (up to homeomorphism) of X into compact spaces of reasonable size. I claim that we can in fact make all the ''distinctions'' we need using maps into the closed unit interval, $[0,1]$! The key fact is that any two points of a compact Hausdorff space K can be separated by a continuous map into $[0,1]$ (Urysohn's Lemma). I will sketch how this is used.

Let X be any topological space, let W denote the set of continuous maps $w\colon X\to[0,1]$, let $u\colon X\to[0,1]^W$ be the map induced by $(w)_{w\in W}$, and let $C\subseteq[0,1]^W$ be the closure of $u(X)$. It is immediate that C has the property

(3.17.6) Every continuous function of X into $[0,1]$ factors uniquely through a function $C\to[0,1]$ (namely, one of the projections of $[0,1]^W$).

To show that C has the universal property of the Stone-Čech compactification of X, let K be a compact Hausdorff space. We can separate points of K by some set S of continuous maps $s\colon K\to[0,1]$, hence we can embed K in a ''cube'' $[0,1]^S$. (The map $K\to[0,1]^S$ given by our separating family of functions is one-to-one; hence it will be a topological embedding by the properties of compact Hausdorff spaces.) Let us therefore assume, without loss of generality, that K is a closed subspace of $[0,1]^S$. Now given any map $v\colon X\to K$, we regard it as a map into the space $[0,1]^S$ containing K, and get a factorization $v=gu$ for a unique map $g\colon C\to[0,1]^S$ by applying (3.17.6) to each coordinate. Because K is compact, it is closed in $[0,1]^S$, so g will take C, the closure of $u(X)$, into K, establishing the required universal property. Cf. [**70**, pp. 152-153].

Another twist: Following the idea of Exercise 2.3:6, we may regard a point c of the Stone-Čech compactification of a space X as determining a function $\tilde{c}$ which associates to every continuous map v of X into a compact Hausdorff space K a point $\tilde{c}(v)\in K$ – namely, the image of c under the unique extension of v to C. This map $\tilde{c}$ will be ''functorial'', i.e., will respect continuous maps $f\colon K_1\to K_2$, in the sense indicated in the diagram below.

$$\begin{tikzcd}
 & K_1 \ni \tilde{c}(v) \arrow[d, "f"] \\
X \arrow[ur, "v"] \arrow[dr, "fv"'] & \\
 & K_2 \ni \tilde{c}(fv)
\end{tikzcd}$$

By Urysohn's Lemma, $\tilde{c}$ will be determined by its behavior on maps $w\colon X\to[0,1]$, hence, more generally, by its behavior on maps w of X into closed intervals $[a,b]\subseteq\mathbf{R}$. We carry this observation further in

Exercise 3.17:8. (i) Show that although $\mathbf{R}$ is not compact, one may obtain from $\tilde{c}$ (by the ''functoriality'' property) a well-defined map from the set $B(X)$ of all *bounded* real-valued continuous functions on X to the real numbers $\mathbf{R}$.

(ii) Show that this map is a ring homomorphism (under the obvious ring structure on $B(X)$).

One can show, further, that every ring homomorphism $B(X)\to\mathbf{R}$ is continuous, and deduce that each such homomorphism is induced by a point of C. So one gets another description of the Stone-Čech compactification C of X, as the space of homomorphisms into $\mathbf{R}$ of the ring $B(X)$ of bounded continuous real-valued functions on X. The topology of C is the function topology on maps of $B(X)$ into $\mathbf{R}$.

Perhaps I have made this approach sound too esoteric. A simpler way of putting it is to note that every bounded continuous real function on X (i.e., every continuous function which has range in a compact subset of $\mathbf{R}$) extends to a bounded continuous real function on its Stone-Čech compactification C, so $B(X)\cong B(C)$; and then to recall that for any compact Hausdorff space

$C,$ the homomorphisms from the function-ring $B(C)$ into $\mathbf{R}$ are just the evaluation functions at points of $C.$

One can use this approach to get another proof of the existence of the Stone-Čech compactification of a topological space [**58**, Chapter 6]. This homomorphism space can also be identified with the space of all *maximal ideals* of $B(X),$ equivalently, of all *prime ideals* that are closed in the topology given by the sup norm.

Exercise 3.17:9. Suppose B' is any $\mathbf{R}$-*subalgebra* of $B(X).$ Let C' denote the set of all maximal ideals of $B'.$ Show that there is a natural map $m\colon C\to C'.$ Show by examples that this map can fail to be one-to-one (even if B' separates points of X), or to be onto. Try to find conditions for it to be one or the other.

In [**77**, §41], the Bohr compactification of a topological group G (Exercise 3.17:7 above) is obtained as the maximal ideal space of a subring of $B(G),$ the subring of ''almost periodic'' functions.

Most often, complex- rather than real-valued functions are used in these constructions.

3.18. Universal covering spaces. Let X be a pathwise connected topological space with a basepoint (distinguished point) $x_0.$

A *covering space* of X means a pair $(Y, c),$ where Y is a pathwise connected space with a basepoint $y_0,$ and c is a continuous basepoint-preserving map $Y\to X,$ such that every $x\in X$ has a neighborhood V for which $c^{-1}(V)$ is a disjoint union of open subspaces, each mapped homeomorphically onto V by $c.$ (Draw a picture!) Such a c will have the *unique path-lifting property*: Given any continuous map $p\colon [0,1]\to X$ taking 0 to $x_0,$ there will exist a unique continuous map $\tilde{p}\colon [0,1]\to Y$ taking 0 to y_0 such that $p = c\tilde{p};$ and further, $\tilde{p}$ depends continuously on p in the appropriate function-space topology.

Given $X,$ consider any covering space (Y, c) of $X,$ and let us ask what points of Y we can ''describe'' in a well-defined manner.

Of course, we have the basepoint, $y_0.$ Further, for any path p in X starting at the basepoint $x_0,$ we know there will be a unique lifting of p to a path $\tilde{p}$ in Y starting from $y_0;$ so Y also has all points of this lifted path. It is enough, however, to note that we have the endpoint $\tilde{p}(1),$ since all the other points of $\tilde{p}$ can be described as endpoints of liftings of ''subpaths'' of $p.$ In fact, every $y\in Y$ will be the endpoint $\tilde{p}(1)$ of a lifted path in $X.$ For Y was assumed pathwise connected, hence for any $y\in Y$ we can find a path q in Y with $q(0) = y_0,$ $q(1) = y.$ Letting $p = cq,$ a path in $X,$ we see that $q = \tilde{p},$ so $y = \tilde{p}(1).$

Suppose p and p' are two paths in $X;$ when will $\tilde{p}(1)$ and $\tilde{p}'(1)$ be the same point of Y? Clearly, a necessary condition is that these two points have the same image in X: $p(1) = p'(1) = x.$ Assuming this condition, note that if p and p' are homotopic in the class of paths in X from x_0 to this point $x,$ then as one smoothly deforms p to p' in this class, the lifted path in Y will vary continuously, hence its endpoint in $c^{-1}(x)$ will vary continuously. But $c^{-1}(x)$ is discrete, so the endpoint must remain constant. Thus, p's being homotopic to p' in the class of paths with these fixed endpoints implies $\tilde{p}(1) = \tilde{p}'(1).$

So in general, we get a point of Y for every homotopy class $[p]$ of paths in X with initial point x_0 and common final point. In a particular covering space $Y,$ there may or may not be further equalities among these points; but we can ask whether, if we write U for the set of such homotopy classes of paths, and u for the map from U to X defined by $u([p]) = p(1),$ we can make U a topological space in such a way that the pair (U, u) is a covering space for $X.$ Under appropriate assumptions on the topology of X (the hypotheses used in [**64**] are: connected, locally pathwise connected, and semi-locally simply connected), this can indeed be done. The resulting covering space U has a unique continuous map onto each covering space Y of X, which respects basepoints and the maps into X. Hence (U, u) is called the *universal covering space* of X.

The universal covering space is a versatile animal – like the direct product of groups, it has, in addition to the above left universal property, a right universal one:

It is not hard to show that U is simply connected. Consider, now, pairs (S, c) where S is a *simply connected* pathwise connected topological space with basepoint, and $c: S \to X$ a basepoint-respecting continuous map. Let us ask, for such a space S, the question that we noted in §3.8 as leading to *right universal* constructions: If s is an arbitrary point of S, what data will it determine that can be formulated in terms of the given space X? Well, obviously s determines the point $c(s) \in X$. To get more information, note that since S is pathwise connected, there will be some path q in S connecting s_0 to s; and since S is *simply* connected, all such paths q are homotopic. Applying c to these paths, we see that s determines a *homotopy class* of paths in X from x_0 to $c(s)$. But as we have just noted, the set of homotopy classes of paths from x_0 to points of X can (under appropriate conditions) itself be made into a simply connected space, the universal covering space of X. One deduces that this space U is right universal among simply connected spaces with basepoint, given with maps into X (diagram below).

$$\begin{tikzcd} U \arrow[r, "u"] & X \\ S \arrow[u, "\exists 1\, d"] \arrow[ur, "\forall c"'] & \end{tikzcd}$$

Exercise 3.18:1. Show that the universal covering space of the pathwise connected space X (if it exists) is, more generally, right universal among pathwise connected spaces S with basepoint, given with basepoint-preserving maps c into X such that the group homomorphism $\pi_1(c)\colon \pi_1(S) \to \pi_1(X)$ is trivial. Give an example showing that such a space S need not be simply connected.

We could also look for a *right* universal covering space for X, or a simply connected space with basepoint having a *left* universal map into X. But these turn out to be uninteresting: They are X itself, and the one-point space.

There are many other occurrences of universal constructions in topology. Some, like the constructions considered in this and the preceding section, can be approached in the same way as universal constructions in algebra. Others used in algebraic topology are rather different, in that one is interested, not in maps being equal, unique, etc., but *homotopic*, unique *up to homotopy*, etc.. We shall see that these conditions can be brought into the same framework as our other universal properties via the formalism of *category theory*. But the tasks of constructing and studying the objects these conditions characterize require different approaches, which we will not treat in this course.

Part II. Basic tools and concepts.

In the next five chapters we shall assemble the tools needed for the development of a general theory of algebras and of universal constructions among them.

We begin with two chapters on ordered sets, lattices, closure operations and related concepts, since these will be used repeatedly. Because of the relation between well-ordering and the Axiom of Choice, we also take this occasion to review briefly the Zermelo-Fraenkel axioms for set theory, and several statements equivalent to the Axiom of Choice.

Clearly, the general context for studying universal constructions should be some model of ‘‘a system of mathematical objects and the maps among them’’. This is provided by the concept of a *category*. We develop the basic concepts of category theory in Chapter 6, and in Chapter 7 we formalize universal properties in category-theoretic terms.

Finally, in Chapter 8 we introduce the categories that will be of special interest to us: the *varieties of algebras*.

Chapter 4. Ordered sets, induction, and the Axiom of Choice.

4.1. Partially ordered sets. We began Chapter 1 by making precise the concept of a group. Let us now do the same for that of *partially ordered set*.

A partial ordering on a set is an instance of a ‘‘relation’’. This is a different sense of the word from that of the last two chapters. In these notes, we will be dealing with both kinds of ‘‘relations’’ extensively; which we mean will generally be clear from context. When there is danger of ambiguity, I will make the distinction explicit, as I do, for instance, in the index.

Intuitively, a relation on a family of sets $X_1, \ldots, X_n$ means a *condition* on n-tuples $(x_1, \ldots, x_n)$ $(x_1 \in X_1, \ldots, x_n \in X_n)$. Since the information contained in the relation is determined by the set of n-tuples that satisfy it, this set is taken to *be* the relation in the formal definition, given below. That the relation is *viewed* as a ‘‘condition’’ comes out in the notation and language used.

Definition 4.1.1. *If* $X_1, \ldots, X_n$ *are sets, a* relation *on* $X_1, \ldots, X_n$ *means a subset* $R \subseteq X_1 \times \ldots \times X_n$. *Relations are often written as predicates; i.e., the condition* $(x_1, \ldots, x_n) \in R$ *may be written* $R(x_1, \ldots, x_n)$, *or* $Rx_1 \ldots x_n$, *or, if* $n = 2$, *as* $x_1 R x_2$.

A relation on $X, \ldots, X$, *i.e., a subset* $R \subseteq X^n$, *is called an n-*ary relation on X.

If R *is an n-ary relation on* X, *and* Y *is a subset of* X, *then the* restriction *of* R *to* Y *means* $R \cap Y^n$, *regarded as an n-ary relation on* Y.

We now recall

Definition 4.1.2. *A* partial ordering *on a set* X *means a binary relation* ‘‘$\leq$’’ *on* X *satisfying the conditions*

$$(\forall\, x \in X) \quad x \leq x \qquad \text{(reflexivity)},$$

$$(\forall\, x, y \in X) \quad x \leq y,\ y \leq x \ \Rightarrow\ x = y \qquad \text{(antisymmetry)},$$

$$(\forall\, x, y, z \in X) \quad x \leq y,\ y \leq z \ \Rightarrow\ x \leq z \qquad \text{(transitivity)}.$$

A total ordering *on* X *means a partial ordering which also satisfies*

$$(\forall\, x, y \in X) \quad x \leq y \ \text{ or } \ y \leq x.$$

A partially (*respectively* totally) ordered set *means a set* X *given with a partial* (*total*) *ordering* $\leq$.

If X *is partially ordered by* $\leq$, *and* Y *is a subset of* X, *then* Y *will be understood to be partially ordered by the restriction of* $\leq$, *which will be denoted by the same symbol unless there is danger of ambiguity. This is called the* induced ordering *on* Y.

A *total ordering* is also called a *linear ordering*. The term *ordered* without any qualifier is used by some authors as shorthand for ‘‘partially ordered’’, and by others for the stronger condition ‘‘totally ordered’’; we will generally specify ‘‘partially’’ or ‘‘totally’’. A subset C of a partially ordered set P which is totally ordered under the induced ordering is called a *chain* in P.

A more formal definition would make a partially ordered set a pair $P = (|P|, \leq)$ where $\leq$ is a partial ordering on $|P|$. But for us, partially ordered sets will in general be tools rather than the objects of our study, and it would slow us down to always maintain the distinction between P and

$|P|$, so we shall usually take the informal approach of understanding a partially ordered set to mean a set P for which we ''have in mind'' a partial ordering relation $\leq$. At times, however, we shall be more precise and refer to the pair $(|P|, \leq)$.

Standard examples of partially ordered sets are the set of real numbers with the usual relation $\leq$, the set $\mathbf{P}(X)$ of subsets of any set X under the partial ordering $\subseteq$, and the set of positive integers under the relation ''$|$'', where $m\,|\,n$ means ''m divides n''.

We remark that in addition to the order-theoretic meaning of ''chain'' noted above, there is a nonspecialized use of the word; for instance, one speaks of a ''chain of equalities $x_1 = x_2 = ... = x_n$''. We shall at times use the term in this nontechnical way, relying on context to avoid ambiguity.

The versions of the concepts of homomorphism and isomorphism appropriate to partially ordered sets are given in

Definition 4.1.3. *If X and Y are partially ordered sets, an* isotone *map from X to Y means a function $f: X \to Y$ such that $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$.*

An invertible isotone map whose inverse is also isotone is called an order isomorphism.

Exercise 4.1:1. Give an example of an isotone map of partially ordered sets which is invertible as a set map, but which is not an order isomorphism.

Some obvious notation: When $\leq$ is a partial ordering on a set X, one commonly writes $\geq$ for the opposite relation; i.e., $x \geq y \Leftrightarrow y \leq x$. Clearly the relation $\geq$ satisfies the same conditions of reflexivity, antisymmetry and transitivity as $\leq$.

This leads to a semantic problem: As long as $\geq$ is just an auxiliary notation used in connection with the given ordering $\leq$, one thinks of an element x as being ''smaller'' (or ''lower'') than an element $y \neq x$ if $x \leq y$. But the preceding observation shows that one can take the opposite relation $\geq$ as a new partial ordering on the set X, i.e., consider the partially ordered set $(X, \geq)$, and one should consider x as ''smaller'' than y in this partially ordered set if the pair (x, y) belongs to this *new* ordering. Such properties as which maps $X \to Y$ are isotone (with respect to a fixed partial ordering on Y) clearly change when one goes from considering X under $\leq$ to considering it under $\geq$.

The set X under the opposite of the given partial ordering is called the *opposite* of the original partially ordered set. When one uses the formal notation $P = (|P|, \leq)$ for a partially ordered set, one can write $P^{\mathrm{op}} = (|P|, \geq)$. One may also replace the symbol $\geq$ by $\leq^{\mathrm{op}}$, writing $P^{\mathrm{op}} = (|P|, \leq^{\mathrm{op}})$. Thus, if x is smaller than y in P, i.e., $x \leq y$, then y is smaller than x in P^{op}, i.e., $y \leq^{\mathrm{op}} x$. (''Dual ordering'' is another term often used, and $*$ is sometimes used instead of $^{\mathrm{op}}$.)

In these notes we shall only rarely make use of the opposite partially ordered set construction. But we remark that (once one has gotten past the notational confusion) the symmetry in the theory of partially ordered sets created by that construction is a very convenient tool.

One also commonly uses $x < y$ as an abbreviation for $(x \leq y) \wedge (x \neq y)$, and of course $x > y$ for $(x \geq y) \wedge (x \neq y)$. These relations do *not* satisfy the same conditions as $\leq$. The conditions that they do satisfy are noted in

Exercise 4.1:2. Show that if $\leq$ is a partial ordering on a set X, then the relation $<$ is transitive and *antireflexive*, i.e., satisfies $(\forall\, x \in X)\ x \not< x$. Conversely, show that any transitive antireflexive binary relation $<$ on a set X is induced in the above way by a unique partial ordering $\leq$.

A relation $<$ with these properties (transitivity and antireflexivity) might be called a ‘‘strict partial order’’. One can thus refer to ‘‘the strict partial order $<$ corresponding to the partial order $\leq$’’, and ‘‘the partial order $\leq$ corresponding to the strict partial order $<$’’. Of course, for a partial ordering denoted by a symbol such as ‘‘ $|$ ’’ (‘‘divides’’), or R (a partial ordering written as a binary relation), there is no straightforward symbol for the corresponding strict partial order.

Exercise 4.1:3. For partially ordered sets X and Y, suppose we call a function $f: X \to Y$ a *strict isotone map* if $x < y \Rightarrow f(x) < f(y)$. Show that

$$\text{one-to-one and isotone} \ \Rightarrow \ \text{strict isotone} \ \Rightarrow \ \text{isotone},$$

but that neither implication is reversible.

In contexts where ‘‘$\leq$’’ already has a meaning, if another partial ordering has to be considered, it is often denoted by a variant symbol such as $\preccurlyeq$. One then uses corresponding symbols $\succcurlyeq$, $\prec$, $\succ$ for the opposite order, the strict order relation, etc.. (However, order-theorists dealing with a partial ordering $\leq$ sometimes write $y \succ x$ to mean ‘‘y covers x’’, that is ‘‘$y > x$ and there is no z between y and x’’. When the symbol is used this way, it cannot be used for the strict relation associated with a second ordering. We shall not use the concept of ‘‘covering’’ in these notes.)

A somewhat confused case is that of symbols for the *subset* relation. Most often, the notation suggested by the above discussion is followed; that is, $\subseteq$ is used for ‘‘is a subset of’’, $\supseteq$ for the opposite relation, and $\subset$, $\supset$ for strict inclusions; and we follow these conventions here. However, many authors, especially in Eastern Europe, write $\subset$ for ‘‘is a subset of’’, a usage based on the view that since this is a more fundamental concept than that of a proper subset, it should be denoted by a primitive symbol and not one obtained by adding an extra mark to the symbol for ‘‘proper subset’’. Those authors use $\subsetneqq$ for ‘‘proper subset’’ (and the reversed symbols for the reversed relations). There was even at one time a movement to make ‘‘$<$’’ mean ‘‘less than or equal to’’, with $\lneqq$ for strict inequality. Together with the above set-theoretic usage, this would have formed a consistent system, but the idea never got off the ground. Finally, many authors, for safety, use a mixed system: $\subseteq$ for ‘‘subset’’ and $\subsetneqq$ for ‘‘proper subset’’. (That was the notation used in the first graduate course I took, and I sometimes follow it in my papers. However, I only rarely need a symbol for explicit strict inclusion, so the question of how to write it seldom comes up.)

Although partially ordered sets are not algebras in the sense in which we shall use the term, many of the kinds of universal constructions we have considered for algebras can be carried out for them. In particular

Definition 4.1.4. *Let $(X_i)_{i\in I}$ be a family of partially ordered sets. Then their* direct product *will mean the partially ordered set having for underlying set the direct product of the underlying sets of the X_i, ordered so that $(x_i)_{i\in I} \leq (y_i)_{i\in I}$ if and only if $x_i \leq y_i$ for all $i\in I$.*

Exercise 4.1:4. (i) Verify that the above relation is indeed a partial ordering on the product set, and that the resulting partially ordered set has the appropriate universal property to be called the direct product of the partially ordered sets X_i.

(ii) Let X be a set and R a binary relation on X. Show that there exists a universal example of a partially ordered set $(Y, \leq)$ with a map $u: X \to Y$ such that for all $(x_1, x_2) \in R$ one has $u(x_1) \leq u(x_2)$ in Y. This may be called the partially ordered set *presented* by the generators X and the relation-set R. (Cf. presentations of groups, monoids, and rings, §§3.3, 3.10, 3.12.) Will the map u in general be one-to-one? Onto?

(iii) Determine whether there exist constructions with the universal properties of the *coproduct* of two partially ordered sets, and of the *free* partially ordered set on a set X. Describe these if they exist.

(iv) Discuss the problem of *imposing* a set R of further relations on a given partially ordered set $(X, \leq)$; i.e., of constructing a universal isotone map of X into a partially ordered set Y such that the images of the elements of X satisfy the relations comprising R, and examine the properties of this construction, if it can be carried out.

We have noted that for any set X, the set $\mathbf{P}(X)$ of subsets of X is partially ordered by $\subseteq$. Given a partially ordered set S, we may look for universal ways of representing S by subsets of a set X. Note that if $f: X \to Y$ is map between sets, then f induces, in natural ways, both an isotone map $\mathbf{P}(X) \to \mathbf{P}(Y)$ and an isotone map $\mathbf{P}(Y) \to \mathbf{P}(X)$, the first taking subsets of X to their images under f, the second taking subsets of Y to their inverse images. Let us call these the ''direction-preserving construction'' and the ''direction-reversing construction'' respectively. Thus, given a partially ordered set S, there are four universal sets we might look for: a set X having an isotone map $S \to \mathbf{P}(X)$ universal in terms of the direction-preserving construction of maps among power sets, a set X with such a map universal in terms of the direction-reversing construction, and sets X with isotone maps in the reverse direction, $\mathbf{P}(X) \to S$, universal for the same two constructions of maps among power sets.

Exercise 4.1:5. (i) Write out the universal properties of the four possible constructions indicated.

(ii) Investigate which of the four universal sets exist, and describe these as far as possible.

Definition 4.1.5. *Let X be a partially ordered set, S a subset of X, and s an element of S. Then s is said to be* minimal *in S if there is no $t \in S$ with $t < s$, while s is said to be* the least *element of S if for all $t \in S$, $s \leq t$. The terms* maximal *and* greatest *are used for the dual concepts.*

There was really no need to refer to X in the above definition, since the properties in question just depended on the set S and the induced order relation on it; but these concepts are often applied to subsets of larger partially ordered sets, so I included this context in the statement.

Part (iii) of the next exercise is a caution against an error that I have too often caught myself making.

Exercise 4.1:6. Let X be a partially ordered set.

(i) Show that if X has a least element x, then x is the unique minimal element of X.

(ii) If X is finite, show conversely that a unique minimal element, if it exists, is a least element.

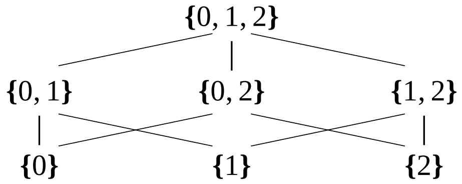
(iii) Give an example showing that if X is not assumed finite, this converse is false.

Exercise 4.1:7. Let $(X, \leq)$ be a partially ordered set. Then the pair $(X, \leq)$ constitutes a presentation of itself as a partially ordered set in the sense of Exercise 4.1:4(i); but of course, there may be proper subsets R of the relation $\leq$ such that (X, R) is a presentation of the same partially ordered set. (I.e., such that R ''generates'' $\leq$ in an appropriate sense.)

(i) If X is finite, show that there exists a *least* subset of R which generates $\leq$.

(ii) Show that this is not in general true for infinite X.

Point (i) of the above exercise is the basis for the familiar way of diagramming finite partially ordered sets. One draws a picture with vertices representing the elements of the set, and edges corresponding to the members of the least relation generating the partial ordering; i.e., the smallest set of order relations from which all the others can be deduced. The higher point on each edge represents the larger element under the partial ordering. For example, the picture below represents the partially ordered set of all nonempty subsets of $\{0,1,2\}$. The relation $\{1\} \leq \{0,1,2\}$ is not shown explicitly, because is a consequence of the relations $\{1\} \leq \{0,1\} \leq \{0,1,2\}$ (and also of $\{1\} \leq \{1,2\} \leq \{0,1,2\}$).



The next definition lists a few more pieces of terminology commonly used in connection with partial orderings.

Definition 4.1.6. *Let* $\leq$ *be a partial ordering on a set* X.

If x, y *are elements of* X *with* $x \leq y$, *then the* interval $[x,y]$ *means the subset* $\{z\in X \mid x \leq z \leq y\}$, *with the induced partial ordering* $\leq$.

Elements x *and* y *of* X *are called* incomparable *if neither* $x \leq y$ *nor* $y \leq x$ *holds. A subset* $Y \subseteq X$ *is called an* antichain *if every pair of distinct elements of* Y *is incomparable.*

An element $x\in X$ *is said to* majorize *a subset* $Y \subseteq X$ *if for all* $y\in Y$, $y \leq x$. *One similarly says* x *majorizes an element* y *if* $y \leq x$.

A subset Y *of* X *is said to be* cofinal *in* X *if every element of* X *is majorized by some element of* Y.

We remark that there are no standard terms for the opposite concepts to ‘‘majorize’’ and ‘‘cofinal’’. One occasionally sees ‘‘minorize’’ and ‘‘coinitial’’, but these are awkward; it seems best to say ‘‘majorizes (or is cofinal) under the opposite ordering’’.

The concept of cofinality defined above probably originated in topology: If s is a point of a topological space S, and $N(s)$ the set of all neighborhoods of s, then a *neighborhood basis* of s means a subset $B \subseteq N(s)$ cofinal in that set, under the ordering by reverse inclusion. The virtue of this concept is that one can verify that a function on S approaches some limit at s by checking its behavior on members of such a B. E.g., one generally checks continuity of a function at a point s of the real line using the cofinal system of neighborhoods $\{(s-\varepsilon,\ s+\varepsilon) \mid \varepsilon > 0\}$.

Exercise 4.1:8. (i) Show that if X is a *finite* partially ordered set, then a subset Y is cofinal in X if and only if it contains all maximal elements of X.

(ii) Show by example this is not true for infinite partially ordered sets. Is one direction true?

Exercise 4.1:9. Let X be a finite partially ordered set. One defines the *height* of X as the maximum of the cardinalities of all chains in X, and the *width* of X as the maximum of the cardinalities of all antichains in X.

(i) Show that $\mathrm{card}(X) \leq \mathrm{height}(X)\cdot\mathrm{width}(X)$.

(That the above result fails for infinite partially ordered sets will be shown in Exercise 4.6:8.)

(ii) Must every (or some) chain in X of maximal cardinality have nonempty intersection with every (or some) antichain of maximal cardinality?

Definition 4.1.7. *Let* $\leq$ *and* $\preccurlyeq$ *be partial orderings on a set* X. *Then one says* $\leq$ *is an* extension *or* strengthening (*or sometimes, a* refinement) *of* $\preccurlyeq$ *if it contains the latter as subsets of* $X \times X$; *that is, if* $x \preccurlyeq y \Rightarrow x \leq y$.

The relation of ‘‘extension’’ is itself a partial ordering on the set of partial orderings on X. This fact can be looked at as follows. If we regard each partial ordering on X as a subset $R \subseteq X \times X$, and partially order the class of all subsets of $X \times X$ by inclusion (the relation $\subseteq$), then the relation of ‘‘extension’’ is the *restriction* of this partial ordering to the subclass of those $R \subseteq X \times X$ which are partial orders. This observation saves us the work of verifying that ‘‘extension’’ satisfies the conditions for a partial order, since we know that the restriction of a partial order on a set to any subset is again a partial order. Many of the partial orderings that arise naturally in mathematics are similarly restrictions of the inclusion relation, or some other natural partial ordering, on a larger set.

Exercise 4.1:10. Let us consider the set of all partial orderings on a set to be partially ordered as above.

(i) Show that the *maximal* elements in the set of all partial orderings on a set X are precisely the *total* orderings.

(ii) How many maximal elements does the set of partial orderings of a set of n elements have?

(iii) How many minimal elements does the set of partial orderings of a set of n elements have?

(iv) Show that every partial ordering on a finite set X is the set-theoretic intersection of a set of total orderings.

(v) Given a partial ordering $\preccurlyeq$ on a set of n elements, what can you say about the smallest number of total orderings that must be intersected to get $\preccurlyeq$? (This is called the ‘‘order dimension’’ of the given partially ordered set.)

Here is an outstanding open problem.

Exercise 4.1:11. Let $(X, \preccurlyeq)$ be a finite partially ordered set. Let N denote the number of total orderings ‘‘$\leq$’’ on X extending $\preccurlyeq$ (‘‘linearizations of $\preccurlyeq$’’) and for $x, y \in X$, let $N_{x,y}$ denote the number of these extensions ‘‘$\leq$’’ which satisfy $x \leq y$.

(i) Prove or disprove, if you can,

Fredman’s conjecture: For any $(X, \preccurlyeq)$ such that $\preccurlyeq$ is not a total order, there exist elements $x, y \in X$ such that

$$1/3 \;\leq\; N_{x,y}/N \;\leq\; 2/3. \tag{4.1.8}$$

If you cannot settle this open question, here are some special cases to look at:

(ii) Let r be a positive integer, and let X be the partially ordered set consisting of a chain of r elements, $p_1 \prec \ldots \prec p_r$, and an element q incomparable with all the p_i. What are N and the $N_{p_i,q}$ in this case? Verify Fredman’s conjecture for this partially ordered set.

(iii) Is the above example consistent with the stronger assertion that if X has no greatest element, then an x and a y satisfying (4.1.8) can be chosen from among the *maximal* elements of X? With the assertion that for every two maximal chains in X, one can choose an x in one of these chains and a y in the other satisfying (4.1.8)? If one or the other of these possible generalizations of Fredman’s Conjecture is not excluded by the above example, can you find an example that does exclude it?

(iv) Let r again be a positive integer, and let X be the set $\{1, \ldots, r\}$ partially ordered by the relation $\preccurlyeq$ such that $i \preccurlyeq j$ if and only if $j - i \geq 2$ (where $\geq$ has the usual meaning for integers). Verify the conjecture in this case as well. How many pairs (i, j) satisfy neither $i \preccurlyeq j$

nor $j \preccurlyeq i$, and of these, how many satisfy (4.1.8)?

(v) If X is any partially ordered set such that the function $N_{x,y}/N$ never takes on the value $1/2$, define a relation $\leq_!$ on X by writing $x \leq_! y$ if either $x = y$, or $N_{x,y}/N > 1/2$. Determine whether this is *always*, *sometimes* or *never* a (total) ordering on X. Show that for any X which is a counterexample to the conjecture of (i), $\leq_!$ is indeed a total ordering on X.

My feeling is that it may be possible to get a proof of Fredman's conjecture by assuming we had a counterexample, and considering the peculiar place the relation $\leq_!$ of point (v) above would have to have among the total orderings on X extending $\preccurlyeq$. One can see something of the structure of the set of all total orderings on a set from the next exercise.

Exercise 4.1:12. Define the *distance* between two total orderings $\leq_i$, $\leq_j$ on a finite set X as

$$d(\leq_i, \leq_j) \ = \ \text{number of pairs of elements } (x,y) \text{ such that } x <_i y, \ x >_j y.$$

Show that d is a metric on the set of all total orderings, and that for any partial ordering $\preccurlyeq$ on X, any two total orderings extending $\preccurlyeq$ can be connected by a chain (*not* meant in the order-theoretic sense!) $\leq_1, \dots, \leq_n$ where each $\leq_i$ is a total ordering extending $\preccurlyeq$, and the distance between successive terms of the chain is 1.

Here is another open question.

Exercise 4.1:13. (*Reconstruction problem for finite partially ordered sets.*) Let P and Q be finite partially ordered sets with the same number $n > 3$ of elements, and suppose they can be indexed $P = \{p_1, \dots, p_n\}$, $Q = \{q_1, \dots, q_n\}$ in such a way that for each i, $P - \{p_i\}$ and $Q - \{q_i\}$ are isomorphic as partially ordered sets. Must P be isomorphic to Q?

(Note that nothing is assumed about what bijections give the isomorphisms $P - \{p_i\} \cong Q - \{q_i\}$. We are definitely not assuming that they are the correspondences $p_j \leftrightarrow q_j$ $(j \neq i)$; if we assumed this, the question would have an immediate positive answer. A way to state the hypothesis without referring to such a correspondence is to say that the families of isomorphism classes of partially ordered $(n-1)$-element subsets of P and of Q, counting multiplicities, are the same. If this is true, then ''one can reconstruct P from its $(n-1)$-element partially ordered subsets'', hence the name of the problem.)

(The corresponding question for *graphs* with $n > 2$ vertices is also open, and perhaps better known.)

Readers interested in ordered sets and related structures should note that there is a journal, *Order*, devoted to these subjects, which regularly includes lists of open questions and reviews of books in the field.

4.2. Digression: preorders. One sometimes encounters binary relations which, like partial orderings, are reflexive and transitive, but which do not satisfy the antisymmetry condition. For instance, although the relation ''divides'' on the positive integers is a partial ordering, the relation ''divides'' on the set of all integers is not antisymmetric, since every n divides $-n$ and vice versa. More generally, on the elements of any commutative integral domain, ''divides'' is a reflexive transitive relation, but for every element x and invertible element u, x and ux each divide the other. Similarly, on a set of *propositions* (sentences in some formal language) about a mathematical situation, the relation $P \Rightarrow Q$ is reflexive and transitive, but not generally antisymmetric: Distinct propositions can each imply the another, i.e., represent equivalent conditions.

Definition 4.2.1. *A reflexive transitive* (*not necessarily antisymmetric*) *binary relation on a set* X *is called a* preorder *on* X.

The concept of a preordered set can be reduced in a natural way to a combination of two other sorts of structure that we already know.

Proposition 4.2.2. *Let* X *be a set. Then the following data are equivalent.*

(i) *A preorder* $\preccurlyeq$ *on* X.

(ii) *An equivalence relation* $\approx$ *on* X, *and a partial ordering* $\leq$ *on the set of equivalence classes,* $X/\approx$.

Namely, to go from (i) *to* (ii), *given the preorder* $\preccurlyeq$, *define* $x\approx y$ *to mean* $x\preccurlyeq y \wedge y\preccurlyeq x$, *and for any two elements* $[x], [y]\in X/\approx$, *write* $[x]\leq[y]$ *in* $X/\approx$ *if and only if* $x\preccurlyeq y$ *in* X.

Inversely, given, as in (ii), *an equivalence relation* $\approx$, *and a partial ordering* $\leq$ *on* $X/\approx$, *one gets a preorder* $\preccurlyeq$ *as in* (i) *by defining* $x\preccurlyeq y$ *to hold in* X *if and only if* $[x]\leq[y]$ *in* $X/\approx$. □

Exercise 4.2:1. Prove the above proposition. (This requires one verification of well-definedness, and some observations showing why the two constructions, performed successively in either order, return the original data.)

This is neat: A reflexive transitive relation (a preorder) decomposes into a reflexive transitive *symmetric* relation (an equivalence relation) and a reflexive transitive *antisymmetric* relation (a partial ordering).

As an example, if we take the set of elements of a commutative ring R, preordered by divisibility, and divide out by the equivalence relation of mutual divisibility, we get a partially ordered set, which can be identified with the set of principal ideals of R partially ordered by reverse inclusion.

In view of the above proposition, there is no need for a *theory* of preorders – that is essentially subsumed in the theory of partial orderings. But it is useful to have the concept available, to refer to such relations when they arise.

The remainder of this section consists of some exercises on preorders which will not be used in subsequent sections. Exercises 4.2:2-4.2:9 concern a class of preorders having applications to ring theory, group theory, and semigroup theory. (The later exercises in that group all depend on 4.2:2 and 4.2:3; also, 4.2:5 is assumed in 4.2:6-4.2:9. If you wish to hand in one of these exercises without writing out the details of others on which it depends, you should begin with a summary of the results from the latter that you will be assuming. You might check this summary with me first.) The last two exercises of the section are independent of that group.

Exercise 4.2:2. If f and g are nondecreasing functions from the positive integers to the nonnegative integers, let us write $f\preccurlyeq g$ if there exists a positive integer N such that for all i, $f(i)\leq g(Ni)$.

(i) Show that $\preccurlyeq$ is a preorder, but not a partial order, on the set of nondecreasing functions.

(ii) On the subset of functions consisting of all polynomials with nonnegative integer coefficients, get an explicit description of $\preccurlyeq$, and determine its ''decomposition'' as in the above proposition.

(iii) Do the same for the set of functions consisting of the polynomials of (ii), together with the exponential functions $i\mapsto n^i$ for all integers $n>1$.

(iv) Show that (when not restricted as in (ii) or (iii)) the partial ordering $\leq$ induced by the preordering $\preccurlyeq$ is not a total ordering.

Exercise 4.2:3. Let S be a monoid and $x_1, \ldots, x_n$ elements of S, and for each positive integer i, let $g_{x_1, \ldots, x_n}(i)$ denote the number of distinct elements of S which can be written as words of length $\leq i$ in $x_1, \ldots, x_n$ (with repetitions allowed). This is a nondecreasing function from the positive integers to the nonnegative integers, the *growth function* associated with $x_1, \ldots, x_n$.

Show that if S is generated by $x_1, \ldots, x_n$, and if $y_1, \ldots, y_m$ is any other finite family of elements of S, then in the notation of the preceding exercise, $g_{y_1, \ldots, y_m} \preccurlyeq g_{x_1, \ldots, x_n}$. Deduce that if $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are two generating sets for the same monoid, then $g_{x_1, \ldots, x_n} \approx g_{y_1, \ldots, y_m}$ (where $\approx$ is the equivalence relation determined as in Proposition 4.2.2 by the preorder $\preccurlyeq$).

Thus, if S is finitely generated, the equivalence class $[g_{x_1, \ldots, x_n}]$ is the same for all finite generating sets $x_1, \ldots, x_n$ of S. This equivalence class is therefore an invariant of the finitely generated monoid S, called its *growth rate*. We see that if a finitely generated monoid S is embeddable in another finitely generated monoid T, then the growth rate of S must be $\leq$ that of T.

Exercise 4.2:4. (i) Determine the growth rates of the *free abelian* monoid on n generators and the *free* monoid on n generators.

(ii) With the help of the result of (i), show that the free abelian monoid on m generators is embeddable in the free abelian monoid on n generators if and only if $m \leq n$.

(iii) Verify that for any positive integer n, the map from the *free monoid* on n generators $x_1, \ldots, x_n$ to the free monoid on 2 generators x, y taking x_i to xy^i $(i = 1, \ldots, n)$ is an embedding. Is this consistent with the results of (i)?

The concept of growth rate is more often studied for *groups* and *rings* than for monoids. Note that elements $x_1, \ldots, x_n$ of a group G generate G as a group if and only if $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ generate G as a monoid, so the group-theoretic growth function of G with respect to $\{x_1, \ldots, x_n\}$ may be defined to be the growth function of G as a monoid with respect to the generating set $\{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\}$. The equivalence class of such growth functions is called the growth rate of the group G, which is thus the same as the growth rate of G as a monoid. The concept of growth rate has been used, in particular, in studying fundamental groups of manifolds [**106**].

If R is an algebra over a field k, then to get the ring theoretic concept of the growth rate of R, one considers, not the *number* of elements which can be written as a product of $\leq i$ generators, but the *dimension* of the k-vector space spanned in R by such products. The remainder of the development is analogous to that of the monoid case.

Though we are digressing a bit from the subject of preorders, let us sketch in the next few exercises an important invariant obtained from these growth rates, and some of its properties.

Exercise 4.2:5. If S is a monoid with finite generating set $x_1, \ldots, x_n$, the *Gel'fand-Kirillov dimension* of S is defined as

$$(4.2.3) \qquad \mathrm{GK}(S) \;=\; \lim\sup_i \,(\ln(g_{x_1, \ldots, x_n}(i)) / \ln(i)).$$

(Here ''ln'' denotes the natural logarithm function, and $\lim\sup_i a(i)$ means $\lim_{j \to \infty} \sup_{i \geq j} a(i)$. Thus, if a is a nonnegative function, $\lim\sup_i a(i)$ will be a nonnegative real number or $+\infty$.)

(i) Show that the right hand side of (4.2.3) is a function only of the *growth rate* $[g_{x_1, \ldots, x_n}]$, hence does not depend on the choice of generators $x_1, \ldots, x_n$, so that the Gel'fand-Kirillov dimension is well defined.

(ii) Determine the Gel’fand-Kirillov dimensions of the free abelian monoid and the free monoid on n generators.

Exercise 4.2:6. (i) In the early literature, it was often stated (in effect) that for monoids S_1, S_2, one had $\mathrm{GK}(S_1 \times S_2) = \mathrm{GK}(S_1) + \mathrm{GK}(S_2)$. Find the fallacy in this claim, and if you can, find a counterexample. (Actually, the statement was made for tensor products of algebras rather than direct products of monoids, but one case can be reduced to the other.)

Exercise 4.2:7. (i) Show that if S is a finitely generated monoid and $\mathrm{GK}(S) < 2$, then $\mathrm{GK}(S) = 0$ or 1.

(ii) Show, on the other hand, that there exist finitely generated monoids having for Gel’fand-Kirillov dimensions all real numbers ≥ 2, and $+\infty$. (Suggestion: Show that for any finite or infinite set S of elements of a free monoid F, one can construct a homomorphic image of F in which all elements not having members of S as subwords are distinct, while all elements that do have subwords in S have a common value, ‘‘0’’.)

(iii) Show that there exist finitely generated monoids with distinct growth rates, but the same finite Gel’fand-Kirillov dimension.

We haven’t seen any exercises on growth rates of k-algebras yet. If, as in the preceding exercise, one is only concerned with what growth rates occur, there is essentially no difference between the cases of k-algebras and of monoids, as shown in

Exercise 4.2:8. Let k be any field.

Show that for every monoid S with generating set $s_1, \ldots, s_n$, there exists a k-algebra R with a generating set $r_1, \ldots, r_n$ such that $g_{r_1, \ldots, r_n} = g_{s_1, \ldots, s_n}$. Similarly, show that for every k-algebra R with generating set $r_1, \ldots, r_n$, there exists a monoid S with a generating set $s_1, \ldots, s_{n+1}$ such that $g_{s_1, \ldots, s_{n+1}} = g_{r_1, \ldots, r_n} + 1$ (where ‘‘1’’ denotes the constant function with value 1).

However, if one is interested in the growth of algebras with particular properties, these do not in general reduce to questions about growth of monoids. For instance, students familiar with the theory of transcendence degree of field extensions might do

Exercise 4.2:9. Show that if k is a field and R a finitely generated commutative k-algebra without zero-divisors, then the Gel’fand-Kirillov dimension of R as a k-algebra equals the transcendence degree over k of the field of fractions of R.

For more on Gel’fand-Kirillov dimension in ring theory, see [**73**].

For students familiar with the definitions of general topology, another important instance of the concept of preorder is noted in:

Exercise 4.2:10. (i) Show that if X is a topological space, and if for $x, y \in X$, we define $y \leq x$ to mean ‘‘the closure of $\{x\}$ contains y’’, then $\leq$ is a preorder on X.

(ii) Show that if X is *finite*, the above construction gives a bijection between topologies and preorders on X.

(iii) Under the above bijection, what classes of preorders correspond to T_0, respectively T_1, respectively T_2 topologies?

(iv) If X is *infinite*, is the above map from topologies to preorders one-to-one? Onto? Can one associate to every preorder on X a topology having a left or right universal property with respect to this construction?

Exercise 4.2:11. The standard topology on the real line $\mathbf{R}$ can be defined in terms of open intervals (a, b), which are in turn defined in terms of the standard ordering of $\mathbf{R}$. Can you generalize this topologization to arbitrary totally ordered, partially ordered, or preordered sets? Is it related to the construction of the preceding exercise?

4.3. Induction and chain conditions. The familiar principle of induction on the natural numbers (nonnegative integers) that one learns as an undergraduate is based on the order properties of that set. In this and the next two sections, we shall examine more general kinds of ordered sets over which one can perform inductive proofs and constructions.

Any students to whom the distinction between ‘‘minimal’’ and ‘‘least’’ elements in a partially ordered set was new should review Definition 4.1.5 before going on.

Lemma 4.3.1. *Let* $(X, \leq)$ *be a partially ordered set. Then the following conditions are equivalent:*

(i) *Every nonempty subset of* X *has a minimal element.*

(ii) *For every descending chain* $x_0 \geq x_1 \geq \ldots \geq x_i \geq \ldots$ *in* X *indexed by the natural numbers, there is some* n *such that* $x_n = x_{n+1} = \ldots$.

(ii′) *Every strictly descending chain* $x_0 > x_1 > \ldots$ *indexed by an initial subset of the natural numbers* (*that is, either by* $\{0, 1, \ldots, n\}$ *for some* n, *or by the set of all nonnegative integers*) *is finite* (*that is, is in fact indexed by* $\{0, 1, \ldots, n\}$ *for some* n).

(ii′′) X *has no strictly descending chains* $x_0 > x_1 > \ldots$ *indexed by the full set of natural numbers.*

Proof. (i)⇒(ii′′)⇔(ii′)⇔(ii) is straightforward. Now assume (ii′′), and suppose we had a nonempty subset $Y \subseteq X$ with no minimal element. Take any $x_0 \in Y$. Since this is not minimal, we can find $x_1 < x_0$. Since this in turn is not minimal, we can find $x_2 < x_1$. Continuing this process, we get a contradiction to (ii′′). □

Definition 4.3.2. *A partially ordered set* X *is said to have* descending chain condition (*abbreviated ‘‘DCC’’; called ‘‘minimum condition’’ by some authors*) *if it satisfies the equivalent conditions of the above lemma.*

Likewise, a partially ordered set X *with the dual condition* (*every nonempty subset has a maximal element, equivalently,* X *has no infinite ascending chains*) *is said to have* ascending chain condition (*‘‘ACC’’ or ‘‘maximum condition’’*).

A well-ordered set *means a totally ordered set with descending chain condition.*

Remark: A *chain* in X, as defined following Definition 4.1.2, is a totally ordered subset, and it is meaningless to call such a subset ‘‘increasing’’ or ‘‘decreasing’’. In the above lemma and definition, the phrases ‘‘descending chain’’ and ‘‘ascending chain’’ are used as shorthand for a totally ordered subset which can be indexed in a descending, respectively in an ascending manner by the natural numbers. (One may consider this a mixture of the order-theoretic meaning of ‘‘chain’’, and the informal meaning, namely a sequence of elements indexed by a set of consecutive integers with a specified relation holding between consecutive terms.) But note that though this shorthand is used in the convenient phrases ‘‘ascending chain condition’’ and ‘‘descending chain condition’’, we made explicit what we meant by such ‘‘chains’’ in Lemma 4.3.1(ii)-(ii′′).

That the natural numbers are well-ordered has been known in one form or another for millennia, but the importance of ACC and DCC for more general partially ordered sets was probably first noted in ring theory, in the early decades of the twentieth century. Rings with these conditions on their sets of *ideals* (partially ordered by inclusion) are called ‘‘Noetherian’’ and ‘‘Artinian’’ respectively, after Emmy Noether and Emil Artin who studied them.

One does not need to formally state a ‘‘principle of induction over partially ordered sets with ACC (or DCC)’’. Rather, when one wishes to prove a result for all elements of a partially ordered set X with, say, DCC, one can simply begin, ‘‘Suppose there are elements of X for which the statement is false. Let x be minimal for this property’’ (since, if the set of such elements is nonempty, it must have a minimal member). Then one knows the statement is true for all $y<x$, and if one can show from this that it is true for x as well, one gets a contradiction, proving the desired result. Since this is a familiar form of argument, one often abbreviates it and says, ‘‘Assume inductively that the statement is true for all $y<x$’’, proves from this that it is true for x as well, and concludes that it is true for all elements of X.

In the most familiar sort of induction on the natural numbers, one starts by proving the desired result for 0 or 1. Why was there no corresponding step in the schema described above? The analog of the statement that our desired result holds for 0 would be the statement that it holds for all *minimal* elements of X. But if one can prove that a statement is true for x whenever it is true for all smaller elements of X, then in particular, one must be able to prove it in the case where the set of smaller elements is empty. Depending on the situation, the proof that a result is true for x if it is true for all smaller elements may or may not involve different arguments in the minimal and nonminimal cases.

Exercise 4.3:1. A noninvertible element of a commutative integral domain C is called *irreducible* if it cannot be written as a product of two noninvertible elements. Give a concise proof that if C is a commutative integral domain with ascending chain condition on ideals (or even just on principal ideals), then every nonzero noninvertible element of C can be written as a product of irreducible elements.

In addition to *proofs* by induction, one often performs *constructions* in which each step requires that a set of preceding steps already have been done. The construction of the Fibonacci numbers f_i $(i = 0, 1, 2, ...)$ from the defining conditions

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+2} = f_n + f_{n+1} \tag{4.3.3}$$

is of this sort. These are called *recursive* definitions or constructions, and we shall now see that, like inductive proofs, they can be carried out over general partially ordered sets with chain conditions.

Let us analyze what such a construction involves, and then show how to justify it. Suppose X is a partially ordered set with DCC, and suppose that we wish to construct a certain function f from X to a set T. To say that for some $x\in X$ the value of f has been determined for all $y<x$ is to say that we have a function $f_{<x}\colon \{y \mid y<x\} \to T$. So ‘‘a rule defining f at each x if it is defined for all $y<x$’’ can be formalized as a T-valued function r on the set of all pairs $(x, f_{<x})$ consisting of an $x\in X$ and a function $f_{<x}\colon \{y \mid y<x\} \to T$. In most applications, our rule defining f at x in terms of the values for $y<x$ actually requires that these values satisfy some good conditions, and we verify these conditions inductively, as the construction is described recursively. But to avoid complicating our abstract formalization, we may assume r defined for *all* pairs $(x, f_{<x})$ where $x\in X$ and $f_{<x}$ is a function $\{y \mid y<x\} \to T$. For if we have a definition of r in ‘‘good’’ cases, we can extend it to other cases in an arbitrary way (e.g., assume $0\in T$ and send $(x, f_{<x})$ to 0 if $f_{<x}$ is not ‘‘good’’). Then the inductive proof that f is

‘‘good’’ can be formally considered to come *after* the recursive construction of f.

We see that the property characterizing an f constructed recursively as above is that for each $x\in X$, $f(x)$ is a certain function of the *restriction* of f to $\{y \mid y<x\}$. Let us recall a common notation for restrictions of functions: If $f: X\to Y$ is a function, and Z is a subset of X, then the restriction of f to Z, a function $Z\to Y$, is denoted $f\,|\,Z$. (A variant symbol which we will not use is $f\restriction Z$.)

We can now justify recursive constructions by proving

Lemma 4.3.4. *Let X be a partially ordered set with descending chain condition, T any set, and r a function associating to every pair $(x, f_{<x})$ such that $x\in X$, and $f_{<x}$ is a function $\{y\in X \mid y<x\}\to T$, an element $r(x, f_{<x})\in T$. Then there exists a unique function $f: X\to T$ such that for all $x\in X$, $f(x) = r(x,\ f\,|\,\{y\,|\,y<x\})$.*

Proof. Let $X'\subseteq X$ denote the set of all $x\in X$ for which there exists a unique function $f_{\leq x}: \{y \mid y\leq x\}\to T$ with the property that

$$(4.3.5)\qquad (\forall y\leq x)\quad f_{\leq x}(y) \;=\; r(y,\ f_{\leq x}\,|\,\{z\,|\,z<y\}).$$

We claim, first, that for any two elements x_0, $x_1\in X'$, the functions $f_{\leq x_0}$, $f_{\leq x_1}$ agree on $\{y \mid y\leq x_0 \wedge y\leq x_1\}$. For if not, choose a minimal y in this set at which they disagree. Then by (4.3.5), $f_{\leq x_0}(y) = r(y,\ f_{\leq x_0}\,|\,\{z\,|\,z<y\})$, and $f_{\leq x_1}(y) = r(y,\ f_{\leq x_1}\,|\,\{z\,|\,z<y\})$. But by choice of y, the restrictions of $f_{\leq x_0}$ and $f_{\leq x_1}$ to $\{z\,|\,z<y\}$ are equal, hence by the above equations, $f_{\leq x_0}(y) = f_{\leq x_1}(y)$, contradicting our choice of y.

Next, suppose that X' were not all of X. Let x be a minimal element of $X-X'$. Since, as we have just seen, the functions $f_{\leq y}$ for $y<x$ agree on the pairwise intersections of their domains, they piece together into one function $f_{<x}$ on the union of their domains. (Formally, this ‘‘piecing together’’ means taking the *union* of these functions, as subsets of $X\times T$.) If we now define $f_{\leq x}$ to agree with this function $f_{<x}$ on $\{y\,|\,y<x\}$, and to have the value $r(x, f_{<x})$ at x, we see that this function satisfies (4.3.5), and is the unique function on $\{y\,|\,y\leq x\}$ which can possibly satisfy that condition. This means $x\in X'$, contradicting our choice of x.

Hence $X'=X$. Now piecing together these functions $f_{\leq x}$ defined on the sets $\{y\,|\,y\leq x\}$, we get the desired function f defined on all of X. □

Example: The Fibonacci numbers are defined recursively by using for X the ordered set of nonnegative integers, and defining $r(n, (f_0, \ldots, f_{n-1}))$ to be 0 if $n = 0$, to be 1 if $n = 1$, and to be $f_{n-2}+f_{n-1}$ if $n\geq 2$.

The next exercise shows that recursive constructions are not in general possible if the given partially ordered set does not satisfy descending chain condition.

Exercise 4.3:2. Show that there does not exist a function f from the interval $[0,1]$ of the real line to the set $\{0,1\}$ determined by the following rules:

(a) $f(0)=0$.

(b) For $x>0$, $f(x)=1$ if for all $y\in[0,x)$, $f(y)=0$; otherwise, $f(x)=0$.

If you prefer, you may replace the interval $[0,1]$ in this example by the countable set $\{0\}\cup\{1/n \mid n = 1, 2, 3, \ldots\}$.

Actually, solving a differential equation with given initial conditions is somewhat like a ‘‘recursive construction over an interval of the real numbers’’. But since the real numbers do not have descending chain condition, the conditions for existence and uniqueness of a solution, and the arguments needed to prove these, are more subtle. (A key fact that often plays a role like induction in such arguments is the connectedness of the real line.)

There is a situation at the very foundation of mathematics which can be interpreted in terms of a partially ordered system with descending chain condition. The Axiom of Regularity of set theory (which will be stated formally in the next section) says that there is no ‘‘infinite regress’’ in the construction of sets; that is, that there are no left-infinite chains of sets under the membership relation:

$$... \in S_n \in ... \in S_2 \in S_1 \in S_0.$$

This is not a difficult axiom to swallow, since if we had a set theory for which it was not true, we could pass to the ‘‘smaller’’ set theory consisting of those sets which admit no such chain to the left of them. The class of such sets is closed under all the constructions required by the remaining axioms of set theory, and the ‘‘new’’ set theory would satisfy the Axiom of Regularity.

To interpret Regularity in the terms we have just been discussing, let us write $A \prec B,$ for sets A and $B,$ if there is a chain of membership-relations, $A = S_0 \in S_1 \in ... \in S_n = B$ $(n > 0).$ This relation is clearly transitive. The Regularity Axiom implies that $\prec$ is antireflexive (if we had $A \prec A,$ then a chain of membership relations connecting A with itself could be iterated to give an infinite chain going to the left), hence $\prec$ is the strict partial ordering corresponding to a partial ordering $\preccurlyeq;$ and Regularity applied again says that this partial ordering has descending chain condition. (Well, almost. We have only defined the concepts of partial ordering and chain condition for *sets*, and the class of all sets is not a set. To get around this problem we can translate these observations more precisely as saying that for each set $A,$ $\{B \mid B \preccurlyeq A\}$ is itself a set, and has descending chain condition under $\preccurlyeq$.) This allows one to prove set-theoretic results inductively, and make set-theoretic definitions recursively.

We had another such situation in Chapter 1, when we talked about the set $T = T_{X,\,\mu,\,\iota,\,e}$ of group-theoretic terms in a set of symbols $X.$ These also satisfy a principle of regularity, in terms of the relation ‘‘s occurs in t’’, which we denoted $t \gtrdot s$ in Exercise 1.7:4. To show this, let T' denote the set of elements of T admitting no infinite descending $\gtrdot$-chains to the right of them. One verifies that T' is closed under the operations of conditions (a) and (b) of the definition of T (in §1.5), and concludes that if T' were properly smaller than $T,$ one would have a contradiction to condition (c) of that definition. We only sketched the construction of T in Chapter 1, but in §8.3 below we will introduce the concept of ‘‘term’’ for general classes of algebras, and the above argument will then allow us to perform formal recursion and induction over such terms.

One can, of course, do inductive proofs and recursive constructions over partially ordered sets with *ascending* as well as descending chain condition. These come up often in ring theory, where Noetherian rings, i.e., rings whose partially ordered set of ideals has ACC, are important. In proving that a property holds for an arbitrary ideal I of such a ring, one may, as we have noted, assume inductively that it is true for all strictly *larger* ideals. To get the result allowing us to perform *recursive constructions* in such situations, i.e., the analog of Lemma 4.3.4 with $>$ replacing $<,$ it is not necessary to repeat the proof of that lemma; we can use duality of partially ordered sets. I will give the statement and sketch the argument this once, to show how an argument by duality works. After this, I shall consider it sufficient to say ‘‘by duality’’, or ‘‘by the dual of Proposition #.#.#’’ etc., if I want to invoke the dual of an order-theoretic result previously given.

Corollary 4.3.6. *Let X be a partially ordered set with ascending chain condition, T any set, and r a function associating to every pair $(x, f_{>x})$ consisting of an element $x \in X$ and a function $f_{>x}: \{y \mid y > x\} \to T$ an element $r(x, f_{>x}) \in T$. Then there exists a unique function f: $X \to T$ such that for all $x \in X$, $f(x) = r(x, f \mid \{y \mid y > x\})$.*

Sketch of Proof. The *opposite* of the partially ordered set X (the structure with the same underlying set but the opposite ordering) is a partially ordered set X^{op} with *descending chain condition*, and r can be considered to be a function r' with exactly the properties required to apply Lemma 4.3.4 to that partially ordered set. That lemma gives us a unique function f' from X^{op} to T satisfying the conclusions of that lemma relative to r', and this is equivalent to a function f from X to T satisfying the desired condition relative to r. □

Exercise 4.3:3. Above, the Fibonacci numbers f_n were defined for $n \geq 0$. Show that there is a unique way of defining f_n for *all* integers n so that, again, $f_0 = 0$, $f_1 = 1$, and so that the equation $f_n = f_{n-2} + f_{n-1}$ now holds for all n.

Often the key to making an inductive argument or a recursive construction work is a careful choice of a parameter over which to carry out the induction or recursion, and an appropriate ordering on the set of values of that parameter. The next definition describes a way of constructing partial orderings that is frequently useful for such purposes. The well-ordered index set I in that definition can be something as simple as $\{0, 1\}$.

Definition 4.3.7. *Let $(X_i)_{i \in I}$ be a family of partially ordered sets, indexed by a* well-ordered *set I. Then* lexicographic order *on $\prod_I X_i$ is defined by declaring $(x_i) \leq (y_i)$ to hold if and only if either $(x_i) = (y_i)$, or for the least $j \in I$ such that $x_j \neq y_j$, one has $x_j < y_j$ in X_j.*

Note that if $I = \{1, \ldots, n\}$ with its natural order, then this construction orders n-tuples $(x_1, \ldots, x_n) \in \prod_I X_i$ by the same ''left-to-right'' principle that is used to arrange words in the dictionary; hence the name of the construction.

Exercise 4.3:4. Let $(X_i)_{i \in I}$ be as in Definition 4.3.7.

(i) Verify that the relation described in that definition is indeed a partial order.

(ii) Show that if each X_i is *totally* ordered, then so is their direct product under that ordering. Show, on the other hand, that this is not true of the *product ordering*, described in Definition 4.1.4.

(iii) Show that if I is finite and each of the X_i has descending (or ascending) chain condition, then so does their product under lexicographic ordering.

(iv) Comparing lexicographic ordering with the product ordering, deduce that a direct product of finitely many partially ordered sets with descending chain condition satisfies descending chain condition under the product ordering as well.

(v) Show that the product of countably many copies of the two-element ordered set $\{0, 1\}$ (with $0 < 1$) does not have descending chain condition under the product ordering. Deduce that lexicographic ordering on products of infinite families of partially ordered sets with descending chain condition also fails, in general, to have descending chain condition.

In the next exercise, lexicographic ordering is used to give a concise proof of a standard result on symmetric polynomials.

Exercise 4.3:5. Let R be a commutative ring, and $R[x_1, \ldots, x_n]$ the polynomial ring in n indeterminates over R. Given any nonzero polynomial $f = \Sigma\, r_{i(1), \ldots, i(n)}\, x_1^{i(1)} \ldots x_n^{i(n)}$ (almost all $r_{i(1), \ldots, i(n)}$ zero), let us define the *leading term* of f to be the nonzero summand in this expression with the largest exponent-string $(i(1), \ldots, i(n))$, under lexicographic ordering on the set of all such strings. (Since the set of nonzero summands is finite, no chain condition is needed to make this definition.)

(i) Let f and g be nonzero elements of $R[x_1, \ldots, x_n]$, and suppose that the coefficient of the leading term of f is not a zero-divisor in R. (E.g., this is automatic if R is an integral domain.) Show that the leading term of fg is the product of the leading terms of f and of g.

An element of $R[x_1, \ldots, x_n]$ is called *symmetric* if it is invariant under the natural action of the group of all permutations of the index set $\{1, \ldots, n\}$ on the indeterminates $x_1, \ldots, x_n$. For $1 \leq d \leq n$, the dth *elementary* symmetric function f_d is defined to be the sum of all products of exactly d distinct members of $\{x_1, \ldots, x_n\}$.

(ii) Show that the following sets are the same: (a) The set of all n-tuples $(i(1), \ldots, i(n))$ of nonnegative integers such that $i(1) \geq \ldots \geq i(n)$. (b) The set of all exponent-strings $(i(1), \ldots, i(n))$ of leading terms $r_{i(1), \ldots, i(n)}\, x_1^{i(1)} \ldots x_n^{i(n)}$ of symmetric polynomials. (c) The set of all exponent-strings of leading terms of *products of elementary* symmetric polynomials, $f_1^{j(1)} \ldots f_n^{j(n)}$. With what coefficient does the leading monomial occur in this product?

(iii) Deduce that any nonzero symmetric polynomial can be changed to a symmetric polynomial with lower exponent-string-of-the-leading-term, or to the zero polynomial, by subtracting a scalar multiple of a product of elementary symmetric polynomials. Conclude, by induction on this exponent-string, that the ring of symmetric polynomials in n indeterminates over R is generated over R by the elementary symmetric polynomials.

(For standard proofs of the above result, see [**26** pp. 252-255], or [**28**, Theorem IV.6.1, p.191]. For some related results on noncommutative rings, see [**43**].)

Exercise 4.3:6. For nonnegative integers i and j, let $n_{i,j}$ be defined recursively as the least nonnegative integer not equal to $n_{i,j'}$ for any $j' < j$, nor to $n_{i',j}$ for any $i' < i$. (What ordering of the set of pairs (i,j) of nonnegative integers can one use to justify this recursion?)

Find and prove a concise description of $n_{i,j}$. (Suggestion: Calculate some values and note patterns. To find the ‘‘pattern in the patterns’’, write numbers to base 2.)

4.4. The axioms of set theory.

4.4. The axioms of set theory. We are soon going to look at some order-theoretic principles equivalent to the powerful Axiom of Choice. Hence it is desirable to review the statement of that axiom, and its status in relation to the other axioms of set theory. For completeness, I will record in this section the whole set of axioms most commonly used by set theorists.

Let us begin with some background discussion. In setting up a rigorous foundation for mathematics, one might expect the theory to require several sorts of ‘‘entities’’: ‘‘primitive’’ elements such as numbers, additional sets formed out of these, ordered pairs, functions, etc.. But as the theory was developed, it turned out that one could get everything one wanted from a single basic concept, that of set, and a single relation among sets, that of membership. The result is a set theory in which the only members of sets are themselves sets.

As an important example of how other ‘‘primitives’’ are reduced to the set concept, we recall the case of the *natural numbers* (nonnegative integers). The first thing we learn in our childhood about these numbers is that they are used to count things; to say how many objects there are in a collection. The early set theorists observed that one can formalize the concept of two sets having the ‘‘same number’’ of elements set-theoretically, as meaning that there exists a bijection between them. This is clearly an equivalence relation on sets. Hence the natural numbers ought be some entities which one could associate to finite sets, so that two sets would get the same entity associated to them if and only if they were in the same equivalence class. Their original idea was to use, as those entities, the equivalence classes themselves, i.e., to *define* the natural numbers $0, 1, 2,$ etc., to be the corresponding equivalence classes. Thus, the statement that a finite set had n elements would mean that it was a *member* of the number n. (Cardinalities of infinite sets were to be treated similarly.) This is good in principle – don't create new entities to index the equivalence classes if the equivalence classes themselves will do. But in this case, the equivalence classes turned out not to be a good choice: they are too big to be sets. So the next idea was to choose one easily described member from each such class, call these chosen elements the natural numbers $0, 1, 2, \ldots,$ and define a set to have n elements if it could be put in bijective correspondence with the ''sample'' set n.

Where would one get these ''sample'' finite sets from, using pure set theory? At least there is no problem getting a sample 0-element set – there is a unique set with 0 elements, the empty set $\varnothing$. Having taken this step, we have *one* set in hand – $\varnothing$. This means that we are in a position to create a sample one-element set, the set with that element as its one member, i.e., $\{\varnothing\}$. Having found these two elements, $\varnothing$ and $\{\varnothing\}$, we can define a 2-element set $\{\varnothing, \{\varnothing\}\}$ to use as our next sample, and so on. After the first few steps, we are not so limited in our options. However, the above approach, of always taking for the next number the set of numbers found so far, due to John von Neumann, is an elegant way of manufacturing one set of each natural-number cardinality, and it is taken as the definition of these numbers by modern set theorists:

$$0 = \varnothing, \qquad 1 = \{\varnothing\}, \qquad 2 = \{\varnothing, \{\varnothing\}\}, \qquad 3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\},$$

$$\ldots$$

$$(4.4.1) \qquad i{+}1 = i \cup \{i\} = \{0, 1, 2, \ldots, i\}$$

$$\ldots$$

Another basic concept which was reduced to the concepts of set and membership is that of *ordered pair*. If X and Y are sets, then one can deduce from the axioms (shortly to be listed) that X and Y can each be determined uniquely from the set $\{\{X\}, \{X, Y\}\}$. Since all one needs about ordered pairs is that they are objects which specify their first and second components unambiguously, one *defines* the ordered pair (X, Y) to mean the set $\{\{X\}, \{X, Y\}\}$.

One then goes on to define the *direct product* of two sets in terms of ordered pairs, binary relations in terms of direct products, functions in terms of relations, etc.. From natural numbers, ordered pairs, and functions, one constructs the integers, the rational numbers, the real numbers, the complex numbers, etc., by well-known techniques, which we won't review here.

(One also wants to define ordered n-tuples. The trick by which ordered pairs were defined turns out not to generalize in an easy fashion; the most convenient approach is to define an ordered n-tuple to mean a function whose domain is the set n. However, this conflicts with the definition of ordered pair! To handle this, a careful development of set theory must use different symbols, say $<X, Y>$ for the concept of ''ordered pair'' first described, and $(X_0, X_1, \ldots, X_{n-1})$ for the ordered n-tuples subsequently defined.)

The above examples should give some motivation for the ''sort'' of set theory described by the axioms which we shall now list. Of course, a text on the foundations of mathematics will first develop language allowing one to state these axioms precisely, and, since a statement in such language is not always easy to understand, it will precede or follow many of the precise statements by intuitive developments. I have tried below to give formulations that make it as clear as possible what the axioms assert, and have added some further remarks after the list. But for a thorough presentation, and for more discussion of the axioms, the student should see a text on the subject. Two recommended undergraduate texts are [**10**] and [**18**]. Written for a somewhat more advanced audience is [**16**].

Here, now, are the axioms of *Zermelo-Fraenkel Set Theory with the Axiom of Choice*, commonly abbreviated ZFC.

Axiom of Extensionality: *Sets are equal if and only if they have the same members, i.e.,* $X = Y$ *if and only if for every set* A, $A\in X \Leftrightarrow A\in Y$.

Axiom of Regularity (or **Well-foundedness**, or **Foundation**): *For every nonempty set* X, *there is a member of* X *which is disjoint from* X.

Axiom of the Empty Set: *There exists a set with no members.* (Common notation: $\varnothing$.)

Axiom of Separation: *If* X *is a set and* P *is a condition on sets, there exists a set* Y *whose members are precisely the members of* X *satisfying* P. (Common notation: $Y = \{A\in X \mid P(A)\}$.)

Axiom of Doubletons (or **Pairs**): *If* X *and* Y *are sets, there is a set* Z *whose only members are* X *and* Y. (Common notation: $Z = \{X, Y\}$.)

Axiom of Unions: *If* X *is a set, there is a set* Y *whose members are precisely all members of members of* X. (Common notation: $Y = \bigcup X$ or $\bigcup_{A\in X} A$.)

Axiom of Replacement: *If* f *is an operation on sets* (*formally characterized by a set-theoretic proposition* $P(A, B)$ *such that for every set* A *there is a unique set* $f(A)$ *such that* $P(A, f(A))$ *holds*) *and* X *is a set, then there exists a set* Y *whose members are precisely the sets* $f(A)$ *for* $A\in X$. (Common notation: $Y = \{f(A) \mid A\in X\}$. *When there is no danger of confusion, this is sometimes abbreviated to* $Y = f(X)$.)

Axiom of the Power Set: *If* X *is a set, there exists a set* Y *whose members are precisely all subsets of* X. (Common notations: $Y = \mathbf{P}(X)$ or 2^X.)

Axiom of Infinity: *There exists a set having* $\varnothing$ *as a member, closed under the construction* $i \mapsto i\cup\{i\}$ (*cf.* (4.4.1)), *and minimal for these properties.* (Common name: The set of natural numbers.)

Axiom of Choice: *If* X *is a set, and* f *is a function associating to every* $x\in X$ *a nonempty set* $f(x)$, *then there exists a function* g *associating to every* $x\in X$ *an element* $g(x)\in f(x)$.

Explanations of some of the names: *Extensionality* means that a set is determined by its *extent*, not its *intent*. *Separation* says that one can form new sets by using any well-defined criterion to ‘‘separate out’’ certain elements of an existing set. The Axiom of *Infinity* is so called because if we did not assume it, the collection of all sets which can be built up from the empty set in finitely many steps would satisfy our axioms, giving an example of a set theory in which all sets are finite. So the axiom is equivalent to the statement that there exists an infinite set.

We described *Regularity* earlier as saying that there was no infinite regress under ‘‘$\in$’’. That formulation requires one to have the set of natural numbers to index such a regress, so we chose a formulation that can be expressed independently of the Axiom of Infinity. In the presence of the other axioms one can prove the two formulations equivalent. (Roughly, if one had an infinite chain $\ldots \in S_2 \in S_1 \in S_0$, then $\{S_i\}$ would be a counterexample to Regularity, while if a set X were a counterexample to Regularity, one could select such a chain from its elements.)

Actually the Axiom of Regularity makes little substantive difference for areas of mathematics other than set theory itself (e.g., see [**18**]). Without it, one can have sets with exotic properties such as being members of themselves, but the properties of set-theoretic concepts used by most of mathematics – bijections, direct products, cardinality arguments, etc. – are little affected. Its absence would simply make it a bit trickier to construct, say, a family of *disjoint* copies of a given set. The Regularity Axiom seems to have crept into the Zermelo-Fraenkel axioms by the back door: It was not in the earlier formulations of those axioms, and still does not appear in some listings, such as that in [**10**]. But it is generally accepted, and we will count it among the axioms here, and rely on the convenience it provides. It gives one a comforting assurance that sets are built up from earlier sets with no ‘‘vicious circles’’ in the process; hence the name ‘‘Well-Foundedness’’. (By extension, set-theorists often call the property of descending chain condition on any partially ordered set ‘‘well-foundedness’’.)

Observe that the Axioms of Extensionality and Regularity essentially clarify what we intend to *mean* by a ‘‘set’’. The next seven axioms each say that certain sets exist; in each case these are sets which are *uniquely determined* by the conditions assumed.

The last axiom, however, that of ‘‘choice’’, asserts the existence of an object not uniquely defined by the given data: a function that chooses, *in an unspecified way,* one element from each of a family of sets. It was very controversial in the early decades of the twentieth century, both because it led to consequences which seemed surprising then (such as the existence of nonmeasurable sets of real numbers), and because of a feeling by some that it represented an unjustifiable assumption that something one could do in the finite case could be done in the infinite case as well. It is a standard assumption in modern mathematics; such basic results as that every vector space has a basis, that a direct product of compact topological spaces is compact, and that a countable union of countable sets is countable cannot be proved without it. But there have been, and still are, mathematicians who reject it: the *intuitionists* of the early part of this century, and the *constructivists* today.

Even accepting the Axiom of Choice, as we shall, it is at times instructive to note whether a result or an argument depends on it, or can be obtained from the other axioms. (This is like the viewpoint that, even if one does not accept the constructivists’ extreme claim that proofs of existence that do not give explicit constructions are *worthless*, one may consider constructive proofs to be desirable when they can be found.)

In the next two sections we shall develop several powerful results about sets whose proofs require the Axiom of Choice, and we will show that these are each, in fact, equivalent to that Axiom, in the presence of the other axioms. Hence, in those sections, we shall not assume the Axiom of Choice without stating this assumption explicitly, and the arguments we give to show this equivalence will all be justifiable in terms of the theory given by the other axioms, called ‘‘Zermelo-Fraenkel Set Theory’’, abbreviated ZF. (However, we shall not in general attempt to show explicitly how the familiar mathematical techniques that we use are justified by those axioms – for that, again see a text in set theory.) In all later chapters, on the other hand, we shall freely use the Axiom of Choice, i.e., we will assume ZFC.

In the handful of results proved so far in this chapter, we have implicitly used the Axiom of Choice just once: in Lemma 4.3.1, in showing (ii$''$)$\Rightarrow$(i). Hence for the remainder of this chapter, we shall forgo assuming that implication, and will consider descending chain condition to be defined by condition (i) of that lemma (which still *implies* (ii)-(ii$''$)).

Let us note explicitly one detail of set-theoretic language we have already used: Since the sets satisfying a given property may not together form a set, one needs a word to refer to ‘‘collections’’ of sets that are not necessarily themselves sets. These are called *classes*. An example is the *class* *of all sets*. One can think of classes which are not sets, not as actually being *mathematical objects*, but as providing a convenient *language* to use in making statements about all sets having one or another property.

Since classes are more general than sets, one may refer to any set as a ''class'', and this is sometimes done for reasons not involving the logical distinction, but just to vary the wording. E.g., rather than saying ''the set of those subsets of X such that ... '', one sometimes says ''the class of those subsets of X such that ... ''. And, for some reason, one always says ''equivalence class'', not ''equivalence set''.

4.5. Well-ordered sets and ordinals. Recall (Definition 4.3.2) that a partially ordered set $(X,\leq)$ is called *well-ordered* if it is totally ordered and has descending chain condition. In a totally ordered set, a minimal element is the same as a least element, so the condition of well-ordering says that every nonempty subset of X has a *least* member.

This condition goes a long way toward completely determining the structure of X. Applied first to X as a subset of itself, it tells us that if X is nonempty, it has a least element, x_0. If X does not consist of x_0 alone, then $X-\{x_0\}$ is nonempty, hence this set has a least element, which we may call x_1. We can go on in this fashion, and, unless X is finite, we will get a uniquely determined sequence of elements $x_0 < x_1 < x_2 < x_3 < \ldots$ at the ''bottom'' of X. This list may exhaust X, but if it does not, there will necessarily be a least element in the complement of the subset so far described, which we may call $x_{1,0}$, and if this still does not exhaust X, there will be a least element greater than it, $x_{1,1}$, etc.. We can construct in this way successive hierarchies, and hierarchies of hierarchies – I will not go into details – on the single refrain, ''If this is not all, there is a least element of the complement''.

A couple of concrete examples are noted in

Exercise 4.5:1. If f and g are real-valued functions on the real line $\mathbf{R}$, let us in this exercise write $f\leq g$ to mean that there exists some real number N such that $f(t)\leq g(t)$ for all $t\geq N$.

(i) Show that this relation $\leq$ is a preordering, that its restriction to the set of polynomial functions is a total ordering, and that on polynomials with *nonnegative integer* coefficients, it in fact gives a well-ordering. Determine, if they exist, the elements $x_0, x_1, \ldots, x_n, \ldots, x_{1,0}, x_{1,1}$ of this set, in the notation of the preceding paragraphs.

(ii) Show that the set consisting of all polynomials with nonnegative integer coefficients, and also the function e^t, is still well-ordered under the above relation.

(iii) Find a subset of the rational numbers which is order-isomorphic (under the standard ordering) to the set described in (ii).

To make precise the idea that the order structure of a well-ordered set is ''unique, as far as it goes'', let us define an ''initial segment'' of any totally ordered set X to mean a subset $I\subseteq X$ such that $x\leq y\in I \Rightarrow x\in I$. Then we have

Lemma 4.5.1. *Let X and Y be well-ordered sets. Then exactly one of the following conditions holds:*

(i) *X and Y are order-isomorphic.*

(ii) *X is order-isomorphic to a proper initial segment of Y.*

(iii) *Y is order-isomorphic to a proper initial segment of X.*

Further, in each case, the isomorphism in question is unique; in particular, in (ii) *and* (iii) *the initial segments in question are unique.*

Proof. We shall construct an order isomorphism of one of these three types by a recursive construction on the well-ordered set X. Let us first see the idea intuitively: We start by pairing the least element of X with the least element of Y; and we go on, at every stage pairing the least not-yet-paired-off element of X with the least not-yet-paired-off element of Y, until we run out of elements of either X or Y or both.

Now in our formulation of recursive constructions in Lemma 4.3.4, we said nothing about ''running out of elements''. But we can use a trick to reduce the approach just sketched to a recursion of the sort characterized by that lemma.

Form a set consisting of the elements of Y and one additional element which we shall denote DONE. Given any $x \in X$, and any function $f_{<x}: \{x' \in X \mid x' < x\} \to Y \cup \{\text{DONE}\}$, we define $r(x, f_{<x}) \in Y \cup \{\text{DONE}\}$ as follows:

> If the image of $f_{<x}$ is a proper initial segment of Y, let $r(x, f_{<x})$ be the least element of Y not in that segment. Otherwise, let $r(x, f_{<x}) = \text{DONE}$.

By Lemma 4.3.4 this determines a function $f: X \to Y \cup \{\text{DONE}\}$. It is straightforward to verify inductively that for those x such that $f(x) \neq \text{DONE}$, the restriction $f_{\leq x}$ of f to $\{x' \in X \mid x' \leq x\}$ will be the only order isomorphism between that initial segment of X and any initial segment of Y. From this we easily deduce that if the range of f does not contain the value DONE, exactly one of conclusions (i) or (ii) holds, but not (iii); while if the range of f contains DONE, (iii) holds but not (i) or (ii). In each case, f determines the unique order isomorphism with the indicated properties. □

Exercise 4.5:2. Give the details of the last paragraph of the above proof.

Since the well-ordered sets fall into such a neat array of isomorphism classes, it is natural to look for a way of choosing one ''standard member'' for each of these classes, just as the natural numbers are used as ''standard members'' for the different sizes of finite sets. Recall that in the von Neumann construction of the natural numbers, (4.4.1), each number arises as the set of all those that precede it, so that we have $i < j$ if and only if $i \in j$, and $i \leq j$ if and only if $i \subseteq j$. In particular, each natural number is a well-ordered set under this ordering. Let us take the von Neumann natural numbers as our standard examples of *finite well-ordered sets*, and see whether we can extend this family in a natural way to get models of infinite well-ordered sets.

Following the principle that each new object should be the set of all that precede, we use *the set of natural numbers* as the standard example chosen from among the well-ordered sets which when listed in the manner discussed at the beginning of this section have the form $X = \{x_0, x_1, \ldots\}$ (with subscripts running over the natural numbers but nothing beyond those). Set theorists denote this object

$$\omega = \{0, 1, 2, \ldots, i, \ldots\}.$$

The obvious representative for those sets having an initial segment isomorphic to ω, and just one element beyond that segment, is denoted

$$\omega{+}1 = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, i, \ldots; \omega\}.$$

We likewise go on to get $\omega{+}2$, $\omega{+}3$ etc.. The element coming after all the $\omega{+}i$'s $(i \in \omega)$ is denoted $\omega{+}\omega$ or $\omega 2$. (We will see later why it is not written ''more naturally'' as 2ω.) After the elements $\omega 2{+}i$ comes $\omega 3$; ... after all the elements of the form ωi $(i \in \omega)$ one has $\omega\omega = \omega^2$. In fact, one can form arbitrary ''polynomials'' in ω with nonnegative integer coefficients, and the set of these has just the order structure that was given to the polynomials with such coefficients in Exercise 4.5:1 (though the integer coefficients in our ‘‘polynomials’’ in ω are written on the right). Then the set of *all* these polynomials in ω is taken as the next standard sample well-ordered set ... .

So far, we have been sketching an idea; let us make it precise. A terminological observation first. If X is any well-ordered set, and α is the ‘‘standard’’ well-ordered set order-isomorphic to it, then we have a natural way to index the elements of X by the members of α – i.e., by the ‘‘sample’’ well-ordered sets smaller than α. Thus, the well-ordered sets less than α serve as generalizations of the sequence of words ‘‘first, second, third, ...’’ which are used in ordinary language to index the elements of finite totally ordered sets. Hence, the term *ordinals*, used by grammarians for those words, is used by mathematicians for the ‘‘standard samples of isomorphism types of well-ordered sets’’. Now for the formal definition.

Definition 4.5.2. *An* ordinal (*or* von Neumann ordinal) *is a set* α *such that* $\gamma\in\beta\in\alpha \Rightarrow \gamma\in\alpha$, *and such that if* $\beta\in\alpha$ *and* $\gamma\in\alpha$, *then either* $\beta = \gamma$, *or* $\beta\in\gamma$, *or* $\gamma\in\beta$.

Observe that the first part of the definition says that the relation ‘‘$\in$’’ on the elements of an ordinal is transitive; by the Regularity Axiom it is antisymmetric, hence it is a ‘‘strict partial order’’ in the sense of the paragraph following Exercise 4.1:2; thus the weakened relation ‘‘$\in$ or =’’ will be a partial order. The second condition in the above definition makes that relation a *total* ordering, and by the Regularity Axiom, this will be a well-ordering. (If one does not assume the Regularity Axiom, one adds to the *definition* of ordinal the two conditions we have just deduced using that axiom.)

The ordinals themselves *almost* form a well-ordered set under the relation ‘‘$\in$ or =’’. The only trouble is that they do not form a set! Here are the basic facts.

Proposition 4.5.3. (i) *The class of all ordinals is not a set.*

(ii) *Every member of an ordinal is an ordinal.*

(iii) *If* α *and* β *are ordinals, then the following conditions are equivalent:*

(a) $\alpha = \beta$ *or* $\alpha\in\beta$,

(b) $\alpha \subseteq \beta$.

(iv) *For any two ordinals* α *and* β *one has either* $\alpha \subseteq \beta$ *or* $\beta \subseteq \alpha$, *and every nonempty class of ordinals has a ‘‘$\subseteq$-least’’ member.* (*In other words, the* class *of ordinals satisfies the analog of the set-theoretic property of well-orderedness under* $\subseteq$.) *In particular, every ordinal, and more generally every set of ordinals, is well-ordered under* $\subseteq$.

(v) *If* α *and* β *are ordinals, and* $\alpha \subseteq \beta$, *then* α *is an initial segment of* β. *If it is a proper initial segment, it is also the least element of* β *not in that initial segment.*

(vi) *The union of any set of ordinals is an ordinal.*

(vii) *Every well-ordered set has a unique order isomorphism with an ordinal.*

Proof. We will put off (i) till near the end. (ii) follows immediately from the definition, as does the implication (a)$\Rightarrow$(b) of (iii). To get the reverse implication, suppose the ordinal α is a proper subset of the ordinal β, and let us show that it is a member of β. We observed above that β is well-ordered under ‘‘$\in$ or =’’, so as α is closed under $\in$, it will form an initial segment of β. Let $\gamma\in\beta$ be the least element not belonging to this initial segment. By definition of the ordering of β, the members of γ are the elements smaller than it. But these are the elements of α, so $\alpha = \gamma$, proving (a). Note that we have also proved (v).

To show the first assertion of (iv), we shall show that given two ordinals α and β, their intersection $\gamma = \alpha \cap \beta$ will coincide with one of them. If it did not, it would be a proper initial segment of each, and by (v), it would then be a member of each, namely the first element not belonging to that initial segment. But this would make γ a member of each of α and β but not a member of $\alpha \cap \beta$, an absurdity. To get the final assertion of the first sentence of (iv), let C be a nonempty class of ordinals, take any $\beta \in C$, note that $C' = \{\alpha \subseteq \beta \mid \alpha \in C\}$ is a *set* of ordinals, and apply Regularity to this set. The last sentence of (iv) is immediate.

From the first assertion of (iv) we easily see that the union of a set of ordinals will satisfy the definition of an ordinal, i.e., (vi) holds. We can now get (i): If there were a set of all ordinals, its union would be an ordinal, hence a member of itself, contradicting Regularity.

To show (vii), let S be a well-ordered set. For convenience, let us form a new ordered set T consisting of the elements of S, ordered as in S, and one additional element z, greater than them all. It is immediate that T will again be well-ordered. I claim that for every $t \in T$, there is a unique order-isomorphism between $\{s \mid s < t\}$ and some (unique) ordinal. Indeed, if not, there would be a least t for which this failed, and it is easy to check that the set of ordinals associated with all the elements $< t$ would then be order-isomorphic to $\{s \mid s < t\}$ and would be the unique ordinal with this property, again by a unique isomorphism, contradicting our assumption. In particular, there is a unique order-isomorphism between $\{s \mid s < z\} = S$ and an ordinal, as required. □

Exercise 4.5:3. State the version of our definition of ordinal that one would use if one did not assume the Regularity Axiom, and show how each use of that axiom in the proof of Proposition 4.5.3 could be eliminated if that definition were assumed.

Exercise 4.5:4. Let α and β be ordinals. Show that if there exists a one-to-one isotone map $f: \alpha \to \beta$, then $\alpha \leq \beta$.

Exercise 4.5:5. If P is a partially ordered set with DCC, let the *height* $\mathrm{ht}(p)$ of an element $p \in P$ be defined, recursively, as the least ordinal greater than the height of every element $q < p$, and define the height of P to be the least ordinal greater than the height of every element of P.

(i) Show that the height function is the *least* strict isotone ordinal-valued function on P, and that it has range precisely $\mathrm{ht}(P)$.

(ii) Show that for every ordinal α there exists a partially ordered set containing no infinite chains, and having height α.

(iii) Suppose we define the *chain height* of P, $\mathrm{chht}(P)$, to be the least ordinal which cannot be embedded in P by an isotone map, and $\mathrm{chht}(p)$ for $p \in P$ as $\mathrm{chht}(\{q \in P \mid q < p\})$. What can you establish about the relation between this function and the height function, defined above?

Since one considers ordinals to be ordered under the relation $\subseteq$, equivalently, ‘‘$\in$ or $=$’’, one has the choice, in speaking about them, between writing $\leq$ and $\subseteq$, and likewise between $<$ and $\in$. Both the order-theoretic and the set-theoretic notation are used, sometimes mixed together.

For every ordinal α, there is a least ordinal greater than α, namely $\alpha \cup \{\alpha\}$. This is called the *successor* of α, and written $\alpha + 1$. ‘‘Most’’ ordinals are successor ordinals. Those, such as 0, ω, $\omega 2$, etc., which are not, are called *limit ordinals*. (Although 0 is, as I have just said, logically a limit ordinal, and I will consider it such here, it is sometimes treated as a special case, neither a successor nor a limit ordinal.)

Exercise 4.5:6. Show that an ordinal is a limit ordinal if and only if it is the least upper bound of all strictly smaller ordinals; equivalently, if and only if, as a set, it is the union of all its members.

Now that we understand why ordinals in general, and natural numbers in particular, are defined so that each ordinal is equal to the set of smaller ordinals, let us rescind the convention we set up in §1.3, where, for the sake of familiarity, we said that an n-tuple of elements of a set S would mean a function $\{1, \dots, n\} \to S$:

Definition 4.5.4. *Throughout the remainder of these notes, n-tuples will be defined in the same way as I-tuples for other sets I. That is, for n a natural number, an n-tuple of elements of a set S will mean a function $n \to S$, i.e., a family $(s_0, s_1, \dots, s_{n-1})$ $(s_i \in S)$. The set of all such functions will be denoted S^n.*

We have referred to ordinals denoted by symbols such as $\omega 2$, ω^2+1, etc.. As this suggests, there is an arithmetic of ordinals. If α and β are ordinals, $\alpha+\beta$ represents the ordinal which has an initial segment α, and the remaining elements of which form a subset order-isomorphic to β. This exists, since by putting an order-isomorphic copy of β ''above'' the ordinal α, one gets a well-ordered set, and we know that there is a unique ordinal order-isomorphic to it. Similarly, $\alpha\beta$ represents an ordinal which is composed of a family of disjoint well-ordered sets, each order-isomorphic to α, one above the other, with the order structure of the set of copies being that of β. These operations are (of course) formally defined by recursion, as we will describe below.

Unfortunately, the formalization of recursion that we proved in Lemma 4.3.4 is not quite strong enough for the present purposes, because in constructing larger ordinals from smaller ones, we will not easily be able to give *in advance* a codomain set corresponding to the T of that lemma, and as a result, we will not be able to precisely specify the function r required by that lemma either. However, there is a version of recursion based on the Replacement Axiom (Fraenkel's contribution to Zermelo-Fraenkel set theory) which gets around this problem. Like that axiom, it assumes we are given a construction which is not necessarily a function, because its range and domain are not assumed to be sets, but which nonetheless uniquely determines one element given another. I will not discuss this concept, but will state the result below. The proof is exactly like that of Lemma 4.3.4, except that the Axiom of Replacement is used to carry out the final step of ''piecing together'' the partial functions.

Lemma 4.5.5 (Cf. [**18**, Theorem 7.1.5, p.74])**.** *Let X be a partially ordered set with descending chain condition, and r a construction associating to every pair $(x, f_{<x})$, where $x \in X$ and $f_{<x}$ is a function with domain $\{y \in X \mid y < x\}$, a uniquely defined set $r(x, f_{<x})$. Then there exists a unique function f with domain X such that for all $x \in X$, $f(x) = r(x, f \,|\, \{y \,|\, y < x\})$.* □

We can now define the operations of ordinal arithmetic. For completeness we start with the (nonrecursive) definition of the successor operation. Note that in each of the remaining (recursive) definitions, the ordinal α is taken as ''constant'', and the ordinal over which we are doing the recursion is written β or $\beta+1$.

Definition of the successor of an ordinal:

(4.5.6) $$\beta+1 = \beta \cup \{\beta\}.$$

Definition of addition of ordinals:

(4.5.7) $$\alpha+0=\alpha,\quad \alpha+(\beta+1)=(\alpha+\beta)+1,\quad \alpha+\beta=\bigcup_{\gamma<\beta}\alpha+\gamma \text{ for } \beta \text{ a limit ordinal } >0.$$

Definition of multiplication of ordinals:

(4.5.8) $$\alpha 0=0,\quad \alpha(\beta+1)=(\alpha\beta)+\alpha,\quad \alpha\beta=\bigcup_{\gamma<\beta}\alpha\gamma \text{ for } \beta \text{ a limit ordinal } >0.$$

Definition of exponentiation of ordinals:

(4.5.9) $$\alpha^0=1,\quad \alpha^{(\beta+1)}=(\alpha^\beta)\alpha,\quad \alpha^\beta=\bigcup_{\gamma<\beta}\alpha^\gamma \text{ for } \beta \text{ a limit ordinal } >0.$$

Exercise 4.5:7. Definitions (4.5.7) and (4.5.8) do not look like the descriptions of ordinal addition and multiplication sketched informally above. Show that they do in fact have the properties indicated there.

Although the operations defined above agree with the familiar ones on the finite ordinals (natural numbers), they have unexpected properties on infinite ordinals. Neither addition nor multiplication is commutative:

$$1+\omega = \omega, \quad \text{but} \quad \omega+1 > \omega,$$

$$2\omega = \omega, \quad \text{but} \quad \omega 2 > \omega.$$

Exponentiation is also different from exponentiation of cardinals (discussed later in this section):

$$2^\omega = \omega.$$

Students who have not seen ordinal arithmetic before should do:

Exercise 4.5:8. Prove the three equalities and two inequalities asserted above.

The formulas (4.5.7)-(4.5.9) define *pairwise* arithmetic operations. We can also define arithmetic operations on families of ordinals indexed by ordinals. Let us record the case of addition, since we will need this later. Given $(\alpha_\gamma)_{\gamma\in\beta}$, the idea is to define $\Sigma_{\gamma\in\beta}\,\alpha_\gamma$ to be the ordinal which, as a well-ordered set, is the union of a chain of disjoint subsets of order types α_γ $(\gamma\in\beta)$, in that order.

Definition of infinite ordinal addition:

(4.5.10) $$\Sigma_{\gamma\in 0}\,\alpha_\gamma=0,\quad \Sigma_{\gamma\in\beta+1}\,\alpha_\gamma=(\Sigma_{\gamma\in\beta}\,\alpha_\gamma)+\alpha_\beta,$$
$$\Sigma_{\gamma\in\beta}\,\alpha_\gamma=\bigcup_{\delta<\beta}\Sigma_{\gamma\in\delta}\,\alpha_\gamma \text{ for } \beta \text{ a limit ordinal } >0.$$

Taking the α_γ's all equal, we see that our recursive definition of $\Sigma_\beta\,\alpha_\gamma$ reduces to our definition of multiplication or ordinals; thus

(4.5.11) $$\Sigma_{\gamma\in\beta}\,\alpha = \alpha\beta.$$

Exercise 4.5:9. (i) Given an ordinal-indexed family of ordinals, $(\alpha_\gamma)_{\gamma\in\beta}$, let δ denote the ordinal $\bigcup_{\gamma\in\beta}\alpha_\gamma$ (the supremum of the α_γ's). Let P be the set $\beta\times\delta$, lexicographically ordered. Show that the ordinal $\Sigma_{\gamma\in\beta}\,\alpha_\gamma$ is isomorphic as a well-ordered set to $\{(\gamma,\varepsilon)\mid \varepsilon\in\alpha_\gamma\}\subseteq P$.

(ii) Deduce from this a description of a well-ordered set isomorphic to the ordinal product $\alpha\beta$ of two arbitrary ordinals.

This description clearly extends inductively to finite products $\Pi_{\gamma\in\beta}\,\alpha_\gamma$ $(\beta<\omega)$, leading, incidentally, to an easy proof of *associativity* of multiplication of ordinals. The extension of these ideas to infinite products will be developed in a later exercise in this section.

We have seen that every well-ordered set is indexed in a canonical way by an ordinal, but we do not yet know whether we can well-order every set. It turns out that we can do so if we assume the Axiom of Choice. This is stated in the second point of the next lemma; the first point gives a key argument (not requiring the Axiom of Choice) used in the proof.

Lemma 4.5.12. *Let X be a set. Then*

(i) *There exists an ordinal α which cannot be put in bijective correspondence with any subset of X; equivalently, such that for any well-ordering ''$\leq$'' of any subset of $Y \subseteq X$, $(Y, \leq)$ is isomorphic to a* proper *initial segment of α.*

(ii) *Assuming the Axiom of Choice, X itself can be well-ordered.*

Proof. The class of well-orderings of subsets of X is easily shown to be a set, hence by the Replacement Axiom, the (unique) ordinals isomorphic to these various well-ordered sets form a set, hence the union of this set is an ordinal β. Take $\alpha = \beta + 1$. By construction, any well-ordering of a subset of X induces a bijection with an initial segment of β, which is a proper initial segment of α, yielding the second formulation of (i). To get the first formulation of (i), note that if α could be put in bijective correspondence with a subset of X, then the ordering of α would induce a well-ordering of that subset, such that α was the unique ordinal isomorphic to that well-ordered set, giving a contradiction to our preceding conclusion.

Assuming the Axiom of Choice, let us now take a function c which associates to every nonempty subset $Y \subseteq X$ an element $c(y) \in Y$. Let us recursively construct a one-to-one map from some initial subset of the ordinal α of part (i) into X as follows: Suppose we have gotten a function f_β from a proper initial segment $\beta \subset \alpha$ into X. If its image is X, we are done. If not, we map the next element of α to $c(X - \mathrm{image}(f_\beta))$. It is easy to verify by induction that each map f_β is one-to-one. If this process went on to give a one-to-one map f_α of α into X, that would contradict (i). So instead, the construction must terminate at some step, which means we must get a bijection between an initial segment of α and X, and hence a well-ordering of X, proving (ii). (As in the proof of Lemma 4.5.1, our use of a recursion that terminates before we get through all of α can be formalized by adjoining to X an element DONE.) □

Let us assume the Axiom of Choice for the rest of this section (though at the beginning of the next section, we will again suspend this assumption).

Recall that two sets are said to have the same *cardinality* if they can be put in bijective correspondence. We have shown that (assuming the Axiom of Choice), every set has the same cardinality as an ordinal. This means we can use an appropriately chosen ordinal as a ''standard example'' of each cardinality. In general, there are more than one ordinals of a given cardinality (e.g., ω, $\omega+1$, $\omega 2$ and ω^2 are all countable), so the ordinal to use is not uniquely determined. The obvious choice is that of the *least* ordinal of the given cardinality; so one makes

Definition 4.5.13. *A* cardinal *is an ordinal which cannot be put into bijective correspondence with a proper initial segment of itself.*

For any set X, the least ordinal with which X can be put in bijective correspondence will be denoted $\mathrm{card}(X)$. *Thus, this is a cardinal, and is the only cardinal with which X can be put in bijective correspondence.*

There is an arithmetic of cardinals: If κ and λ are cardinals, $\kappa+\lambda$ is defined as the cardinality of the union of any two disjoint sets one of which has cardinality κ and the other cardinality λ, $\kappa\lambda$ as the cardinality of the direct product of a set of cardinality κ and a set of cardinality λ, and κ^{λ} as the cardinality of the set of all functions from a set of cardinality λ to a set of cardinality κ. Unfortunately, if we consider the class of cardinals as a subset of the ordinals, these are different operations from the *ordinal arithmetic* we have just defined! To compare these arithmetics, let us temporarily use the notations $\alpha +_{\mathrm{ord}} \beta$, $\alpha \cdot_{\mathrm{ord}} \beta$ and $\alpha^{\mathrm{ord}\,\beta}$ for ordinal operations, and $\kappa +_{\mathrm{card}} \lambda$, $\kappa \cdot_{\mathrm{card}} \lambda$ and $\kappa^{\mathrm{card}\,\lambda}$ for cardinal operations. A positive statement we can make is that for cardinals κ and λ, the computation of their cardinal sum and product can be reduced to that of their ordinal sum and product, by the formulas

(4.5.14) $\qquad \kappa +_{\mathrm{card}} \lambda \ = \ \mathrm{card}(\kappa +_{\mathrm{ord}} \lambda) \qquad$ and $\qquad \kappa \cdot_{\mathrm{card}} \lambda \ = \ \mathrm{card}(\kappa \cdot_{\mathrm{ord}} \lambda).$

These are, in fact, cases of an equality true for any family of ordinals $(\alpha_\gamma)_{\gamma\in\beta}$:

(4.5.15) $$\Sigma^{\mathrm{card}}_{\gamma\in\beta}\ \mathrm{card}(\alpha_\gamma) \ = \ \mathrm{card}(\Sigma^{\mathrm{ord}}_{\gamma\in\beta}\, \alpha_\gamma).$$

On the other hand, the cardinality of an *infinite ordinal product* of ordinals is not in general equal to the cardinal product of the cardinalities of these ordinals; in particular, cardinal exponentiation does not in any sense agree with ordinal exponentiation: $2^{\mathrm{card}\,\omega}$ gives the cardinality of the continuum, which is uncountable, while $2^{\mathrm{ord}\,\omega} = \omega$. There is no standard notation for distinguishing ordinal and cardinal arithmetic; authors either introduce ad hoc notations, or say in words whether cardinal or ordinal arithmetic is meant, or rely on context to show this.

Exercise 4.5:10. In this exercise we shall extend the results of Exercise 4.5:9, which characterized the order-types of general sums and finite products of ordinals, to general products. (I have put this off until now mainly so that we would have the notation to distinguish the ordinal product $\Pi^{\mathrm{ord}}\, \alpha_\gamma$ from the set-theoretic product.) We will also note at the end a relation with cardinal arithmetic. We need to begin with a generalization of lexicographic ordering.

Suppose $(X_i)_{i\in I}$ is a family of partially ordered sets, indexed by a totally ordered set I; and let each X_i have a distinguished element, denoted 0 (or 0_i if there is danger of ambiguity). Define the *support* of $(x_i)\in\Pi_I\, X_i$ as $\{i\in I \mid x_i \neq 0\}$, and let $\Pi_I^{\mathrm{w.o.s.}}\, X_i$ denote the set of elements of $\Pi_I\, X_i$ having *well-ordered* support. Similarly, let $\Pi_I^{\mathrm{f.s.}}\, X_i$ denote the set of elements of *finite* support.

(i) Show that lexicographic order, which in Definition 4.3.7 was defined on $\Pi_I\, X_i$ only for I well-ordered, may be defined on $\Pi_I^{\mathrm{w.o.s.}}\, X_i$ for arbitrary totally ordered I, and that the resulting ordering is total if each X_i is totally ordered.

(ii) Show that if I is reverse-well-ordered (has ascending chain condition) and if each X_i has descending chain condition, and has 0 as *least* element, then $\Pi_I^{\mathrm{f.s.}}\, X_i$ has descending chain condition under lexicographic ordering.

(iii) Let us now be given an ordinal-indexed family of ordinals, $(\alpha_\gamma)_{\gamma\in\beta}$. Write down the definition of $\Pi^{\mathrm{ord}}_{\gamma\in\beta}\, \alpha_\gamma$ analogous to (4.5.10). Verify that if any α_γ is 0, your definition gives the ordinal 0. In the contrary case, show that this ordinal is isomorphic to $\Pi^{\mathrm{f.s.}}_{\gamma\in\beta^{\mathrm{op}}}\, \alpha_\gamma$. (Here β^{op} denotes the set β, but with its ordering – used in defining lexicographic order on our product – reversed. Note that ‘‘$\gamma\in\beta^{\mathrm{op}}$’’ means the same as ‘‘$\gamma\in\beta$’’. For the elements 0 in the definition of $\Pi^{\mathrm{f.s.}}$, we take the ordinal $0\in\alpha_\gamma$, which is why we need to assume all α_γ nonzero.)

(iv) Deduce a description of the order-type of $\alpha^{\mathrm{ord}\,\beta}$, and conclude that $\mathrm{card}(\alpha^{\mathrm{ord}\,\beta}) \leq$

$\alpha^{\operatorname{card}\beta}$.

You might also want to do

(v) Show by examples that (ii) above fails if any of the three hypotheses is deleted.

The concept of cardinality historically antedates the construction of the ordinals, so there is a system of names for cardinals independent of their names as ordinals. The finite cardinals are, of course, denoted by the traditional symbols 0, 1, ... The least infinite cardinal is denoted $\aleph_0$, the next $\aleph_1$, etc.. From our description of the cardinals as a subclass of the ordinals, we see that the class of cardinals is ''well-ordered'' (which we write in quotes, as we did for the class of ordinals, because this class is not a set). Hence today, having the concept of ordinal, one continues the above set of symbols using ordinal subscripts: The αth cardinal after $\aleph_0$ is written $\aleph_\alpha$.

There is a further notation for cardinals ''regarded as ordinals''. Each $\aleph_\alpha$, regarded as an ordinal, is written ω_α. Thus one writes $\aleph_0 = \omega_0 = \omega$, $\aleph_1 = \omega_1$, etc..

Let us recall, without repeating the proofs here, some well-known properties of cardinal arithmetic, though we will use them only occasionally.

Theorem 4.5.16. *Letting* κ, λ, *etc., denote cardinals, and letting arithmetic notation denote* cardinal arithmetic*, the following statements are true.*

(i) *For all* κ, λ, μ,

$$\kappa+\lambda = \lambda+\kappa, \quad \kappa\lambda = \lambda\kappa, \quad (\kappa+\lambda)\mu = \kappa\mu+\lambda\mu, \quad \kappa^{\lambda+\mu} = \kappa^{\lambda}\kappa^{\mu}, \quad \kappa^{\lambda\mu} = (\kappa^{\lambda})^{\mu}.$$

(ii) *For sets* X_i $(i\in I)$, $\operatorname{card}(\bigcup_I X_i) \leq \Sigma \operatorname{card}(X_i)$.

(iii) *If* $\kappa_\beta \leq \lambda_\beta$ *for all* $\beta\in\alpha$, *then*

$$\Sigma_\alpha \kappa_\beta \leq \Sigma_\alpha \lambda_\beta, \qquad \kappa_0\kappa_1 \leq \lambda_0\lambda_1, \qquad \textit{and, if } \kappa_0 > 0, \quad \kappa_0^{\kappa_1} \leq \lambda_0^{\lambda_1}.$$

(iv) *If* $\kappa \leq \lambda$ *and* λ *is infinite, then* $\kappa+\lambda = \lambda$. *If also* $\kappa > 0$, *then* $\kappa\lambda = \lambda$. *In particular,* $\omega\omega = \omega$, *hence by* (ii) *and* (iii)*, a countable union of countable sets is countable.*

(v) $2^\kappa > \kappa$. *Equivalently, the power set of any set* X *has strictly larger cardinality than* X.

Proof. See [**26**, pp. 17-21], or [**28**, appendix 2, §1 and exercises at the end of that appendix]. □

It is interesting that while the statement $\omega\omega = \omega$ is easy to prove without the Axiom of Choice (by describing an explicit bijection), its consequence, ''a countable union of countable sets is countable'', requires that axiom, to enable us to choose bijections between the infinitely many given countable sets, and the set ω.

Turning from arithmetic back to order properties, let me define a concept of interest in the general study of ordered sets, and note a specific application to cardinals.

Definition 4.5.17. *If* X *is a partially ordered set, then the* cofinality *of* X *means the least cardinality of a cofinal subset* $Y \subseteq X$ (*Definition 4.1.6*).

A cardinal κ *is called* regular *if, as an ordinal, it has cofinality* κ. *A cardinal that is not regular is called* singular.

Exercise 4.5:11. Show that if a partially ordered set X has cofinality κ, then every cofinal subset $Y \subseteq X$ also has cofinality κ.

Exercise 4.5:12. Prove:

(i) Every cardinal of the form $\aleph_{\alpha+1}$ (i.e., every cardinal indexed by a successor ordinal) is regular.

(ii) The first infinite singular cardinal is $\aleph_\omega$.

The next exercise examines the class of regular cardinals within the class of ordinals.

Exercise 4.5:13. Let us call an ordinal α regular if there is no set map from an ordinal $<\alpha$ onto a cofinal subset of α.

(i) Show that regular ordinals are ''sparse'', by verifying that the only regular ordinals are 0, 1, and the regular infinite *cardinals*.

(ii) On the other hand, we saw in point (i) of the preceding exercise that within the set of infinite cardinals, the singular cardinals are sparse: They must be limit cardinals, i.e., cardinals ω_α such that α is a limit ordinal. (Nothing for you to do here – this point is numbered (ii) only to show the sequence.)

(iii) Show that among the *limit* cardinals, *regular* cardinals are again sparse, by showing that if ω_α is regular and α is a limit ordinal, then α must be a cardinal; in fact, 0 or a cardinal κ satisfying

$$\kappa = \omega_\kappa.$$

Show that the first cardinal κ satisfying that equation is the supremum of the chain $\kappa(i)$ $(i\in\omega)$ defined by $\kappa(0) = 0$, $\kappa(i+1) = \omega_{\kappa(i)}$, but that this cardinal is still *not* regular.

(Regular limit cardinals will come up again in §6.4.)

4.6. Zorn's Lemma. Ordinals, together with the Axiom of Choice, give a powerful tool for constructing non-uniquely-determined objects in most areas of mathematics. Consider the following approach to such constructions:

Hoping to construct a certain kind of object, one considers ''partial constructions''. One verifies that these form a set; hence there exists an ordinal α of greater cardinality than that of this set. One then recursively maps an initial segment of α into the set of partial constructions. The setting up of this recursion involves three tasks:

(i) Getting an ''initializing'' partial construction to which to map 0.

(ii) Specifying what to do at a successor ordinal: If one has built up one's partial construction through the stage indexed by α, and it is still not ''finished'', one shows that it can be extended further, to give an $\alpha+1$'st stage. The Axiom of Choice lets one choose, for each ''unfinished'' construction, an extension to use.

(iii) Specifying what to do at a nonzero limit ordinal α. In this case, one has a chain of preceding partial constructions each extending the one before, and it is usually easy to verify that their ''union'' (in the appropriate sense) is a (possibly partial) construction extending all of them.

Now note that if the resulting recursion did not lead to a ''finished'' construction at some step, one would get a one-to-one map from α into the set of partial constructions, contradicting the choice of α. Hence a finished construction must be obtained at some stage, as desired!

(Example: To show that an arbitrary vector space V has a basis B, one considers as ''partial constructions'' arbitrary linearly independent subsets of V. One can begin with the linearly independent subset $B_0 = \varnothing$. If the subset B_α one has obtained at a given stage does not span V, there will be some element $v\in V$ outside the span of B_α, and we can take $B_{\alpha+1}$ to be $B_\alpha \cup \{v\}$, and verify that this is linearly independent. If α is a limit ordinal, we let B_α be the set-theoretic union of the chain of subsets B_β $(\beta\in\alpha)$, and verify linear independence for this set. The preceding argument then shows that we will eventually get a linearly independent subset which cannot be extended, i.e., which spans V, as desired.)

In view of the ubiquity of this pattern, it is natural to look for a lemma that will do the repetitious part once and for all, and show us what needs to be proved separately for each case. In formulating this lemma, let us render the set of all ''partial constructions'' by a partially ordered set $(X, \leq)$, where $\leq$ is thought of as the relation of one construction being a ''part of'' another. The condition we need to initialize our recursion ((i) above) is that X be nonempty. To say that we can extend a partial construction further if it is not yet ''finished'' ((ii) above) is to say that if X has any *maximal* element, this is an object of the sort we desire. Finally, the condition we need to be able to continue at steps indexed by limit ordinals, namely, that given a chain of partial constructions we can pass to one which includes them all ((iii) above), is made the content of a definition:

Definition 4.6.1. *A partially ordered set* X *is called* inductive *if for every nonempty chain* $Y\subseteq X$, *there is an element* $z\in X$ *majorizing* Y (*i.e.,* $\geq$ *all elements of* Y).

We can now state the desired result, *Zorn's Lemma*, and show that it and a number of other statements are equivalent to the Axiom of Choice.

Theorem 4.6.2. *Assuming the axioms of Zermelo-Fraenkel set theory* (*but not the Axiom of Choice*), *the following four statements are equivalent:*

(i) **The Axiom of Choice:** *If* X *is a set, and* f *is a function associating to every* $x\in X$ *a nonempty set* $f(x)$, *then there exists a function* g *associating to every* $x\in X$ *an element* $g(x)\in f(x)$. (*Equivalently: the direct product of any family of nonempty sets is nonempty.*)

(ii) **Zorn's Lemma:** *Every nonempty inductive partially ordered set* $(X, \geq)$ *has a maximal element.*

(iii) **The Well-ordering Principle:** *Every set can be well-ordered.* (*Equivalently: every set can be put in bijective correspondence with an ordinal.*)

(iv) **Comparability of Cardinalities:** *Given any two sets* X *and* Y, *one of the sets can be put in bijective correspondence with a subset of the other.* (*Loosely: the class of* cardinalities *is totally ordered.*)

Proof. The scheme of proof will be (iv)$\Leftrightarrow$(iii)$\Leftrightarrow$(i)$\Leftrightarrow$(ii). That the parenthetical restatement of (iii) is equivalent to the main statement follows from Proposition 4.5.3(vii).

(iv)$\Leftrightarrow$(iii): Assuming (iv), let X be any set and α an ordinal with the property stated in the first assertion of Lemma 4.5.12. By (iv), there is either a bijection between X and a subset of α, or vice versa. By choice of α, the latter case cannot occur, so there is a bijection between X and a subset $S\subseteq\alpha$. Since α is well-ordered, so is every subset, and the well-ordering of S induces a well-ordering of X, proving (iii). Assuming (iii), statement (iv) follows from the comparability of ordinals (Proposition 4.5.3(iv)).

(iii)$\Leftrightarrow$(i): We proved (i)$\Rightarrow$(iii) in Lemma 4.5.12. Conversely, assume (iii). Given X and f as in (i), statement (iii) tells us that we can find a well-ordering $\leq$ on the set $\bigcup_{x\in X} f(x)$. We now define g to take each x to the $\leq$-least element of $f(x)$. (In terms of the axioms, we are using the Replacement Axiom to construct $\{(x, y) \mid x\in X$ and y is the least element of $f(x)\}$. This set of pairs, regarded as a function, is the desired g.)

(i)⇒(ii): Let $(X, \leq)$ be a nonempty inductive partially ordered set, and let us choose as in Lemma 4.5.12 an ordinal α which cannot be put in bijective correspondence with any subset of X. Note that the combination of conditions ''inductive'' and ''nonempty'' is equivalent to saying that for *every* chain $C \subseteq X$, including the empty chain, there is an element $\geq$ all members of C. By (i), we may choose a function g associating to every nonempty subset of X a member. We will now recursively define an isotone map $f : \alpha \to X$. Assuming that for some $\beta \in \alpha$ we have defined an isotone map $f_{<\beta} : \beta \to X$, observe that its image will be a chain $C_\beta \subseteq X$. If the set Y_β of elements of X greater than all members of C_β is nonempty, we define $f(\beta) = g(Y_\beta)$. In the contrary case, the hypothesis that X is inductive still tells us that there is an element $\geq$ all members of C_β. We conclude that such an element must be equal to some member of C_β, which means that the chain has a largest element, c. In this case, we take $f(\beta) = c$. Note that in this case c must be maximal in X, for if not, any element of X greater than it would be greater than all elements of C_β, contradicting our assumption that Y_β was empty.

By choice of α, the map f we have constructed cannot be one-to-one, but by the nature of our construction, the only situation in which one-one-ness fails is if at some point we get a maximal element of X. Thus X has a maximal element, as claimed.

(ii)⇒(i): This will be a typical application of Zorn's Lemma. Let X and f be given as in (i). Let P be the set of all maps defined on *subsets* $Y \subseteq X$ and carrying each $x \in Y$ to an element of $f(x)$. Partially order P by setting $g_1 \geq g_0$ if g_1 is an *extension* of the map g_0. P is nonempty because it contains the empty mapping; it is easy to see that given any chain C of elements of P under the indicated partial ordering, the union of C will be an element of P that is $\geq$ all elements of C, hence P is inductive. Thus it has a maximal element g. This maximal element must be a function defined on all of X (otherwise we could extend it further), completing the proof of (i). □

Convention 4.6.3. *Throughout the remainder of these notes, we shall assume the Axiom of Choice along with the other axioms of ZFC, and thus freely use any of the equivalent statements of the preceding theorem.*

Of these equivalent formulations, Zorn's Lemma is usually the most convenient.

Note that in the last paragraph of the above proof, our verification that P was nonempty was by the same method used to show that every nonempty chain had an upper bound: To show the latter, we used the union of the chain, while to get an element of P we took the empty function, which is the union of the empty chain. It is my experience that in *most* proofs by Zorn's Lemma, the verification of nonemptiness may be achieved by the same construction that shows every *nonempty* chain has an upper bound; i.e., the assumption ''nonempty'' is not really used in that verification. Hence my personal preference would be to use a definition of ''inductive'' that required *every* chain to have an upper bound, and eliminate ''X nonempty'' as a separate hypothesis of Zorn's Lemma. (Of course, in some exceptional cases, the verification that all chains have upper bounds may have to treat empty and nonempty chains separately. But in fact, I notice that even in the common situation where the same verification works for both cases, many authors are apparently embarrassed to use the most trivial example to show the set is nonempty, and unnecessarily give a more complicated one instead.) For conformity with common usage, I have stated Zorn's Lemma above in terms of the standard definition of ''inductive''. But we may, at times, skip a separate verification that our inductive set is nonempty, and instead observe that some construction gives an upper bound for any chain, empty or nonempty.

Exercise 4.6:1. We saw in Exercise 4.1:10 that the maximal partial orderings on a set X were the total orderings. Deduce now for arbitrary X (as we were able to deduce there for finite X) that

(i) Every partial ordering can be extended to a total ordering.

(ii) Every partial ordering is an intersection of total orderings.

Exercise 4.6:2. (i) If X is a totally ordered set, show that X has a subset Y well-ordered under the induced ordering, and cofinal in X (Definition 4.1.6).

(ii) Show that the Y of (i) can be taken order-isomorphic to a regular cardinal (Exercise 4.5:12), and that this cardinal is unique. However show that, even under these conditions, the set Y is not in general unique, and that if the condition of regularity is dropped, uniqueness of the cardinal is also lost.

(iii) Prove that every partially ordered set has a cofinal subset with descending chain condition.

Exercise 4.6:3. For a partially ordered set $X,$ show that the following conditions are equivalent:

(i) X has no maximal element.

(ii) X has two disjoint cofinal subsets.

The next exercise is an example where the ''obvious'' Zorn's Lemma proof does not work. The simplest valid proof in this case is by the well-ordering principle, which is not surprising since it is a result about well-orderability. However, this can also be turned into a Zorn's Lemma proof, if one is careful.

Exercise 4.6:4. Let X be a set, let P be the set of partial order relations on $X,$ partially ordered by inclusion as in Exercise 4.1:10, and let $Q \subseteq P$ consist of those partial orderings having descending chain condition.

(i) Show that the maximal elements of Q (under the partial ordering induced from P) are the well-orderings of $X.$

(ii) Show that Q is *not* inductive.

(iii) Prove nonetheless that every element of Q is majorized by a maximal element, and deduce that every partial ordering with DCC on a set X is an intersection of well-orderings. (Hint: Take an appropriate ordinal α and construct an indexing of the elements of X by an initial segment of $\alpha,$ in a way ''consistent'' with the partial order.)

The next three exercises, though not closely related to Zorn's Lemma, explore further the relation between partially ordered sets and their well-ordered subsets.

Exercise 4.6:5. Let S be an infinite set, and $\mathbf{P}(S)$ the set of all subsets of $S,$ partially ordered by inclusion. Show by example that $\mathbf{P}(S)$ can contain chains of cardinality $> \mathrm{card}(S),$ but prove that $\mathbf{P}(S)$ can never contain a well-ordered chain of cardinality $> \mathrm{card}(S).$

Exercise 4.6:6. (i) Show that every infinite *totally* ordered set has either a subset order-isomorphic to ω or a subset order-isomorphic to $\omega^{\mathrm{op}}.$

(ii) Show that every infinite partially ordered set P contains either a subset order-isomorphic to $\omega,$ a subset order-isomorphic to $\omega^{\mathrm{op}},$ or an infinite antichain (Definition 4.1.6). (Suggestion: If P has no infinite antichain, obtain a finite antichain $B \subseteq P$ maximal for the property that the set S of elements incomparable with all elements of B is infinite; then study the properties this S must have. Alternatively, do the same thing with the roles of comparable and incomparable elements reversed.)

This family of three partially ordered sets is essentially unique for the above property:

(iii) Show that a set F of infinite partially ordered sets has the property that every infinite partially ordered set contains an isomorphic copy of a member of F if and only if F contains a partially ordered set order-isomorphic to $\omega,$ a partially ordered set order-isomorphic to $\omega^{\mathrm{op}},$

and a countable antichain.

An application of the preceding exercise is

Exercise 4.6:7. Let P be a partially ordered set.

(i) Show that the following conditions are equivalent:

(i.a) P contains no chains order-isomorphic to ω^{op}.

(i.b) Every infinite subset of P contains either a subset order-isomorphic to ω, or an infinite antichain.

(i.c) P satisfies the descending chain condition.

(ii) It is clear from (i) above that conditions (ii.a)-(ii.c) below are equivalent. Show that they are also equivalent to (ii.d):

(ii.a) P contains no chains order-isomorphic to ω^{op}, and no infinite antichains.

(ii.b) Every infinite subset of P contains a subset order-isomorphic to ω.

(ii.c) P has descending chain condition, and contains no infinite antichains.

(ii.d) Every total ordering extending the ordering of P is a well-ordering.

A partially ordered set P with the equivalent properties of (ii) is sometimes called ''partially well-ordered''.

The first part of the next exercise notes that for uncountable cardinalities, things are more complicated.

Exercise 4.6:8. (i) Deduce from Exercise 4.6:5 that one can have a totally ordered set P of some infinite cardinality κ which contains no well-ordered or reverse-well-ordered subset of cardinality κ.

(ii) Suppose P is as in (i), and φ is a bijection between P and a well-ordered set Q of cardinality κ. Consider $\{(p, \varphi(p)) \mid p \in P\}$, under the partial ordering induced by the product ordering on $P \times Q$. Show that this has neither chains nor antichains of cardinality κ (in contrast to the result of Exercise 4.1:9 for finite partially ordered sets).

But perhaps one can repair this deficiency. (I have not thought hard about the next question.)

(iii) Exercise 4.1:9 was based on defining the ''height'' of a partially ordered set as the supremum of the cardinalities of its chains; but a different concept of ''height'' was introduced for partially ordered sets with descending chain condition in Exercise 4.5:5. Can this definition be extended in some way to general partially ordered sets, or otherwise modified, so as to get an analog of Exercise 4.1:9 for partially ordered sets of arbitrary cardinality? (Or can the definition of ''width'' be so modified?)

For a curious application of the well-ordering principle to the study of abelian groups, see the first section of [**36**].

4.7. Some thoughts on set theory. I have mentioned that when the Axiom of Choice and various equivalent principles were first considered, they were the subject of a heated controversy.

The Axiom of Choice is now known to be *independent* of the other axioms of set theory; i.e., it has been proved that, assuming the consistency of the Zermelo-Fraenkel axioms without Choice, both the full set of axioms including Choice, and the Zermelo-Fraenkel axioms plus the *negation* of the Axiom of Choice are consistent. And there are further statements (for instance the Continuum Hypothesis, saying that $2^{\aleph_0} = \aleph_1$) which have been shown independent of Zermelo-Fraenkel set theory *with* the Axiom of Choice, and which there do not seem to be any reasons either for accepting or rejecting. This creates the perplexing question of what is the ''true'' set theory.

Alongside Zermelo-Fraenkel Set Theory with and without Choice, etc., there are further contenders for the ‘‘correct’’ foundations of mathematics. The Intuitionists objected not only to the Axiom of Choice, but to the ‘‘law of the excluded middle’’, the logical principle that every meaningful statement is either true or false. They claimed (if I understand correctly) that an assertion such as Fermat’s Last Theorem (the statement that there are no nontrivial integer solutions to $x^n + y^n = z^n$, $n > 2$, which was unproven at the time) could be said to be false if a counterexample were found, or true if an argument could be found (using forms of reasoning acceptable to them) that proved it, but that it would be neither true nor false if neither a counterexample nor a proof existed. They maintained that the application of the law of the excluded middle to statements which involve infinitely many cases, and which thus cannot be checked case by case, was a fallacious extension to infinite sets of a method correct only for finite sets; in their words, that one cannot so reason about an infinite set such as the set of natural numbers, because it cannot be regarded as a ‘‘completed totality’’.

Although this viewpoint is not current, note that the distinction between sets and proper classes, which got mathematics out of the paradoxes that came from considering ‘‘the set of all sets’’, leaves us wondering whether the *class* of all sets is ‘‘a real thing’’; and indeed one current textbook on set theory refers to this in terms of the question of whether mathematicians can consider such classes as ‘‘completed totalities’’.

During a painfully protracted correspondence with someone who insisted he could show that Zermelo-Fraenkel set theory was inconsistent, and that the fault lay in accepting infinite sets, which he called ‘‘mere phantasms’’, I was forced to think out my own view of the matter, and the conclusion I came to is that all sets, finite and infinite, are ‘‘phantasms’’; that none of mathematics is ‘‘real’’, so that there is no true set theory; but that this does not invalidate the practice of mathematics, or the usefulness of choosing a ‘‘good’’ set theory.

To briefly explain this line of thought, let us understand the physical world to be ‘‘real’’. (If your religious or philosophical beliefs say otherwise, you can nevertheless follow the regression to come.)

Our way of perceiving the world and interacting with it leads us to partition it into ‘‘objects’’. This partitioning is convenient, but is not a ‘‘real thing’’.

To deal intelligently with objects, we think about families of objects, and, as our thinking gets more sophisticated, families of such families. Though I do not think the families, and families of families, are ‘‘real things’’, they are useful, as descriptions of the way we classify the world.

Consider in particular our system of numbers, which are themselves not ‘‘real things’’, but which give models that allow us to use one coherent arithmetic system to deal with the various things in the world that one can count. Note that in spite of this motivation in terms of things one can count, in developing the numbers we use a system that is *not* bounded by the limitations of how high a person could count in a lifetime. A system with such a limitation arbitrarily imposed would be *more* difficult to define, learn, and work with than our system, in which the behavior of arithmetic is uniform for arbitrarily large values! Moreover, our unbounded system turns out to have applications to situations that a system bounded in that way would not be able to deal with: to demographic, geographical, astronomical and other data, which we compute from observations and theoretical models of our world, though no one human being could have counted the numbers involved unit by unit.

Now in thinking about our system of numbers, we are dealing with the concept of ‘‘all the numbers in the system’’ – even those who refuse to call that family a ‘‘completed totality’’ do reason about it – so, if possible, we want our set theory to be able to cover such concepts. Just as we found it natural to extend the system of numbers beyond the sizes of sets a real person could count, so we may extend our system of ‘‘sets’’ beyond finite sets. This is not as simple as with the number-concept. Some plausible approaches turned out to lead to contradictions, e.g., those that allowed one to speak of ‘‘the set of all sets’’. Among those approaches that do not lead to contradictions, some are more convenient than others. I think we are justified in choosing a more convenient system to work in – one in which the ‘‘unreal objects’’ that we are considering are easier to understand and generalize about.

It may seem pointless to work in a set theory which is to some extent ‘‘arbitrary’’, and to which we do not ascribe absolute ‘‘truth’’. But observe that as long as we use a system consistent with the laws of finite arithmetic, any statements we can prove in our system about arithmetic models of aspects of the real world, and which can in principle be confirmed or disproved in each case by a finite calculation, will be correct; i.e., as valid as those models are. This is what I see as the ‘‘justification’’ for including the Axiom of Choice and other convenient axioms in our set theory.

Note also that making one choice among set theories or systems of reasoning does not consign all others to oblivion. Logicians *do* consider which statements hold if the Axiom of Choice is assumed and which hold if its negation is assumed. (E.g., [**63**] shows that in a model of ZF without Choice, one can have commutative rings with properties contradicting several standard theorems of ZFC ring theory.) Even intuitionistic logic is still studied – not, nowadays, as a preferred mode of reasoning, but as a formal system, related to objects called Brouwerian lattices (cf. [**4**]) in the same way standard logic is related to Boolean algebras.

Chapter 5. Lattices, closure operators, and Galois connections.

5.1. Semilattices and lattices. Many of the partially ordered sets P we have seen have a further valuable property: that for any two elements of P, there is a least element $\geq$ both of them, and a greatest element $\leq$ both of them, i.e., a *least upper bound* and a *greatest lower bound* for the pair. In this section we shall study partially ordered sets with this property. To get a better understanding of the subject, let us start by looking separately at the properties of having least upper bounds and of having greatest lower bounds.

Recall that an element x is said to be *idempotent* with respect to a binary operation $*$ if $x*x = x$. The binary operation $*$ itself is often called idempotent if $x*x = x$ holds for all x.

Lemma 5.1.1. *Suppose X is a partially ordered set in which every two elements $x, y \in X$ have a* least upper bound*; that is, such that there exists a least element which majorizes both x and y. Then if we write this least upper bound as $x \vee y$, and regard $\vee$ as a binary operation on X, this operation will satisfy the identities*

$$
\begin{array}{rll}
(\forall x) & x \vee x = x & (\textit{idempotence}),\\
(\forall x, y) & x \vee y = y \vee x & (\textit{commutativity}),\\
(\forall x, y, z) & (x \vee y) \vee z = x \vee (y \vee z) & (\textit{associativity}).
\end{array}
$$

Conversely, given a set X with a binary operation $\vee$ satisfying the above three identities, there is a unique partial order relation $\leq$ on X for which $\vee$ is the least upper bound operation. This relation $\leq$ may be recovered from the operation $\vee$ in two ways: It can be constructed as

$$\{(x, x \vee y) \mid x, y \in X\},$$

or characterized as the set of elements satisfying an equation:

$$\{(x, y) \mid y = x \vee y\}. \quad \square$$

Exercise 5.1:1. Prove the non-obvious part of the above lemma, namely that every idempotent commutative associative binary operation on a set arises from a partial ordering with least upper bounds. Show that uniqueness of this partial ordering follows from one of the ‘‘straightforward’’ parts of the lemma.

Hence we make

Definition 5.1.2. *An* upper semilattice *means a pair $S = (|S|, \vee)$, where $|S|$ is a set, and $\vee$ (read ‘‘join’’) is an idempotent commutative associative binary operation on $|S|$. Loosely, the term ‘‘upper semilattice’’ will also be used for the equivalent structure of a partially ordered set in which every pair of elements has a least upper bound.*

Given an upper semilattice $(|S|, \vee)$, we shall consider $|S|$ as partially ordered by the unique ordering which makes $\vee$ the least upper bound operation (characterized in two equivalent ways in the above lemma). The set $|S|$ with this partial ordering is sometimes called the ‘‘underlying partially ordered set’’ of the upper semilattice S.

The join of a finite nonempty family of elements x_i $(i \in I)$ (which by the associativity and commutativity of the join operation $\vee$ makes sense without specification of an order or bracketing *for the elements, and which is easily seen to give the least upper bound of* $\{x_i\}$ *in the natural partial ordering*) *is denoted* $\bigvee_{i\in I} x_i$.

The confusion caused by the symmetry of the partial order concept is now ready to rear its head! Observe that in a partially ordered set in which every pair of elements x, y has a *greatest lower bound* $x\wedge y$, the operation $\wedge$ will also be idempotent, commutative and associative (it is simply the operation $\vee$ for the opposite partially ordered set), though the partial ordering is recovered from it in the opposite way, by defining $x\leq y$ if and only if x can be written $y\wedge z$, equivalently, if and only if $x = x\wedge y$. We have no choice but to make a formally identical definition for the opposite concept (first half of the first sentence below):

Definition 5.1.3. *A* lower semilattice *means a pair* $S = (|S|, \wedge)$, *where* $|S|$ *is a set and* $\wedge$ (*read ''meet''*) *is an idempotent commutative associative binary operation on* $|S|$; *or loosely, the equivalent structure of a partially ordered set in which every pair of elements has a greatest lower bound. If* $(|S|, \wedge)$ *is such a pair, regarded as a lower semilattice, then* $|S|$ *will be considered partially ordered in the unique way which makes* $\wedge$ *the greatest lower bound operation.*

The notation for the meet of a finite nonempty family of elements is $\bigwedge_{i\in I} x_i$.

A partially ordered set $(X, \leq)$ in which every pair of elements x and y has both a least upper bound $x\vee y$ and a greatest lower bound $x\wedge y$ is clearly determined by the 3-tuple $L = (X, \vee, \wedge)$. We see that a 3-tuple consisting of a set, an upper semilattice operation, and a lower semilattice operation arises in this way if and only if these operations are compatible, in the sense that the unique partial ordering for which $\vee$ is the least-upper-bound operation coincides with the unique partial ordering for which $\wedge$ is greatest-lower-bound operation.

Is there a nice formulation for this compatibility condition? The statement that for any two elements x and y, the element y can be written $x\vee z$ for some z if and only if the element x can be written $y\wedge w$ for some w would do, but it is awkward. If, instead of using as above the descriptions of how to *construct* all pairs (x,y) with $x\leq y$ in terms of the operations $\vee$ and $\wedge$, we use the formulas that characterize them as solution-sets of equations, we get the condition that for all elements x and y, $y = x\vee y \Leftrightarrow x\wedge y = x$. But the best expression for our condition – one that does not use any ''can be written''s or ''$\Leftrightarrow$''s – is obtained by playing off one description of $\vee$ against the other description of $\wedge$. This is the fourth pair of equations in

Definition 5.1.4. *A* lattice *will mean a 3-tuple* $L = (|L|, \vee, \wedge)$ *satisfying the following identities for all* $x, y, z\in|L|$:

$$\begin{array}{lll} x\vee x = x & x\wedge x = x & (\textit{idempotence}),\\ x\vee y = y\vee x & x\wedge y = y\wedge x & (\textit{commutativity}),\\ (x\vee y)\vee z = x\vee(y\vee z) & (x\wedge y)\wedge z = x\wedge(y\wedge z) & (\textit{associativity}),\\ x\wedge(x\vee y) = x & x\vee(x\wedge y) = x & (\textit{compatibility}), \end{array}$$

in other words, such that $(|L|, \vee)$ *is an upper semilattice,* $(|L|, \wedge)$ *is a lower semilattice, and these two semilattices have the same natural partial ordering. Loosely, the term will also be used for the equivalent structure of a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound.*

Given a lattice $(|L|, \vee, \wedge)$, *we shall consider* $|L|$ *partially ordered by the unique partial ordering* (*characterizable in four equivalent ways*) *which makes its join operation the least upper* *bound and its meet operation the greatest lower bound. The set $|L|$ with this partial ordering is sometimes called the ''underlying partially ordered set of L''.*

Examples: If S is a set, then the power set $\mathbf{P}(S)$ (the set of all subsets of S), partially ordered by the relation of inclusion, has least upper bounds and greatest lower bounds, given by the union and intersection operations on sets; hence $(\mathbf{P}(S), \cup, \cap)$ is a lattice. Since the definition of Boolean algebra was modeled on the structure of the power set of a set, every Boolean algebra $(|B|, \cup, \cap, {}^{c}, 0, 1)$ gives a lattice $(|B|, \cup, \cap)$ on dropping the last three operations; and since we know that Boolean rings are equivalent to Boolean algebras, every Boolean ring $(|B|, +, \cdot, -, 0, 1)$ becomes a lattice under the operations $x \vee y = x+y+xy$ and $x \wedge y = xy$.

Every totally ordered set – for instance, the real numbers – is a lattice, since the larger and the smaller of two elements will respectively be their least upper bound and greatest lower bound. The set of real-valued functions on any set X may be ordered by writing $f \leq g$ if $f(x) \leq g(x)$ for all x, and this set is a lattice under *pointwise* maximum and minimum.

Under the partial ordering by divisibility, the set of positive integers has least upper bounds and greatest lower bounds, called ''least common multiples'' and ''greatest common divisors''. Note that if we represent a positive integer by its prime factorization, and consider such a factorization as a function associating to each prime a nonnegative integer, then least common multiples and greatest common divisors reduce to pointwise maxima and minima of these functions.

Given a group G, if we order the set of subgroups of G by inclusion, then we see that for any two subgroups H and K, there is a largest subgroup contained in both, gotten by intersecting their underlying sets, and a smallest subgroup containing both, the subgroup *generated by* the union of their underlying sets. So the set of subgroups of G forms a lattice, called the *subgroup lattice* of G. This observation goes over word-for-word with ''group'' replaced by ''monoid'', ''ring'', ''vector space'', etc..

Some writers use ''ring-theoretic'' notation for lattices, writing $x+y$ for $x \vee y$, and xy for $x \wedge y$. Note, however, that a nontrivial lattice is never a ring (its join operation cannot be a group structure). We will not use such notation here.

Although one can easily draw pictures of partially ordered sets and semilattices which are not lattices, it takes a bit of thought to find naturally occurring examples. The next exercise notes a couple of these.

Exercise 5.1:2. (i) If G is an infinite group, show that within the lattice of subgroups of G, the finitely generated subgroups form an upper semilattice under the induced order, but not necessarily a lower semilattice, and the finite subgroups form a lower semilattice but not necessarily an upper semilattice. (For partial credit you can verify the positive assertions; for full credit you must find examples establishing the negative assertions as well.)

(ii) Let us partially order the set of *polynomial* functions on the unit interval $[0, 1]$ by pointwise comparison ($f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in [0,1]$). Show that this partially ordered set is neither an upper nor a lower semilattice.

Exercise 5.1:3. Give an example of a 3-tuple $(|L|, \vee, \wedge)$ which satisfies all the identities defining a lattice except for *one* of the two compatibility identities. If possible, give a systematic way of constructing such examples. Can you determine for which upper semilattices $(|L|, \vee)$ there will exist operations $\wedge$ such that $(|L|, \vee, \wedge)$ satisfies all the lattice identities except the specified one? (The answer will depend on which identity you leave out; you can try to solve the problem for one or both cases.)

Exercise 5.1:4. Show that the two compatibility identities in Lemma 5.1.1 together imply the two idempotence identities.

Exercise 5.1:5. Show that an element of a lattice is a *maximal* element if and only if it is a *greatest* element. Is this true in every upper semilattice? In every lower semilattice?

A *homomorphism* of lattices, upper semilattices, or lower semilattices means a map of their underlying sets which respects the lattice or semilattice operations. If L_1 and L_2 are lattices, one can speak loosely of an ''upper semilattice homomorphism $L_1 \to L_2$'', meaning a map of underlying sets which respects joins but not necessarily meets; this is really a homomorphism $(L_1)_\vee \to (L_2)_\vee$, where $(L_i)_\vee$ denotes the upper semilattice $(|L_i|, \vee)$ gotten by forgetting the operation $\wedge$; one may similarly speak of ''lower semilattice homomorphisms'' of lattices. Note that if $f: |L_1| \to |L_2|$ is a lattice homomorphism, or an upper semilattice homomorphism, or a lower semilattice homomorphism, it will be an isotone map with respect to the natural order-relations on $|L_1|$ and $|L_2|$, but in general, an isotone map f need not be a homomorphism of any of these sorts.

A *sublattice* of a lattice L is a lattice whose underlying set is a subset of $|L|$ and whose operations are the restrictions to this set of the operations of L. A *subsemilattice* of an upper or lower semilattice is defined similarly, and one can speak loosely of an upper or lower subsemilattice of a *lattice* L, meaning a subsemilattice of $L_\vee$ or $L_\wedge$.

Exercise 5.1:6. (i) Give an example of a subset S of the underlying set of a lattice L such that every pair of elements of S has a least upper bound and a greatest lower bound in S under the induced ordering, but such that S is not the underlying set of either an upper or a lower subsemilattice of L.

(ii) Give an example of an upper semilattice homomorphism between lattices that is not a lattice homomorphism.

(iii) Give an example of a bijective isotone map between lattices which is not an upper or lower semilattice homomorphism.

(iv) Show that a bijection between lattices *is* a lattice isomorphism if either (a) it is an upper (or lower) semilattice homomorphism, or (b) it and its inverse are both isotone.

Exercise 5.1:7. Let k be a field. If V is a k-vector space, then the *cosets of subspaces* of V, together with the empty set, are called the *affine subspaces* of V.

(i) Show that the affine subspaces of a vector space form a lattice.

(ii) Suppose we map the set of affine subspaces of the vector space k^n into the set of *vector subspaces* of k^{n+1} by sending each affine subspace $A \subseteq k^n$ to the vector subspace of k^{n+1} spanned by $\{(1, x_0, \ldots, x_{n-1}) \mid (x_0, \ldots, x_{n-1}) \in A\}$. We may ask whether this map respects meets and/or joins. Show that it respects one of these, and respects the other in ''most but not all'' cases, in a sense you should make precise.

(The study of the affine subspaces of k^n is called n-dimensional *affine geometry*. By the above observations, the geometry of the vector subspaces of k^{n+1} may be regarded as a slight extension of n-dimensional affine geometry; this is called n-dimensional *projective geometry*. In view of the relation with affine geometry, the 1-dimensional subspaces of k^{n+1} are called ''points'' of projective n-space, the 2-dimensional subspaces, regarded as sets of points, are called ''lines'', etc..)

The methods introduced in Chapters 2 and 3 can clearly be used to establish the existence of *free* lattices and semilattices, and of lattices and semilattices presented by *generators and relations*. As in the case of semigroups, a ''relation'' means a statement equating two terms formed from the given generators using the available operations – in this case, the lattice or semilattice operations.

Exercise 5.1:8. (i) If P is a partially ordered set, show that there exist universal examples of an upper semilattice, a lower semilattice, and a lattice, with isotone maps of P into their underlying partially ordered sets, and that these may be constructed as semilattices or lattices presented by appropriate generators and relations.

(ii) Show likewise that given any upper or lower semilattice S, there is a universal example of a lattice with an upper, respectively lower semilattice homomorphism of S into it.

(iii) If the S of point (ii) above ''is a lattice'' (has both least upper bounds and greatest lower bounds), will this universal semilattice homomorphism be an isomorphism?

(iv) Show that the universal maps of (i) and (ii) above are in general not surjective, and investigate whether each of them is in general one-to-one.

Exercise 5.1:9. Determine a normal form or other description for the free upper semilattice on a set X. Show that it will be finite if X is finite.

There exists something like a normal form theorem for free lattices [**4**, §VI.8], but it is much less trivial than the result for semilattices referred to in the above exercise, and we will not develop it here. However, the next exercise develops a couple of facts about free lattices.

Exercise 5.1:10. (i) Determine the structures of the free lattices on 0, 1, and 2 generators.

(ii) Show for some positive integer n that the free lattice on n generators is infinite. (One approach: In the lattice of affine subsets of the plane $\mathbf{R}^2$ (Exercise 5.1:7), consider the sublattice generated by the five lines $x = 0$, $x = 1$, $x = 2$, $y = 0$, $y = 1$.)

Exercise 5.1:11. (i) Recall (Exercise 4.1:5) that a set map $X \to Y$ induces maps $\mathbf{P}(X) \to \mathbf{P}(Y)$ and $\mathbf{P}(Y) \to \mathbf{P}(X)$. Show that one of these is always, and the other is not always a lattice homomorphism.

(ii) If L is a lattice (respectively, an upper semilattice, a lower semilattice, a partially ordered set), show that there exists a universal example of a set X and a lattice homomorphism $L \to (\mathbf{P}(X), \cup, \cap)$ (respectively an upper semilattice homomorphism $L \to (\mathbf{P}(X), \cup)$, a lower semilattice homomorphism $L \to (\mathbf{P}(X), \cap)$, an isotone map $L \to (\mathbf{P}(X), \subseteq)$). (First formulate the proper universal properties, bearing in mind the answer to part (i).) Describe the set X as explicitly as you can in each case.

(iii) In which of the above cases can you show the map $|L| \to \mathbf{P}(X)$ one-to-one? In the case(s) where you cannot prove this, can you find an example in which it is not one-to-one?

In Exercise 4.6:3, we saw that any partially ordered set without maximal elements has two disjoint cofinal subsets. Let us examine what similar results hold for lattices.

Exercise 5.1:12. Let L be a lattice without greatest element.

(i) If L is *countable*, show that it contains a cofinal chain, that this chain will have two disjoint cofinal subchains, and that these will be disjoint cofinal sublattices of L.

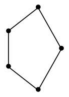
(ii) Show that in general, L may not have a cofinal chain.

(iii) Must L have two disjoint cofinal sublattices? (I don't know the answer.)

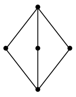
(iv) Show that L will always contain two disjoint upper sub*semi*lattices, each cofinal in L.

I could not end an introduction to lattices without showing you the concepts introduced in the next two exercises, though this brief mention, and the results developed in the two subsequent exercises, will hardly do them justice. I will refer in these exercises to the following two 5-element lattices:

N_5:



M_5:



Exercise 5.1:13. (i) Show that the following conditions on a lattice L are equivalent:

(a) For all $x, y, z \in |L|$ with $x \leq z$, one has $x \vee (y \wedge z) = (x \vee y) \wedge z$.

(b) L has no sublattice isomorphic to N_5 (shown above).

(c) For every pair of elements $x, y \in |L|$, the intervals $[x \wedge y, y]$ and $[x, x \vee y]$ are isomorphic, the map in one direction being given by $z \mapsto x \vee z$, in the other direction by $z \mapsto z \wedge y$.

(ii) Show that condition (a) is equivalent to an *identity*, i.e., a statement that a certain equation in n variables and the lattice-operations holds for all n-tuples of elements of L. (Condition (a) as stated fails to be an identity, because it refers only to 3-tuples satisfying $x \leq z$.)

(iii) Show that the lattice of subgroups of an abelian group satisfies the above equivalent conditions.

Deduce that the lattice of submodules of a module over a ring will satisfy the same conditions. For this reason, a lattice satisfying these conditions is called *modular*.

(iv) Determine, as far as you can, whether each of the following lattices is in general modular: the lattice of all subsets of a set; the lattice of all subgroups of a group; the lattice of all normal subgroups of a group; the lattice of all ideals of a ring; the lattice of all subrings of a ring; the lattice of all subrings of a Boolean ring; the lattice of elements of a Boolean ring under the operations $x \vee y = x + y + xy$ and $x \wedge y = xy$; the lattice of all sublattices of a lattice; the lattice of all closed subsets of a topological space; the lattices associated with n-dimensional affine geometry and with n-dimensional projective geometry (Exercise 5.1:7 above).

Exercise 5.1:14. (i) Show that the following conditions on a lattice L are equivalent:

(a) For all $x, y, z \in |L|$, one has $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

(a*) For all $x, y, z \in |L|$, one has $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(b) L has no sublattice isomorphic either to M_5 or to N_5.

Note that if one thinks of $\vee$ as ‘‘addition’’ and $\wedge$ as ‘‘multiplication’’, then (a*) has the form of the distributive law of ring theory. (Condition (a) is also a distributive law, though this identity does not hold in any nonzero ring.) Hence lattices satisfying the above equivalent conditions are called *distributive*.

(ii) Show that the lattice of subsets of a set is distributive.

(iii) Determine, as far as you can, whether lattices of each of the remaining sorts listed in parts (iii) and (iv) of the preceding exercise are always distributive.

(iv) Show that every finitely generated distributive lattice is finite.

Exercise 5.1:15. Let V be a vector space over a field k, let $S_1, \ldots, S_n$ be subspaces of V, and within the lattice of all subspaces of V, let L denote the sublattice generated by $S_1, \ldots, S_n$.

(i) Show that if V has a basis B such that each S_i is spanned by a subset of B, then L is distributive.

Below we will prove the converse to (i); so for the remainder of this exercise, suppose L is distributive. Clearly, it will suffice to prove that V contains a direct sum of subspaces, with the property that each S_i is the sum of some subfamily thereof; so this is what we will aim for. You may assume the last result of the preceding exercise, that every finitely generated distributive lattice is finite.

(ii) Let $T = S_1 + \ldots + S_n$, the largest element of L. Assuming L has elements other than T, let W be maximal among these. Show that there is a *least* element $U \in L$ not contained in W.

(iii) Let E be a subspace of V such that $U = (U \cap W) \oplus E$. (Why does one exist?) Show that every member of L is either contained in W, or is the direct sum of E with a member of L contained in W.

(iv) Writing L' for the sublattice of L consisting of members of L contained in W, show that the lattice of subspaces of V generated by $\{S_1, \ldots, S_n, E\}$ is isomorphic to $L' \times \{0, E\}$.

(v) Conclude by induction that there exists a family of subspaces $E_1, \ldots, E_r \subseteq V$ such that every member of L, and hence in particular, each of $S_1, \ldots, S_n$, is the direct sum of a subset of this family. Deduce that V has a basis B such that each S_i is spanned by a subset of B.

Exercise 5.1:16. Let us show that the result of the preceding exercise fails for infinite families $(S_i)_{i\in I}$. Our example will be a chain of subspaces, so

(i) Verify that every chain in a lattice is a distributive sublattice.

Now let k be a field, and V the k-vector-space of all k-valued functions on the nonnegative integers. You may assume the standard result that V is uncountable-dimensional. For each nonnegative integer n, let $S_n = \{ f\in V \mid f(i) = 0 \text{ for } i < n\}$.

(ii) Show that V does not have a basis B such that each S_i is spanned by a subset of B. (Hint: Each S_i has codimension 1 in the preceding, and their intersection is $\{0\}$, but $\dim(V)$ is uncountable.)

Incidentally, the analog of Exercise 2.3:2, with the finite lattice N_5 in place of the finite group S_3, is worked out for $n=3$ in [**103**].

5.2. 0, 1, and completeness. We began this chapter with the observation that many natural examples of partially ordered sets have the property that every *pair* of elements has a least upper bound and a greatest lower bound. But in fact, most of these examples have the stronger property that such bounds exist for every *set* of elements. E.g., in the lattice of subgroups of a group, one can take the intersection, or the subgroup generated by the union, of an arbitrary set of subgroups. The property that every subset $\{x_i \mid i\in I\}$ has a least upper bound (denoted $\bigvee_I x_i$) and a greatest lower bound (denoted $\bigwedge_I x_i$) defines the class of nonempty *complete* lattices, which we shall consider in this section.

Note that in an ordinary lattice, where every *pair* of elements x, y has a least upper bound $x\vee y$, it is also true for all positive integers n that every family of n elements $x_0, \ldots, x_{n-1}$ has a least upper bound, namely, $\bigvee x_i = x_0 \vee \ldots \vee x_{n-1}$. Hence, to get least upper bounds for *all* families, we need to bring in the additional cases of *infinite* families, and the *empty* family.

Now every element of a lattice is an upper bound for the empty family, so a *least upper bound* for the empty family means a *least element* in the lattice. Such an element is often written 0, or when there is a possibility of ambiguity, 0_L. Likewise, a greatest lower bound for the empty family means a greatest element, commonly written 1 or 1_L.

It is not hard to see that the two conditions of existence of pairwise joins and of existence of a least element (a join of the empty family) are independent: A partially ordered set can satisfy either, or neither, or both. On the other hand, existence of pairwise joins and existence of infinite joins (joins indexed by infinite families, with repetition allowed just as in the case of pairwise joins) are not independent; the latter condition implies the former. However, we may ask whether the property ‘‘existence of infinite joins’’ can somehow be decomposed into the conjunction of existence of pairwise joins, and some useful condition which *is* independent thereof. The next result shows that it can, and more generally, that for any cardinal α, the condition ‘‘there exist joins of families of cardinality $\leq\alpha$’’ can be so decomposed.

Lemma 5.2.1. *Let P be a partially ordered set, and α an infinite cardinal. Then the following conditions are equivalent:*

(i) *Every nonempty subset of P with $\leq\alpha$ elements has a least upper bound in P.*

(ii) *Every* pair *of elements of P has a least upper bound, and every nonempty* chain *in P with $\leq\alpha$ elements has a least upper bound.*

The dual statements concerning greatest lower bounds are likewise equivalent to one another.

Proof. (i)⇒(ii) is clear.

Conversely, assuming (ii) let us take any set X of $\leq\alpha$ elements of P, and index it by an ordinal $\beta\leq\alpha$: $X=\{x_\varepsilon \mid \varepsilon<\beta\}$. We shall prove inductively that for $0<\gamma\leq\beta$, there exists a least upper bound $\bigvee_{\varepsilon<\gamma} x_\varepsilon$. Because we have not assumed a least upper bound for the empty set, this need not be true for $\gamma=0$, so we start the induction by observing that for $\gamma=1$, the set $\{x_\varepsilon \mid \varepsilon<1\}=\{x_0\}$ has least upper bound x_0. Now let $\gamma>1$ and assume our result is true for all positive $\delta<\gamma$. If γ is a successor ordinal, $\gamma=\delta+1$, then we apply the existence of pairwise least upper bounds in P and see that $(\bigvee_{\varepsilon<\delta} x_\varepsilon)\vee x_\delta$ will give the desired least upper bound $\bigvee_{\varepsilon<\gamma} x_\varepsilon$. On the other hand, if γ is a limit ordinal, then the elements $\bigvee_{\varepsilon<\delta} x_\varepsilon$ where δ ranges over all nonzero members of γ will form a chain, which by hypothesis has a least upper bound in P, and this gives the desired element $\bigvee_{\varepsilon<\gamma} x_\varepsilon$. So by induction, $\bigvee_{\varepsilon<\beta} x_\varepsilon$ exists, proving (i).

The final statement follows by duality. □

Definition 5.2.2. *Let* α *be a cardinal. Then a lattice or an upper semilattice* L *in which every nonempty set of* $\leq\alpha$ *elements has a least upper bound is said to be* upper α-semicomplete. *A lattice or a lower semilattice satisfying the dual condition is said to be* lower α-semicomplete. *A lattice satisfying both conditions is called* α-complete.

When these conditions hold for all nonzero cardinals α, *one calls* L upper semicomplete, *respectively* lower semicomplete, *respectively* complete.

Note that in a partially ordered set (e.g., a lattice) with *ascending chain condition*, all nonempty chains have least upper bounds – since they in fact have greatest elements. Likewise in a partially ordered set with descending chain condition, all chains have greatest lower bounds.

Exercise 5.2:1. Suppose β and γ are infinite cardinals, and X a set of cardinality $\geq$ both β and γ. Let $L=\{S\subseteq X \mid \mathrm{card}(S)<\beta \text{ or } \mathrm{card}(X-S)<\gamma\}$. Verify that L is a lattice, and investigate for what cardinals α this lattice is upper, respectively lower α-semicomplete.

The upper and lower semicompleteness conditions, when not restricted as to cardinality, have an unexpectedly close relation.

Proposition 5.2.3. *Let* L *be a partially ordered set. Then the following conditions are equivalent:*

(i) *Every subset of* L *has a least upper bound; i.e.,* L *is* (*the underlying partially ordered set of*) *an upper semicomplete upper semilattice with least element.*

(i*) *Every subset of* L *has a greatest lower bound; i.e.,* L *is* (*the underlying partially ordered set of*) *a lower semicomplete lower semilattice with greatest element.*

(ii) L *is* (*the underlying partially ordered set of*) *a nonempty complete lattice.*

Proof. To see the equivalence of the two formulations of (i), recall that a least upper bound for the empty set is a least element, while the existence of least upper bounds for all other subsets is what it means to be an upper semicomplete upper semilattice.

To show (i)⇒(ii), observe that the existence of a least element shows that L is nonempty, and the upper complete upper semilattice condition gives half the condition to be a complete lattice. It remains to show that any nonempty subset X of L has a greatest lower bound u. In fact, the least *upper* bound of the set of *all lower* bounds for X will be the desired u; the reader should verify that it has the required property.

Conversely, assuming (ii), we have by definition least upper bounds for all *nonempty* subsets of L. A least upper bound for the empty set is easily seen to be given by the greatest lower bound of all of L. (How is the nonemptiness condition of (ii) used?)

Since (ii) is self-dual and equivalent to (i), it is also equivalent to (i*). □

Exercise 5.2:2. If T is a topological space, show that the open sets in T, partially ordered by inclusion, form a complete lattice. Describe the meet and join operations (finite and arbitrary) of this lattice. Translate these results into statements about the set of closed subsets of T.

(General topology buffs may find it interesting to show that, on the other hand, the partially ordered set $\{\text{open sets}\} \cup \{\text{closed sets}\}$ is not in general a lattice, nor is the partially ordered set of *locally closed* sets.)

Exercise 5.2:3. Which ordinals, when considered as ordered sets, form complete lattices?

Exercise 5.2:4. (i) Show that every isotone map from a nonempty complete lattice into itself has a fixed point.

(ii) Can you prove the same result for a larger class of partially ordered sets?

Exercise 5.2:5. Let L be a complete lattice.

(i) Show that the following conditions are equivalent: (a) L has no chain order-isomorphic to an uncountable cardinal. (b) For every subset $X \subseteq |L|$ there exists a countable subset $Y \subseteq X$ such that $\bigvee Y = \bigvee X$.

(ii) Let a be any element of L. Are the following conditions equivalent? (a) L has no chain order-isomorphic to an uncountable cardinal and having join a. (b) Every subset $X \subseteq L$ with join a contains a countable subset Y also having join a.

When we were motivating the statement of Zorn's Lemma in the preceding chapter, we said that in the typical construction where one calls on it, if one has a chain of partial constructions, one can ''put them together'' to get a partial construction extending them all. This means that the set of partial constructions is a partially ordered set in which every chain has not merely an *upper bound* but a *least* upper bound. This leads to the following question: Suppose we state a ''weakened'' form of Zorn's Lemma, saying only that partially ordered sets with *this* property have maximal elements – which is virtually all one every uses. Is this equivalent to the full form of Zorn's Lemma? This is answered in

Exercise 5.2:6. Show, without assuming the Axiom of Choice, that the statement ''If P is a nonempty partially ordered set such that all nonempty chains in P have least upper bounds, then P has a maximal element'', implies the full form of Zorn's Lemma. (If possible, make your proof self-contained, i.e., avoid using the equivalence of Zorn's Lemma with Axiom of Choice etc..)

Our proof in Lemma 5.2.1 that the existence of least upper bounds of *chains* made a lattice upper semicomplete really only used well-ordered chains, i.e., chains order-isomorphic to ordinals. In fact, one can do still better:

Exercise 5.2:7. Recall from Exercise 4.6:2 that every totally ordered set has a cofinal subset order-isomorphic to a regular cardinal.

(i) Deduce that for P a partially ordered set and α an infinite cardinal, the following two conditions are equivalent:

(a) Every chain in P of cardinality $\leq \alpha$ has a least upper bound.

(b) Every chain in P which is order-isomorphic to a regular cardinal $\beta \leq \alpha$ has a least upper bound.

(ii) With the help of the above result, extend Lemma 5.2.1, adding a third equivalent condition.

There are still more ways than those we have seen to decompose the condition of being a complete lattice, as shown in point (ii) of

Exercise 5.2:8. (i) Show that following conditions on a partially ordered set L are equivalent:

(a) Every nonempty subset of L having an upper bound has a least upper bound.

(b) Every nonempty subset of L having a lower bound has a greatest lower bound.

(c) L satisfies the *complete interpolation property*: Given two nonempty subsets X, Y of L, such that every element of X is $\leq$ every element of Y, there exists an element $z \in L$ which is $\geq$ every element of X and $\leq$ every element of Y.

(ii) Show that L is a nonempty complete lattice if and only if it has a greatest and a least element, and satisfies the above equivalent conditions.

(iii) Give an example of a partially ordered set which satisfies (a)-(c) above, but is not a lattice.

(iv) Give an example of a partially ordered set with greatest and least elements, which has the *finite interpolation property*, i.e., satisfies (c) above for all finite nonempty families X and Y, but which is not a lattice.

This condition-splitting game is carried still further in

Exercise 5.2:9. If σ and τ are properties of sets of elements of partially ordered sets, let us say that a partially ordered set L has the (σ, τ)*-interpolation property* if for any two subsets X and Y of L such that X satisfies σ, Y satisfies τ, and all elements of X are $\leq$ all elements of Y, there exists an element $z \in L$ which is $\geq$ every element of X and $\leq$ every element of Y. Now consider the *nine* conditions on L gotten by taking for σ and τ all combinations of the three properties ‘‘is empty’’, ‘‘is a pair’’ and ‘‘is a chain’’.

(i) Find simple descriptions for as many of these nine conditions as you can. (Note cases that are equivalent to conditions we have already named.)

(ii) Show that L is a nonempty complete lattice if and only if it satisfies all nine of these conditions.

(iii) How close to independent are these nine conditions? To answer this, determine as well as you can which of the $2^9 = 512$ functions from the set of these conditions to the set $\{$true, false$\}$ can be realized by appropriate choices of L. (Remark: A large number of these combinations *can* be realized, so to show this, you will have to produce a large number of examples. I therefore suggest that you consider ways that examples with certain *combinations* of properties can obtained from examples of the separate properties.)

Exercise 5.2:10. (i) We saw in Exercise 5.1:2(ii) that the set of real polynomials on the unit interval $[0,1]$, partially ordered by the relation $(\forall x \in [0,1])\ f(x) \leq g(x)$, does not form a lattice. Show, however, that it has the finite interpolation property. (This gives a solution to Exercise 5.2:8, but far from the easiest solution. The difficulty in proving this result arises from the possibility that some members of X will be tangent to some members of Y.)

(ii) Can you obtain similar results for the partially ordered set of real polynomials on a general compact set $K \subseteq \mathbf{R}^n$?

Although we write the least upper bound and greatest lower bound of a set X in a complete lattice as $\bigvee X$ or $\bigvee_{x \in X} x$ and $\bigwedge X$ or $\bigwedge_{x \in X} x$, and call these the meet and join of X, these ‘‘meet’’ and ‘‘join’’ are not *operations* in quite the sense we have been considering so far. An operation is supposed to be a map $S^n \to S$ for some n. One may allow n to be an infinite cardinal (or other set), but when we consider complete lattices, there is no fixed cardinality to use. Presumably, we should consider each of the symbols $\vee$ and $\wedge$ to stand for a *system* of operations, of varying finite and infinite arities. But how large is this system? In a given complete lattice L, all meets and joins reduce (by dropping repeated arguments) to meets and joins of families of cardinalities $\leq \mathrm{card}(L)$. But if we want to develop a general theory of complete lattices, then meets and joins of families of arbitrary cardinalities will occur, so this ''system of operations'' will not be a *set* of operations. We shall eventually see that as a consequence of this, though complete lattices are in many ways like algebras, not all of the results that we prove about algebras will be true for them (Exercise 7.10:5(iii)).

Another sort of complication in the study of complete lattices comes from the equivalence of the various conditions in Proposition 5.2.3: Since these lattices can be characterized in terms of different systems of operations, there are many natural kinds of ''maps'' among them: maps which respect arbitrary meets, maps which respect arbitrary joins, maps which respect both, maps which respect meets of all *nonempty* sets and joins of all *pairs*, etc.. The term ''homomorphism of complete lattices'' will mean a map respecting meets and joins of all nonempty sets, but the other kinds of maps are also of interest. These distinctions are brought out in:

Exercise 5.2:11. (i) Show that every complete lattice can be embedded, by a map which respects arbitrary *joins* (including the join of the empty set), in a power set $\mathbf{P}(S)$, for some set S, and likewise may be embedded by a map which respects arbitrary *meets* in a power set.

(ii) On the other hand, deduce from Exercise 5.1:14(ii) that the finite lattices M_5 and N_5 considered there cannot be embedded by any *lattice homomorphism*, i.e., any map respecting *both* finite meets and finite joins, in a power set $\mathbf{P}(S)$.

An interesting pair of invariants related to point (i) above is examined in

Exercise 5.2:12. (i) Show that for any complete lattice L, the following cardinals are equal: (a) the least α such that L can be embedded, by a map respecting arbitrary *meets*, in the power set $\mathbf{P}(S)$ of a set of cardinality α, (b) the least cardinality of a subset $T \subseteq L$ such that every element of L is the *join* of a subset of T.

Let us call the invariant α characterized above the *upward generating number* of L (because of the relation with generation by joins). We dually define the *downward* generating number.

(ii) Find a *finite* lattice L for which these two generating numbers are not equal.

We saw in the preceding section that the concept of lattice, though motivated by properties of certain partially ordered sets, could be formalized purely in terms of two operations and some identities. The concepts of (α-)complete lattice and semilattice can be similarly formalized in terms of operations and identities; the interested reader will not find it hard to write down the details. Just how the infinite associative and commutative laws are stated will depend on the way one describes the infinitary operations, but once this is settled, the statements are straightforward. The compatibility laws only need to be stated for meets and joins of pairs (why?), and so do not need to be modified.

To motivate the next definition, let us consider the following situation. Suppose L is the complete lattice of all subgroups of a group G, and let $K \in L$ be a *finitely generated* subgroup of G, generated by elements $g_1, \ldots, g_n$. Suppose K is majorized by the join of a family of subgroups H_i $(i \in I)$, i.e., is contained in the subgroup generated by the H_i. Then each of $g_1, \ldots, g_n$ can be expressed by a group-theoretic term in elements of $\bigcup |H_i|$. But any group-theoretic term involves only finitely many elements; hence K will actually be contained in the subgroup generated by *finitely many* of the H_i. The converse also holds: If K is a *non*-finitely generated subgroup of G, then K equals (and hence is contained in) the join of all the cyclic subgroups it contains, but is *not* contained in the join of any finite subfamily thereof.

The property we have just shown to characterize the finitely generated subgroups in the lattice of all subgroups of G is analogous to the property characterizing *compact* subsets in a topological space – that if they are covered by a family of open subsets, they are covered by some finite subfamily. Hence one makes

Definition 5.2.4. *An element k of a complete lattice (or more generally, of a complete upper semilattice) L is called* compact *if every set of elements of L with join $\geq k$ has a finite subset with join $\geq k$.*

By the preceding observations, the compact elements of the subgroup lattice of a group are precisely the finitely generated subgroups. We will be able to generalize this observation when we have a general theory of algebraic objects.

We noted in Exercise 5.1:2(i) that the finitely generated subgroups of a group form an upper subsemilattice of the lattice of all subgroups. This suggests

Exercise 5.2:13. Do the compact elements of a complete lattice L always form an upper subsemilattice?

Exercise 5.2:14. Show that a complete lattice L has ascending chain condition if and only if all elements of L are compact.

There seems to be no standard name for an element of a complete lattice having the dual property to compactness. Sometimes such elements are called *co-compact*.

We examined in Exercise 5.2:11 the embedding of semilattices and lattices in power sets $\mathbf{P}(S)$ (and found that though there were embeddings that respected meets, and embeddings that respected joins, there were not in general embeddings that respected both). Let us look briefly at another very fundamental sort of complete lattice, and the problem of embedding arbitrary lattices therein.

If X is a set, and $\approx_0$ and $\approx_1$ are two equivalence relations on X, let us say $\approx_1$ *extends* $\approx_0$ if it contains it, as a subset of $X \times X$, and write $\approx_0 \leq \approx_1$ in this situation. Let $\mathbf{E}(X)$ denote the set of equivalence relations on X, partially ordered by this relation $\leq$. (One could use the reverse of this order, saying that $\approx_0$ is a *refinement* of $\approx_1$ when the latter extends the former, and could justify considering the refinement to be ‘‘bigger’’ because it gives ‘‘more’’ equivalence classes. So our choice of the sense to give to our ordering is somewhat arbitrary; but let us stick with the ordering based on set-theoretic inclusion.)

Exercise 5.2:15. (i) Verify that the partially ordered set $\mathbf{E}(X)$ forms a complete lattice. Identify the elements $0_{\mathbf{E}(X)}$ and $1_{\mathbf{E}(X)}$.

(ii) Let L be any nonempty complete lattice, and $f: L \to \mathbf{E}(X)$ a map respecting arbitrary meets (a complete lower semilattice homomorphism respecting greatest elements). Show that for any $x, y \in X$, there is a *least* $d \in L$ such that $(x, y) \in f(d)$. Calling this element $d(x, y)$, verify that the map $d: X \times X \to L$ satisfies the following conditions for all $x, y, z \in X$:

$$(\mathrm{a}_0)\quad d(x,x) = 0_L,$$

$$(\mathrm{b})\quad d(x,y) = d(y,x),$$

$$(\mathrm{c})\quad d(x,z) \leq d(x,y) \vee d(y,z).$$

(iii) Prove the converse, i.e., that given a complete lattice L and a set X, any function $d: X \times X \to L$ satisfying (a_0)-(c) arises as in (ii) from a unique complete lower semilattice-homomorphism $f: L \to \mathbf{E}(X)$ respecting greatest elements.

In the remaining parts, we assume that $f: L \to \mathbf{E}(X)$ and $d: X \times X \to L$ are maps related as in (ii) and (iii).

(iv) Show that the map f respects least elements, i.e., that $f(0_L) = 0_{\mathbf{E}(X)}$, if and only if d satisfies

(a) $d(x,y) = 0_L \Leftrightarrow x = y$ (cf. (a_0) above).

(v) Show that f respects joins of finite nonempty families if and only if d satisfies

(d) whenever $d(x,y) \leq p \vee q$ ($x, y \in X$, $p, q \in |L|$), there exists a finite ''path'' from x to y in X, i.e., a sequence $x = z_0, z_1, \ldots, z_n = y$, such that for each $i < n$, either $d(z_i, z_{i+1}) \leq p$ or $d(z_i, z_{i+1}) \leq q$.

A function d which satisfies (a)-(c) above might be called an ''L-valued metric on X'', and (d) might be called ''path connectedness'' of the L-valued metric space X. Two other properties of importance are noted in

(vi) Assuming that f respects finite nonempty joins, show that it respects arbitrary nonempty joins if and only if

(e) for all $x, y \in X$, $d(x,y)$ is a compact element of L.

(vii) Show that f is one-to-one if and only if

(f) L is generated under (not necessarily finite) joins by the elements $d(x,y)$ ($x, y \in X$).

Thus, to embed a complete lattice L in a lattice of the form $\mathbf{E}(X)$, it suffices to construct a set X with an appropriate L-valued metric.

Exercise 5.2:16. (i) Given a complete lattice L and an upper subsemilattice K of L containing 0_L, show how to get a set X with an L-valued metric d such that $\{d(x,y) \mid x, y \in X\} = K$. (Hint: Consider a tree, in the graph-theoretic sense, with edges labeled by elements of K.)

(ii) Given L, K, X and d as in (i), show that if either $K = |L|$ or K is the set of compact elements of L and generates L under not-necessarily-finite joins, then X can be embedded in a ''path-connected'' L-valued metric space Y, with the extended function d still having values in K. (Idea: First show that X can be embedded in an X' such that condition (d) holds in X' for all $x, y \in X$, then iterate this construction.) If possible, replace the pair of alternative conditions I have assumed on K by some simple condition which is satisfied in both cases.

(iii) Deduce that a complete lattice L can be embedded in a lattice $\mathbf{E}(X)$ by a map respecting arbitrary meets and joins if and only if every element of L is a (possibly infinite) join of compact elements. (Cf. Exercise 5.2:14.) Also deduce that *every* complete lattice can be embedded in a lattice $\mathbf{E}(X)$ by a map respecting arbitrary meets and *finite* joins.

We shall see in the next section that any lattice can be embedded by a lattice homomorphism in a complete lattice, so by the result of the above exercise, any lattice can be embedded by a lattice homomorphism in a lattice of equivalence relations.

If the lattice L is finite, the construction of the preceding exercise gives, in general, a countable, but not a finite set X. It was for a long time an open question whether every finite lattice could be embedded in the lattice of equivalence relations of a finite set. This was finally proved in 1980 by P. Pudlák and J. Tůma [**90**]. However, good estimates for the *size* of an X such that even a quite small lattice L (e.g., the 15-element lattice $\mathbf{E}(4)^{\mathrm{op}}$) can be embedded in $\mathbf{E}(X)$ remain to be found. The least m such that $\mathbf{E}(n)^{\mathrm{op}}$ embeds in $\mathbf{E}(m)$ has been shown by Pudlák to grow at least exponentially in n; the first *upper* bound obtained for it was $2^{2^{\cdot^{\cdot^{\cdot}}}}$ with n^2 exponents! For subsequent better results see [**75**] and [**66**, in particular p.16, top].

5.3. Closure operators. We introduced this chapter by noting certain properties common to the partially ordered sets of all subsets of a set, of all subgroups of a group, and similar examples. But so far, we seem to have made a virtue of abstractness, defining semilattice, lattice, etc., without reference to systems of subsets of sets. Neither abstractness nor concreteness is everywhere a virtue; each makes its contribution, and it is time to turn to an important class of concrete lattices.

Lemma 5.3.1. *Let S be a set. Then the following data are equivalent:*

(i) *A lower semicomplete lower subsemilattice of $\mathbf{P}(S)$ which contains S, that is, a system C of subsets of S closed under taking arbitrary intersections, including the empty intersection, S itself.*

(ii) *A function $\mathrm{cl}: \mathbf{P}(S) \to \mathbf{P}(S)$ with the properties:*

$$\begin{aligned} &(\forall\, X \subseteq S)\ \ \mathrm{cl}(X) \supseteq X && (\mathrm{cl}\ \ \textit{is increasing}),\\ &(\forall\, X,\, Y \subseteq S)\ \ X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y) && (\mathrm{cl}\ \ \textit{is isotone}),\\ &(\forall\, X \subseteq S)\ \ \mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X) && (\mathrm{cl}\ \ \textit{is idempotent}). \end{aligned}$$

Namely, given C, one defines cl *as the operator taking each $X \subseteq S$ to the intersection of all members of C containing X, while given* cl, *one defines C as the set of $X \subseteq S$ satisfying* $\mathrm{cl}(X) = X$, *equivalently, as the set of subsets of S of the form* $\mathrm{cl}(Y)$ $(Y \subseteq S)$. □

Exercise 5.3:1. Verify the above lemma. That is, show that the procedures described do carry families C with the properties of point (i) to operations cl with the properties of point (ii) and vice versa, and are inverse to one another, and also verify the assertion of equivalence in the final clause.

Definition 5.3.2. *An operator* cl *on the class of subsets of a set S with the properties described in point* (ii) *of the above lemma is called a* closure operator *on S. If* cl *is a closure operator on S, the subsets $X \subseteq S$ satisfying* $\mathrm{cl}(X) = X$, *equivalently, the subsets of the form* $\mathrm{cl}(Y)$ $(Y \subseteq S)$, *are called the* closed subsets *of S under* cl.

We see that virtually every mathematical construction commonly referred to as ‘‘the ... generated by’’ (fill in the blank with subgroup, normal subgroup, subring, sublattice, submonoid, ideal, congruence, etc.) is an example of a closure operator on a set. The operation of topological closure on subsets of a topological space is another example. Some cases are called by other names: the *convex hull* of a set of points in Euclidean n-space, the *span* of a subset of a vector space (i.e., the vector subspace it generates), the set of *derived operations* of a set of operations on a set (§1.6). Incidentally, the constructions of subgroup and subring generated by a set illustrate the fact that the closure of the empty set need not be empty.

A very common way of obtaining a closure operator on a set S, which includes most of the above examples, can be abstracted as follows: One specifies a subset

$$G \ \subseteq\ \mathbf{P}(S) \times S, \tag{5.3.3}$$

and then defines a subset $X \subseteq S$ to be *closed* if for all $(A, x) \in G$, $A \subseteq X \Rightarrow x \in X$. It is straightforward to verify that the class of ‘‘closed sets’’ under this definition is closed under arbitrary intersections, and so by Lemma 5.3.1, corresponds to a closure operator cl on S.

For example, if K is a group, the operator ‘‘subgroup generated by’’ on subsets of $|K|$ is of this form. One takes for (5.3.3) the set of all pairs of the forms

(5.3.4) $$(\{x,y\},\ xy),\qquad (\{x\},\ x^{-1}),\qquad (\varnothing,\ e)$$

where x and y range over $|K|$. To get the operator ‘‘*normal* subgroup generated by –’’, we use the above pairs, supplemented by the further family of pairs $(\{x\},\ yxy^{-1})$ $(x,y\in|K|)$. Clearly, each of the ‘‘... generated by’’ constructions we mentioned above can be characterized similarly. For a non-algebraic example, the operator giving the topological closure of a subset of the real line $\mathbf{R}$ can be obtained by taking G to consist of all pairs (A,x) such that A is the set of points of a convergent sequence, and x is the limit of that sequence.

Exercise 5.3:2. Show that for any closure operator cl on a set S, there exists a subset $G\subseteq \mathbf{P}(S)\times S$ which determines cl in the sense we have been discussing.

Exercise 5.3:3. If T is a set, display a subset $G\subseteq \mathbf{P}(T\times T)\times(T\times T)$ such that the *equivalence relations* on T are precisely the subsets of $T\times T$ closed under the operator cl determined by G. (The previous exercise gives us a way of doing this ‘‘blindly’’. But what I want here is an explicit set, which one might show to someone who didn’t know what ‘‘equivalence relation’’ meant, to provide a characterization of the concept.)

In Chapter 2 we contrasted the approaches of obtaining sets one is interested in ‘‘from above’’ as intersections of systems of larger sets, and of building them up ‘‘from below’’. We have constructed the closure operator associated with a family (5.3.3) by noting that the class of subsets of S we wish to call closed is closed under arbitrary intersections; so we have implicitly obtained these closures ‘‘from above’’. The next exercise constructs them ‘‘from below’’.

Exercise 5.3:4. Let S be a set and G a subset of $\mathbf{P}(S)\times S$. For X a subset of S and α any ordinal, let us define $\mathrm{cl}_G^{(\alpha)}(X)$ recursively by:

$$\mathrm{cl}_G^{(0)}(X) = X,$$
$$\mathrm{cl}_G^{(\alpha+1)}(X) = \mathrm{cl}_G^{(\alpha)}(X)\cup\{x\mid (\exists\, A\subseteq \mathrm{cl}_G^{(\alpha)}(X))\ \ (A,x)\in G\}.$$
$$\mathrm{cl}_G^{(\alpha)}(X) = \textstyle\bigcup_{\beta\in\alpha}\mathrm{cl}_G^{(\beta)}(X)\ \text{ if } \alpha \text{ is a limit ordinal.}$$

(i) Show that (for any S, G as above) there exists an ordinal α such that for all $\beta>\alpha$ and all $X\subseteq S$, $\mathrm{cl}_G^{(\beta)}(X)=\mathrm{cl}_G^{(\alpha)}(X)$, and that $\mathrm{cl}_G^{(\alpha)}(X)$ is then $\mathrm{cl}(X)$ in the sense of the preceding discussion. (Cf. the construction in §2.2 of the equivalence relation R on group-theoretic terms as the union of a chain of relations R_i.)

(ii) If for all $(A,x)\in G$, A is finite, show that the α of part (i) can be taken to be ω.

(iii) For each ordinal α, can you find an example of a set S and a $G\subseteq\mathbf{P}(S)\times S$ such that α is the least ordinal having the property of part (i)?

We have seen that there are restrictions on the sorts of lattices that can be embedded by lattice homomorphisms into lattices $(\mathbf{P}(S),\cup,\cap)$ (Exercise 5.1:14), or into lattices of submodules of modules (Exercise 5.1:13). In contrast, we have

Lemma 5.3.5. (i) *Every* complete *lattice* L *is isomorphic to the lattice of closed sets of a closure operator* cl *on some set* S.

(ii) Every *lattice* L *is isomorphic to a* sublattice *of the lattice of closed sets of a closure operator* cl *on some set* S.

Proof. (i): Take $S=|L|$, and for each $X\subseteq S$, define $\mathrm{cl}(X)=\{y\mid y\le\bigvee X\}$. Then L is isomorphic to the lattice of closed subsets of S, by the map $x\mapsto\{y\mid y\le x\}$.

(ii): Again take $S = |L|$, but since joins of arbitrary families may not be defined in L, define $\mathrm{cl}(X)$ to be the set of all elements majorized by joins of finite subsets of X. Embed L in the lattice of cl-closed subsets of S by the same map as before. □

Exercise 5.3:5. Verify that the above constructions give closure operators, and that the induced maps are respectively an isomorphism of complete lattices and a lattice embedding.

The second of the two closure operators used in the above proof can be thought of as closing a set X in $|L|$ under forming joins of pairs of its elements, and forming meets of its elements with arbitrary elements of L. In the notation that denotes join by $+$ and writes meet as ''multiplication'', this is has the same form as the definition of an ideal of a ring. So lattice-theorists often call sets of elements in a lattice closed under these operations ''ideals''. In particular, $\{y \mid y \leq x\}$ is called the *principal ideal* generated by x.

Exercise 5.3:6. (i) Show that assertion (ii) of the preceding lemma can also be proved by taking the same S and the same map, but taking $\mathrm{cl}(X) \subseteq S$ to be the intersection of all principal ideals of L containing X.

(ii) Will the complete lattices generated by the images of L under these two constructions in general be isomorphic?

Exercise 5.3:7. Can the representation of a (complete) lattice L by closed sets of a closure operator given in Lemma 5.3.5(ii) and/or that given in Exercise 5.3:6 be characterized by any universal properties?

Exercise 5.3:8. Show that a lattice L is complete and nonempty if and only if every intersection of principal ideals of L (including the intersection of the empty family) is a principal ideal.

The concept of closure operation is not only general enough to allow representations of all lattices, it is a convenient tool for constructing examples. For example, recall that Exercise 5.1:10(ii), if solved by the hint given, shows that a lattice generated by 5 elements can be infinite. With more work, that method could have been made to give an infinite lattice with 4 generators, but one can show that any 3-generator sublattice of the lattice of affine subspaces of a vector space is finite. However, we shall now give an ad hoc construction of a closure operator whose lattice of closed sets has an infinite 3-generator sublattice.

Exercise 5.3:9. Let $S = \omega \cup \{x, y\}$, where ω is regarded as the set of nonnegative integers, and x, y are two elements not in ω. Let $G \subseteq \mathbf{P}(S) \times S$ consist of all pairs

$$(\{x, 2m\},\ 2m+1), \qquad (\{y, 2m+1\},\ 2m+2),$$

where m ranges over ω in each case. Let L denote the lattice of closed subsets of S under the induced closure operator, and consider the sublattice generated by $\{x\}$, $\{y, 0\}$, and ω. Show by induction that for every $n \geq 0$, this sublattice of L contains the set $\{0, \dots, n\}$. Thus, this 3-generator lattice is infinite.

Exercise 5.3:10. The lattice of the above exercise contains an infinite chain. Does there exist a 3-generator lattice which is infinite but does not contain an infinite chain?

Exercise 5.3:11. If A is an abelian group, can a finitely generated sublattice of the lattice of all subgroups of A contain an infinite chain?

We now turn to a property which distinguishes the sort of closure operators commonly occurring in algebra from those arising in topology and analysis.

Lemma 5.3.6. *Let* cl *be a closure operator on a set* S. *Then the following conditions are equivalent:*

(i) *For all* $X \subseteq S$, $\mathrm{cl}(X) = \bigcup_{\text{finite } X_0 \subseteq X} \mathrm{cl}(X_0)$.

(ii) *The union of every chain of closed subsets of* S *is closed.*

(iii) *The closure of each singleton* $\{s\} \subseteq S$ *is compact in the lattice of closed subsets.*

(iv) cl *is the closure operator determined by a set* $G \subseteq \mathbf{P}(S) \times S$ *having the property that the first component of each of its members is finite.* □

Exercise 5.3:12. Prove Lemma 5.3.6.

Definition 5.3.7. *A closure operator satisfying the equivalent conditions of the above lemma is called* finitary.

This is because the lattice of subalgebras of an algebra A has this property if the operations of A are all *finitary*, i.e., have finite *arity* (§1.4). (Some authors call such closure operators ‘‘algebraic’’ instead of ‘‘finitary’’.)

Exercise 5.3:13. (i) Show that an abstract complete lattice L is isomorphic to the lattice of all closed sets under a finitary closure operator if and only if every element of L is a join of compact elements.

(ii) For what complete lattices is it true that *every* closure operator cl, on any set, whose lattice of closed sets is isomorphic to L is finitary?

Exercise 5.3:14. Show that a closure operator cl is finitary if and only if the compact elements in the lattice of its closed subsets are precisely the closures of finite sets. (For a not necessarily finitary closure operator, what is the relation between these two classes of closed sets?)

Exercise 5.3:15. Consider the following three conditions on a closure operator cl on a set S. (a) cl is finitary. (b) The union of any two cl-closed subsets of S is cl-closed. (c) Every singleton subset of S is cl-closed.

For each subset of this set of three properties, find an example of a closure operator that has the properties in that subset, but not any of the others. (Thus, 8 examples are asked for.) Where possible, use familiar or important examples.

Exercise 5.3:16. Show that a closure operator cl on a set S is the operation of topological closure with respect to some topology on S if and only if it satisfies (b) above, and: (c_0) $\varnothing$ is cl-closed in S. Assuming S has more than one element, show that cl is closure with respect to a Hausdorff topology if and only if it satisfies (b) and (c).

(Since the operation of topological closure determines the topology, this shows that topologies on a space are equivalent to closure operations satisfying the indicated conditions.)

Exercise 5.3:17. It is well known that if a group K is generated by $\leq \gamma$ elements (γ a cardinal), then $\mathrm{card}(|K|) \leq \gamma + \aleph_0$.

(i) Deduce this fact from simple properties of the set $G \subseteq \mathbf{P}(|K|) \times |K|$ defined in (5.3.4).

(ii) Try to generalize (i) to a result on the way the cardinalities of sets increase when a closure operator cl obtained as above from a set G is applied to them, in terms of the properties G.

When we described how to construct a closure operator cl from a subset $G \subseteq \mathbf{P}(S) \times S$, it would have been tempting to call cl ‘‘the closure operator generated by G’’. This would not quite have made sense, because a closure operator is not itself a subset of $\mathbf{P}(S) \times S$. However, we can show what this is ‘‘trying to say’’ by setting up a correspondence between closure operators on

S and certain subsets of $\mathbf{P}(S)\times S$:

Exercise 5.3:18. If cl is a closure operator on S, let us write $\sigma(\mathrm{cl}) = \{(A,x) \mid A \subseteq S,\, x\in\mathrm{cl}(A)\}$ and let us call a subset $H \subseteq \mathbf{P}(S)\times S$ a closure *system* on S if $H = \sigma(\mathrm{cl})$ for some closure operator cl on S.

(i) If S is a set, show that closure systems on S are precisely the subsets of $\mathbf{P}(S)\times S$ closed under a certain closure operator, $\mathrm{cl}_{\mathrm{sys}}^{S}$ (which you should describe).

(ii) Show that for any subset $G \subseteq \mathbf{P}(S)\times S$, if we let cl be the closure operator determined by G in the sense discussed earlier, then $\sigma(\mathrm{cl}) = \mathrm{cl}_{\mathrm{sys}}^{S}(G)$.

So although we cannot call cl the closure operator generated by G, it is the operator corresponding to the *closure system* generated by G.

Of course, we cannot resist adding

(iii) Describe $\mathrm{cl}_{\mathrm{sys}}^{S}$ as the closure operator on $\mathbf{P}(S)\times S$ determined (''generated'') by an appropriate set G_{sys}^{S} (of elements of what set?)

We now have three ways of looking at closure data on a set S: as certain families of subsets of S, as certain operators on subsets of S, and as certain ''systems'' contained in $\mathbf{P}(S)\times S$. We take a global look at this data in:

Exercise 5.3:19. Let S be a set. Call the set of all families of subsets of S that are closed under arbitrary intersections $\mathrm{Clofam}(S)$, and order this set by inclusion. Call the set of all closure operators on S $\mathrm{Clop}(S)$, and order it by putting $\mathrm{cl}_1 \leq \mathrm{cl}_2$ if for all X, $\mathrm{cl}_1(X) \leq \mathrm{cl}_2(X)$. Call the set of closure systems on S (in the sense of the preceding exercise) $\mathrm{Closys}(S)$, and order it by inclusion.

Verify that $\mathrm{Clofam}(S)$, $\mathrm{Clop}(S)$ and $\mathrm{Closys}(S)$ are all complete lattices. Do the natural correspondences between the three types of data constitute lattice isomorphisms? If not, state precisely the relationships involved. Interpret in terms of closure operators the meet and join operations of these lattices.

Exercise 5.3:20. Investigate the subset of *finitary* closure operators within the set $\mathrm{Clop}(S)$ defined in the preceding exercise. Will it be closed under meets (finite? arbitrary?) – joins (ditto)? Given any $\mathrm{cl}\in\mathrm{Clop}(S)$, will there be a least finitary closure operator containing cl? A greatest finitary closure operator contained in cl?

Descending from the abstruse to the elementary, here is a problem on closure operators that could be explained to a bright High School student, but which has so far defied solution:

Exercise 5.3:21. (Péter Frankl's question) Let S be a finite set, and cl a closure operator on S such that $\mathrm{cl}(\varnothing) \neq S$. Must there exist an element $s\in S$ which belongs to *not more than half* of the sets closed under cl?

(I usually state this conjecture in terms of ''a system C of subsets of S which is closed under pairwise intersections, and contains at least one proper subset of S''. There are still other formulations; for instance, as asking whether every nontrivial finite lattice has an element which is join-irreducible (not a join of two smaller elements) and which is majorized by no more than half the elements of the lattice.)

One occasionally encounters the dual of the type of data defining a closure operator – a system U of subsets of a set S closed under forming arbitrary *unions*; equivalently, an operator f on subsets of S which is *decreasing*, idempotent, and isotone. These conditions on f mean that the operator $X \mapsto {}^{\mathrm{c}}(f({}^{\mathrm{c}}X))$ (where ${}^{\mathrm{c}}$ denotes complement relative to S) is a closure operator, whose closed subsets are the complements of the members of U. Thus, when such an operator is discovered, it is often convenient to change viewpoints and work with the dual operator ${}^{\mathrm{c}}f^{\mathrm{c}}$, to which one can apply the theory of closure operators. However, U and f may be more natural in some situations than the dual family and map. In such cases one may refer to f as an *interior operator* (though the term is not widely used), since in a topological space, the complement of the closure of the complement of X is called the interior of X. Clearly, every result about closure operators gives a dual result on interior operators.

(Péter Frankl's question, introduced in the last exercise, is often stated in dual form, asking whether, given a system C of subsets of a finite set S which is closed under pairwise unions and contains at least one nonempty subset of S, there must exist a member of S belonging to at least half the members of C. As such, it is called the ''union-closed set'' question, and papers on the topic can be found by searching for the keyword ''union-closed''.)

5.4. Digression: a pattern of threes. It is curious that many basic mathematical definitions involve similar systems of three parts. A *group structure* on a set is given by (1) a neutral element, (2) an inverse-operation and (3) a multiplication; these must satisfy (1) the neutral-element laws, (2) the inverse laws and (3) the associative law. A *partial ordering* on a set is a binary relation that is (1) reflexive, (2) antisymmetric and (3) transitive, while an *equivalence relation* is (1) reflexive, (2) symmetric and (3) transitive. The operation of a *semilattice* is (1) idempotent, (2) commutative and (3) associative. A *closure operator* is (1) increasing, (2) isotone and (3) idempotent. In a *metric space*, the metric satisfies (1) a condition on when distances are 0, (2) symmetry and (3) the triangle inequality.

This parallelism is not just numerical. The general pattern seems to be that the simplest condition or operation has to do with the relation of an element to itself; the intermediate one tells us, if we know how two elements relate in one order, how they relate in the reverse order; and the strongest tells us how to use the relation of one element to a second and this second to a third to get a relation between the first and the third.

Let us see this in the examples listed above. We must distinguish in some cases between the abstract structures and the ''concrete'' structures that motivated them.

The concrete situation motivating the concept of a group is that of a group of permutations of a set. For a set of permutations to form a group, (1) it should contain the permutation that takes every element of the set to itself, (2) if it contains a permutation x, it should also contain the permutation x^{-1} which carries q to p whenever x carries p to q, and (3) along with any permutations x and y it should contain the permutation xy, which carries p to r whenever y carries p to q and x carries q to r. So this fits the pattern described.

When we look at the definition of an *abstract* group G, the above *closure conditions* are replaced by *operations* of neutral element, inverse, and composition. The conditions on these operations needed to mimic the internal properties of permutation groups say that when G acts on itself by left or right multiplication, the three operations of G actually behave like the constructions they are modeled on: left or right multiplication by the neutral element leaves all elements of $|G|$ fixed, left or right multiplication by x is ''reversed'' by the action of x^{-1}, and left or right multiplication by x followed by multiplication on the same side by y is equivalent to multiplication by yx, respectively xy. These are the neutral-element, inverse and associative laws (slightly reformulated). Finally, when we *return* from this abstract concept to its concrete origins via the concept of a G-set X, we again have three conditions, saying that the actions of the neutral element, of inverses of elements, and of composites of elements of G behave on X in the proper manner. (However, the condition for inverses is a consequence of the other two plus the group identities of G, and so can be, and usually is, omitted from the definition of a G-set.)

In the definitions of *partial ordering*, of *equivalence relation*, and of *metric*, we do not have an abstraction of a structure on a set, but such a structure itself. The reader can easily verify that these 3-part definitions each have the form we have described.

In the cases of *semilattices* and *closure operators*, one can say roughly that closure operators are the concrete origins and semilattices the abstraction. My general characterization of the three components of these definitions does not, as we shall see, give quite as good a fit in this case. The condition that a closure operator be idempotent, $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$, may be considered a ‘‘transitivity’’ type condition, since it says that if you can get some elements from elements of X, and some further elements from these, then you get those further elements from X itself. The ‘‘reflexivity’’ type condition is the one saying $\mathrm{cl}(X) \supseteq X$, since it means that what one gets from X includes all of X itself. But I cannot see a way of interpreting the remaining condition, $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$, as describing the relation between elements considered in two different orders.

In the abstracted concept, that of a semilattice, the three conditions of idempotence, commutativity, and associativity of the operation $\vee$ do fit the pattern described, but they do not seem to come in a systematic way from the corresponding properties of closure operations.

When one looks at important weakenings of the concepts of group etc., the operation or condition relating pairs of elements taken in one and the other order seems to be the one most naturally removed: Monoids are a useful generalization of groups, and preorders are a useful generalizations both of partial orders and of equivalence relations.

The folklorist Alan Dundes (Department of Anthropology, UC Berkeley) has argued that the number ‘‘three’’ holds a fundamental place in the culture of Western civilization, in ways ranging from traditional stories (three brothers go out to seek their fortune; Goldilocks and the three bears), superstitions (‘‘third time’s a charm’’), verbal formulas (‘‘Tom, Dick and Harry’’) etc., to our 3-word personal names. (See essay in [**52**].) He has raised the challenge of how many of the ‘‘threes’’ occurring in science (archeologists’ division of each epoch into an ‘‘early’’, a ‘‘middle’’ and a ‘‘late’’ period; the three-stage polio vaccination; the three dimensions of physics, etc.) represent circumstances given to us by nature, and how many we have imposed on nature through cultural prejudice!

In the situation we have been discussing, I would argue that the pattern is natural. I don’t claim that the way modern mathematics describes instances of this pattern is the *only* way possible. For instance, if all basic textbooks first defined ‘‘monoid’’, and then defined a group as a monoid with an inverse operation, and similarly first defined ‘‘preorder’’, then defined partial orders and equivalence relations as preorders with certain properties, and so on, then, though we might still find a recurring pattern, it would not appear as a pattern of ‘‘threes’’. In a different direction, if we defined composition in a group or monoid as taking each ordered *n-tuple* of elements $(n \geq 0)$ to its product, and formulated the associative law accordingly, then the neutral element would simply be the empty product, and the neutral-element law a case of the associative law; and again, no ‘‘threes’’ would be apparent. But I don’t think the choices that lead to the way we currently present these things are consciously or unconsciously aimed at bringing in the number ‘‘three’’; they are certain decisions about pedagogy and motivation that happen to give a genuine natural pattern this form.

Let me close this discussion by noting that many of the more complicated objects of mathematical study arise by combining one structure that fits, or partially fits, the pattern we have noted, with another. Thus, a lattice is a set with two *semilattice* structures that satisfy compatibility identities; a ring is given by an *abelian group*, together with a bilinear binary operation on this group under which it is a *monoid*. Interestingly, various refinements of the concept of ‘‘ring’’ involve adding one (or more!) conditions that can be thought of as filling in the missing slot in the multiplicative monoid structure, concerning ‘‘how elements relate in opposite orders’’: a multiplicative inverse operation on nonzero elements gives a *division ring* structure; *commutativity* of multiplication determines the favorite class of rings of contemporary algebra; both together give the class of fields. Still another important ring-theoretic concept which can be thought of in this way is that of a an *involution* on a (not necessarily commutative) ring, that is, an abelian group automorphism $*: |R| \to |R|$ satisfying $x^{**} = x$ and $(xy)^* = y^* x^*$. The complex numbers have all three structures: multiplicative inverses, commutativity, and the involution of complex conjugation.

The concept of a *closure operator* also has an important special case gotten by imposing an additional condition on ‘‘how elements relate in opposite orders’’, namely the ‘‘exchange property’’:

$$(5.4.1) \qquad y \notin \mathrm{cl}(X),\ \ y \in \mathrm{cl}(X \cup \{z\}) \ \Rightarrow\ z \in \mathrm{cl}(X \cup \{y\}) \qquad (X \subseteq S,\ \ y, z \in S).$$

This is the condition which in the theory of vector spaces allows one to conclude that bases have unique cardinalities, and in the theory of transcendental field extensions yields a similar result for transcendence bases. Closure operators satisfying (5.4.1) are called (among other names) *matroids*; cf. [**104**].

The reader familiar with the definition of a Lie algebra over a commutative ring R will note similarly that it is an R-module (a concept which fits into the above pattern in the same way as that of G-set), with an R-bilinear operation, the Lie bracket, which satisfies the alternating identity (which tells *both* the result of bracketing an element with itself, and the relation between bracketings in opposite orders), and the Jacobi identity (which describes how the bracket of an element with the bracket of two others can be described in terms of the operations of bracketing with those elements successively).

I do not attach great importance to the observations of this section. But I have noticed them for years, and thought this would be a good place to mention them.

5.5. Galois connections. Let us introduce this very general concept using the case from which it gets its name:

Galois theory deals with the situation where one is given a field F and a finite group G of automorphisms of F. Given any subset A of F, let A^* denote the set of elements of the group G fixing all elements of A, and given any subset B of G, likewise let B^* be the set of elements of the field F fixed by all members of B. It is not hard to see that in these situations, A^* is always a subgroup of G, and B^* a subfield of F. The Fundamental Theorem of Galois Theory says that the groups A^* are *all* the subgroups of G, and similarly that the sets B^* are *all* the fields between the fixed field of G in F and the whole field F, and gives further information on the relation between corresponding subgroups and subfields.

Some parts of the proof of this theorem use arguments specific to fields and their automorphism groups; but certain other parts can be carried out without even knowing what the words mean. For instance, the result, ‘‘If A is a set of elements of the field F, and A^{**} is the set of elements of F fixed by all automorphisms in G that fix all elements of A, then $A^{**} \supseteq A$’’ is clearly true independent of what is meant by a ‘‘field’’, an ‘‘automorphism’’, or ‘‘to fix’’!

This suggests that one should look for a general context to which the latter sort of arguments apply. Replacing the set of elements of our field F by an arbitrary set S, the set of elements of the group G by any set T, and the condition of elements of F being fixed by elements of G by any relation $R \subseteq S \times T$, we get the following set of observations:

Lemma 5.5.1. *Let S, T be sets, and $R \subseteq S \times T$ a relation. For $A \subseteq S$, $B \subseteq T$, let us write*

(5.5.2) $$A^* \;=\; \{t \in T \mid (\forall\, a \in A)\ aRt\} \;\subseteq\; T,$$

$$B^* \;=\; \{s \in S \mid (\forall\, b \in B)\ sRb\} \;\subseteq\; S,$$

thus defining two operations written $*$, *one from* $\mathbf{P}(S)$ *to* $\mathbf{P}(T)$ *and the other from* $\mathbf{P}(T)$ *to* $\mathbf{P}(S)$. *Then for* $A, A' \subseteq S$, $B, B' \subseteq T$, *we have*

(i) $A \subseteq A' \Rightarrow A^* \supseteq A'^*$ $\quad B \subseteq B' \Rightarrow B^* \supseteq B'^*$ $\quad$ ($*$ *reverses inclusions*),

(ii) $A^{**} \supseteq A$ $\quad B^{**} \supseteq B$ $\quad$ ($**$ *is increasing*),

(iii) $A^{***} = A^*$ $\quad B^{***} = B^*$ $\quad$ ($*** = *$).

(iv) $**\colon \mathbf{P}(S) \to \mathbf{P}(S)$ *and* $**\colon \mathbf{P}(T) \to \mathbf{P}(T)$ *are closure operators on* S *and* T *respectively.*

(v) *The sets* A^* $(A \subseteq S)$ *are precisely the closed subsets of* T, *and the sets* B^* $(B \subseteq T)$ *are precisely the closed subsets of* S *with respect to these closure operators* $**$.

(vi) *The maps* $*$, *restricted to closed sets, give an antiisomorphism* (*an order-reversing, equivalently,* $\vee$*-and-*$\wedge$*-interchanging, bijection*) *between the complete lattices of* $**$*-closed subsets of* S *and of* T.

Proof. (i) and (ii) are immediate. We shall prove the remaining assertions from those two.

If we apply $*$ to both sides of (ii), so that the inclusions are reversed by (i), we get $A^{***} \subseteq A^*$, $B^{***} \subseteq B^*$; but if we put B^* for A and A^* for B in (ii) we get $B^{***} \supseteq B^*$, $A^{***} \supseteq A^*$. Together these inclusions give (iii). To get (iv), note that by (i) applied twice, the operators $**$ are inclusion-preserving, by (ii) they are increasing, and by applying $*$ to both sides of (iii) we find that they are idempotent. To get (v) note that by (iii) every set B^* respectively A^* is closed, and of course every closed set X has the form Y^* for $Y = X^*$. (vi) now follows from (v), (iii) and (i). □

If for each $t \in T$ we consider the relation $-Rt$ as a condition satisfied by some elements $s \in S$, then for $A \subseteq S$ we can interpret A^{**} as ‘‘the set of elements of S which satisfy all conditions (of this sort) that are satisfied by the elements of A’’. From this interpretation, the fact that $**$ is a closure operation is intuitively understandable.

Definition 5.5.3. *If* S *and* T *are sets, then a pair of maps* $*\colon \mathbf{P}(S) \to \mathbf{P}(T)$ *and* $*\colon \mathbf{P}(T) \to \mathbf{P}(S)$ *satisfying conditions* (i) *and* (ii) *of Lemma 5.5.1* (*and hence the consequences* (iii)-(vi)) *is called a* Galois connection *between the sets* S *and* T.

Exercise 5.5:1. Show that every Galois connection between sets S and T arises from a relation R as in Lemma 5.5.1, and that this relation R is in fact unique.

Thus, a Galois connection on a pair of sets S, T can be characterized either abstractly, by Definition 5.5.3, or as a structure arising from some relation $R \subseteq S \times T$. In all naturally occurring cases that I know of, the relation R is what we start with, and the Galois connection is obtained from it. On the other hand, the characterization as in Definition 5.5.3 has the advantage that it can be generalized by replacing $\mathbf{P}(S)$ and $\mathbf{P}(T)$ by arbitrary partially ordered sets.

Here is another order-theoretic characterization of Galois connections:

Exercise 5.5:2. If S and T are sets, show that a pair of maps $*: \mathbf{P}(S) \to \mathbf{P}(T)$, $*: \mathbf{P}(T) \to \mathbf{P}(S)$ is a Galois connection if and only if for $X \subseteq S$, $Y \subseteq T$, one has

$$X \subseteq Y^* \quad \Leftrightarrow \quad Y \subseteq X^*.$$

More generally, you can show that given two partially ordered sets $(|P|, \leq)$ and $(|Q|, \leq)$, and a pair of maps $*$: $|P| \to |Q|$, $*$: $|Q| \to |P|$, these maps will satisfy conditions (i)-(ii) of Lemma 5.5.1 if and only if they satisfy the above condition (with ‘‘$\leq$’’ in place of ‘‘$\subseteq$’’ throughout).

Exercise 5.5:3. Show that for every closure operator cl on a set S, there exists a set T and a Galois connection between S and T, such that the closure operator $**$ on S induced by the Galois connection is cl. Can one in fact take for T any set given with any closure operator whose lattice of closed subsets is antiisomorphic to the lattice of cl-closed subsets of S?

A Galois connection between two sets S and T becomes particularly valuable when the $**$-closed subsets have characterizations of independent interest. Let us give a number of examples, beginning with the one that motivated our definition. (The reader should not worry if he or she is not familiar with all the concepts and results mentioned in these examples.)

Example 5.5.4. Take for S the underlying set of a field F, and for T the underlying set of a finite group G of automorphisms of F. For $a \in F$ and $g \in G$ let $a R g$ mean that g *fixes* a, that is, $g(a) = a$. If we write $K \subseteq F$ for the subfield G^*, then, as noted earlier, the Fundamental Theorem of Galois Theory tells us that the closed subsets of F are precisely the subfields of F containing K, while the closed subsets of G are all the subgroups of G. One finds that the properties of the field extension F/K are closely related to the properties of the group G, and can be studied with the help of group theory ([**26**, Chapter V], [**28**, Chapter VI]). These important further relations between group structure and field structure are not, of course, part of the general theory of Galois connections. That theory gives the underpinnings, over which these further results are built.

Example 5.5.5. Let us take for S a vector space over a field K, for T the dual space $\mathrm{Hom}_K(S, K)$, and let us take $x R f$ to mean $f(x) = 0$. In this case, one finds that the closed subsets of S are precisely all its vector subspaces, while those of T are the vector subspaces that are closed in a certain topology. In the finite-dimensional case, this topology is discrete, and so the closed subsets of T are *all* its subspaces. The resulting correspondence between subspaces of a finite-dimensional vector space and of its dual space is a basic tool which is taught (or should be!) in undergraduate linear algebra. Some details of the infinite-dimensional case are developed in an exercise below.

Example 5.5.6. A superficially similar example: Let $S = \mathbf{C}^n$ (complex n-space), $T = \mathbf{Q}[x_0, \ldots, x_{n-1}]$, the polynomial ring in n indeterminates over the rationals, and let $(a_0, \ldots, a_{n-1}) R f$ mean $f(a_0, \ldots, a_{n-1}) = 0$. This case is the starting-point for classical algebraic geometry, and still the underlying inspiration for much of the modern theory. The closed subsets of $\mathbf{C}^n$ are the solution-sets of systems of polynomial equations, while the Nullstellensatz says that the closed subsets of $T = \mathbf{Q}[x_0, \ldots, x_{n-1}]$ are the ‘‘radical ideals’’.

Example 5.5.7. Let S be a finite-dimensional vector space over the real numbers $\mathbf{R}$, T the set of pairs (f, a) where f is a linear functional on S and $a\in\mathbf{R}$, and define $xR(f, a)$ to mean $f(x)\leq a$. Then the closed subsets of S turn out to be the closed *convex* sets.

If we restrict a to the value 1, so that we can regard T simply as the dual space of S, and write xRf for the condition $f(x)\leq 1$, we get a Galois connection between S and its dual space, under which the closed subsets, on each side, are the closed convex subsets containing 0. For instance, if we take $S=\mathbf{R}^3$ and identify it with its dual via the natural inner product, we find that the dual of a cube centered at the origin is an octahedron centered at the origin. The regular dodecahedron and icosahedron are similarly dual.

Example 5.5.8. Let M be an abelian group (or more generally, a module over a commutative ring k), and $S=T=$ the ring of endomorphisms of M (as an abelian group, respectively a k-module). Let sRt denote the condition $st=ts$. It is easy to verify that the subsets of $S=T$ closed under the resulting Galois connection are certain subrings (respectively k-subalgebras). For every subring X, the subring X^* is called by ring-theorists the *commutant* of X. If, in this situation, we regard M as an X-module, then X^* is the ring (respectively, k-algebra) of X-module endomorphisms of M. The ring $X^{**}\supseteq X$, the commutant of the commutant, is called the *bicommutant* of X.

Example 5.5.9. Let S be a set of mathematical objects, T a set of propositions about an object of this sort, and sRt the relation ‘‘the object s satisfies the proposition t’’ (in logician’s notation, $s\models t$). Then the closed subsets of S are those classes of objects *definable* by sets of propositions from T, which model theorists call *axiomatic classes*, while the closed subsets of T are what they call *theories*. The theory B^{**} generated by a set B of propositions consists of those members of T that are *consequences* of the propositions in B, in the sense that they hold in all members of S satisfying the latter.

(Actually, in the naturally occurring cases of this example, S is often a proper class rather than a *set* of mathematical objects; e.g., the class of all groups. We will see how to deal comfortably with such situations in the next chapter.)

There are, of course, cases where it is preferable to use symbols other than ‘‘*’’ for the operators of a Galois connection. In Example 5.5.5, it is usual to write the set obtained from a set A as $\mathrm{Ann}(A)$ or A^{o} or $A^{\perp}$ (the *annihilator* or *null space* of A) because ‘‘*’’ is commonly used for the dual space. More seriously, whenever $S=T$ but R is not a symmetric relation on S, the two constructions $\{s' \mid (\forall\, s\in A)\ s'Rs\}$ and $\{s' \mid (\forall\, s\in A)\ sRs'\}$ will be distinct, so one must denote them by different symbols, such as A^* and A_*. An example of such a case is

Exercise 5.5:4. If $S=T=\mathbf{Q}$, the set of rational numbers, and R is the relation $\leq$, characterize the two systems of closed subsets of $\mathbf{Q}$. Describe in as simple a way as possible the structure of the lattices of closed sets.

The next exercise gives, as promised, some details on the infinite-dimensional case of Example 5.5.5. The one that follows it is related to Example 5.5.8.

Exercise 5.5:5. Let K be a field, S a K-vector-space, and T its dual space.

(i) Show that the subsets of S closed under the Galois connection of Example 5.5.5 are indeed all the vector subspaces of S.

To characterize the subsets of T closed under this connection, let us, for each $s\in S$ and $c\in K$, define $U_{s,c}=\{t\in T \mid t(s)=c\}$, and topologize T by making the $U_{s,c}$ a subbasis of open sets.

(ii) Show that the resulting topology is the weakest such that for each $s\in S$, the evaluation map $t \mapsto t(s)$ is a continuous map from T to the discrete topological space K.

(iii) Show that the subsets of T closed under the Galois connection described above are the vector subspaces of T closed in the above topology.

(There is an elegant characterization of the class of topological vector spaces that arise in this way. They are called *linearly compact* vector spaces. See [**76**, Chapter II, 27.6 and 32.1], or for a summary, [**2**, first half of §24].)

Exercise 5.5:6. Let M be the underlying abelian group of the polynomial ring $\mathbf{Q}[t]$ in one indeterminate t, let x: $M \to M$ be the abelian group endomorphism given by *multiplication by* t, and d: $M \to M$ the endomorphism given by *differentiation with respect to* t. Find the commutant and bicommutant (as defined in Example 5.5.8) of each of the following subrings of $\mathrm{End}(M)$:

(i) $\mathbf{Z}[x]$.

(ii) $\mathbf{Z}[x^2, x^3]$.

(iii) $\mathbf{Z}<x, d>$ (the ring generated by x and d. Angle brackets are used to indicate generators of not necessarily commutative rings.)

Exercise 5.5:7. If G is a group and X a subset of G, then $\{g\in G \mid (\forall\, x\in X)\ gx = xg\}$ is called the *centralizer* of X in G, often denoted $C_G(X)$. This is easily seen to be a subgroup of G.

(i) Show that if H is a subgroup of a group G then the following conditions are equivalent: (a) H is commutative, and is the centralizer of its centralizer. (b) H is the intersection of some nonempty family of maximal commutative subgroups of G.

(ii) Give a result about Galois connections of which the above is a particular case.

(You may either state and prove in detail the result of (i), and then for (ii) formulate a general result which can clearly be proved the same way, in which case you need not repeat the argument; or do (ii) in detail, then note briefly how to apply your result to get (i).)

We recall that for a general closure operator on a set S, the union of two closed subsets of S is not in general closed; their join in the lattice of closed sets is the *closure* of this union. However, if we consider the Galois connection between a set of objects and a set of propositions, and if these propositions are the sentences in a language that contains the operator ‘‘*or*’’, then the set of objects satisfying the proposition $(s\ \mathit{or}\ t)$ will be precisely the union of the set of objects satisfying s and the set satisfying t:

$$\{(s\ \mathit{or}\ t)\}^* = \{s\}^* \cup \{t\}^*.$$

Likewise, if the language contains the operator ‘‘*and*’’, then

$$\{(s\ \mathit{and}\ t)\}^* = \{s\}^* \cap \{t\}^*.$$

In fact, the choice of the symbols $\vee$ and $\wedge$ (modifications of $\cup$ and $\cap$) by logicians to represent the operators ‘‘or’’ and ‘‘and’’ was probably suggested by these properties of the sets of objects satisfying such relations. (At least, so I thought when I wrote this. But a student told me he had heard a different explanation: that $\vee$ is an abbreviation of Latin *vel* ‘‘or’’, and $\wedge$ was formed by inverting it.) If we look at closed sets of propositions rather than closed sets of objects, these are, of course, ordered in the reverse fashion: The set of propositions implied by a proposition $s\vee t$ is the *intersection* of those implied by s and those implied by t, while the set implied by $s\wedge t$ is the *closure of the union* of the sets implied by s and by t. Thus the use of the words ‘‘and’’ (which implies something ‘‘bigger’’) and ‘‘or’’ (which suggests a weakening) is based on the proposition-oriented viewpoint, while the choice of symbols $\wedge$ and $\vee$ corresponds to the object viewpoint.

The contrast between these two viewpoints explains the problem students in freshman math courses have when they are asked, say, to describe by inequalities the set of real numbers x satisfying $x^2 \geq 1$. We want the answer ‘‘$x \leq -1$ or $x \geq 1$’’, meaning $\{x \mid x \leq -1 \text{ or } x \geq 1\}$. But they often put ‘‘$x \leq -1$ and $x \geq 1$’’. What they have in mind could be translated as ‘‘$\{x \mid x \leq -1\}$ and $\{x \mid x \geq 1\}$’’. We can hardly tell them that their difficulty arises from the order-reversing nature of the Galois connection between propositions and objects! But the more thoughtful students might be helped if, without going into the formalism, we pointed out that there is a kind of ‘‘reverse relation’’ between statements and the things they refer to: the larger a set of statements, the smaller the set of things satisfying it; the larger a set of things, the smaller the set of statements they all satisfy; so that ‘‘*and*’’ for sets of real numbers translates to ‘‘*or*’’ among formulas defining them.

I generally spend fifteen minutes talking about this ‘‘reverse relation’’ when discussing mathematical notation at the beginning of Berkeley’s undergraduate algebra course. Whether this helps, I don’t know.

Logicians often write the propositions $(\forall x \in X)\ P(x)$ and $(\exists x \in X)\ Q(x)$ as $\bigwedge_{x \in X} P(x)$ and $\bigvee_{x \in X} Q(x)$. Here the universal and existential quantifications are being represented as (possibly infinite) conjunctions and disjunctions, corresponding to intersections and unions respectively of the classes of models defined by the given families of conditions $P(x)$ and $Q(x)$, as x ranges over X.

We have noted that for many naturally arising types of closure operators, the closure of a set X can be constructed both ‘‘from above’’ and ‘‘from below’’ – either by taking the intersection of all closed sets containing X, or by ‘‘building’’ elements of $\mathrm{cl}(X)$ from elements of X by iterating some procedure in terms of which cl was defined. Closure operators determined by Galois connections, however, are born with only a construction ‘‘from above’’: For $X \subseteq S$, $\mathrm{cl}(X)$ is the intersection of those sets $\{t\}^*$ $(t \in T)$ which contain X; the definition of a Galois connection does not provide any way of constructing this set ‘‘from below’’. Rather, this is a recurring type of mathematical problem for the particular Galois connections of mathematical interest! Typically, given such a Galois connection, one looks for operations that all the sets $\{t\}^*$ are closed under, and when one suspects one has found enough of these, one seeks to prove that for every X, $\mathrm{cl}(X)$ is the closure of X under these operations. For instance, the fixed set of an automorphism of a field extension F/K is easily seen to contain all elements of K and to be closed under the field operations; the Fundamental Theorem of Galois Theory says that under appropriate hypotheses, the closed subsets of F are precisely the subsets closed under these operations. When one considers mathematical objects and propositions, then the problem of finding a way to ‘‘build up’’ the closure of a set of *propositions* is that of finding an adequate set of *rules of inference* for the type of proposition under consideration, while to construct the closure operation on *objects* is to characterize intrinsically the axiomatic model classes.

The definition of Galois connection is unfortunately seldom presented in courses, and many mathematicians who discover examples of it have not heard of the general concept. Of course, Lemma 5.5.1 is a set of easy observations which can be verified in any particular case without referring to a general result. But it is useful to have the general concept as a guide, and having proved the lemma, we can skip those trivial verifications from now on.

Summary of §6.1 (read down each column, referring to headings at left, then compare across)

Consider:	All automorphisms of a mathematical object X.	All endomorphisms of a mathematical object X.	All subobjects of a mathematical object X.	All homomorphisms between two mathematical objects X and Y.
Structure:	Set with composition, inverse operation, identity element.	Set with composition and identity element.	Set with relation $\subseteq$.	Four sets $\lvert S\rvert_{00}$, $\lvert S\rvert_{01}$, $\lvert S\rvert_{10}$, $\lvert S\rvert_{11}$ with composition maps $\lvert S\rvert_{jk} \times \lvert S\rvert_{ij} \to \lvert S\rvert_{ik}$ and identity elements id_0, id_1.
Properties:	$(ab)c = a(bc)$ $a^{-1}a = \mathrm{id} = a\,a^{-1}$ $a\,\mathrm{id} = a = \mathrm{id}\;a$.	$(ab)c = a(bc)$ $a\,\mathrm{id} = a = \mathrm{id}\;a$.	transitive, antisymmetric, reflexive.	$(ab)c = a(bc)$ (when defined); $a\,\mathrm{id}_i = a = \mathrm{id}_j\;a$ $(a \in \lvert S\rvert_{ij})$.
Abstract definition:	*group* (above properties, but not assumed to arise as above)	*monoid* (above properties, but not assumed to arise as above)	*partially ordered set* (above properties, but not assumed to arise as above)	*“bimonoid”* (above properties, but not assumed to arise as above)
Other examples:	$(\mathbf{Z}, +, -, 0)$, $\pi_1(X, x_0)$, Z_n, etc.	$(\mathbf{N}, \cdot, 1)$, $(\mathbf{N}, \max, 0)$ $(\boldsymbol{\{}\mathit{f.g.\ ab.\ gps.}\boldsymbol{\}}, \otimes, \mathbf{Z})$	$(\mathbf{Z}, \leq)$, $\Rightarrow$, genealogies, etc.	“$\pi_1(X;\, x_0, x_1)$”, Morita contexts, matrices.
Can be represented by:	permutations of a set (Cayley’s Theorem).	maps of a set into itself.	subsets of a set, under $\subseteq$.	maps between two sets.

Chapter 6. Categories and functors.

6.1. What is a category? Let us lead up to the concept of category by first recalling the motivations for some more familiar mathematical concepts:

(a) *Groups.* The definition of a group is motivated by considering the structure on the set $\mathrm{Aut}(X)$ of all automorphisms of a mathematical object X. Given $a, b \in \mathrm{Aut}(X)$, the *composite* map ab lies in $\mathrm{Aut}(X)$; for every $a \in \mathrm{Aut}(X)$, its *inverse* a^{-1} is a member of $\mathrm{Aut}(X)$, and, of course, the *identity map* id_X always belongs to $\mathrm{Aut}(X)$. Thus, $\mathrm{Aut}(X)$ is a set with a binary operation of composition, a unary operation ''${}^{-1}$'', and a zeroary operation id_X. When one examines the conditions these operations satisfy, one discovers the associative law, the inverse laws, and the neutral-element laws.

These laws and their consequences turn out to be fundamental to a wide class of considerations involving automorphisms, so one makes a general definition: A 4-tuple $G = (|G|, \cdot, {}^{-1}, 1)$, where $|G|$ is a set and $\cdot\,, {}^{-1}, 1$ are operations on $|G|$ satisfying the above laws, is called a *group*.

Let me point out something which is obvious today, but took getting used to for the first generation to see the above definition: The definition does not say that G actually consists of automorphisms of an object X – only that it has certain properties we have abstracted from that context. In fact, systems with these properties are also found to arise in other ways:

The additive structures of the sets of integers, rational numbers, and real numbers form groups.

If (X, x_0) is a topological space with basepoint, the set of homotopy classes of closed curves beginning and ending at x_0 forms a group, $\pi_1(X, x_0)$.

And there are groups that are familiar, not because of a particular way they occur, but because of their importance as basic components in the study of groups in general. The finite cyclic groups $\mathbf{Z}_n$ are the simplest examples.

Despite our abstract definition, and the existence of groups arising in these varied ways, the original motivation of the group concept should not be forgotten. A natural question is: *Which* abstract groups can be represented *concretely*, that is, as families of permutations of a set X under the operations of composition, inverse map, and identity permutation? As we learn in undergraduate algebra, the answer is that *every* group can be so represented (Cayley's Theorem). Let us rederive the well-known proof.

The idea is to use the simplest nontrivial construction of a G-set X: Introduce a single generating element $x \in X$, and let all the elements gx $(g \in |G|)$ be distinct. Formally we may define X to be the set of symbols ''gx'', where x is a fixed symbol and g ranges over $|G|$. We let G act on X in the appropriate way to make this a G-action, namely by the law

$$h(gx) = (hg)x \qquad (g, h \in |G|).$$

The permutations of the set X given by the elements of G are seen to form a ''concrete'' group isomorphic to G. One then observes that the symbol ''x'' is irrelevant to the proof. Stripping it away, we get the textbook proof: ''Let G act on $|G|$ by left multiplication ...''. ([**20**, p. 62], [**24**, p. 9], [**26**, p. 90], [**29**, p. 52].)

(b) *Monoids.* Suppose we consider not just the *automorphisms* of a mathematical object X but all its *endomorphisms*, that is, homomorphisms into itself. The set $\mathrm{End}(X)$ is closed under composition and contains the identity map, but there is no inverse operation. The operations of composition and identity still satisfy associative and neutral-element laws, and one calls any set with a binary operation and a distinguished element 1 satisfying these laws a *monoid*. Like the definition of a group, this definition does *not* require that a monoid actually consist of endomorphisms of an object X.

And indeed, there are again examples which arise in other ways than the one which motivated the definition. The nonnegative integers form a monoid under *multiplication* (with 1 as neutral element), and also under the operation $\max$ (with 0 as neutral element). Isomorphism classes of (say) finitely generated abelian groups form a monoid under the operation induced by ‘‘$\oplus$’’, or alternatively under the operation induced by ‘‘$\otimes$’’. (One may remove some set-theoretic difficulties from this example by restricting oneself to a set of finitely generated abelian groups with exactly one member from each isomorphism class.)

One has the precise analog of Cayley’s Theorem: Every monoid S is isomorphic to a monoid of maps of a set into itself, and this is proved the same way, by letting S act on $|S|$ by left multiplication.

(c) *Partially ordered sets.* Again let X be any mathematical object, and now let us consider the set $\mathrm{Sub}(X)$ of all *subobjects* of X.

In general, we do not have a way of defining interesting *operations* on this set. (There are often operations of ‘‘least upper bound’’ and ‘‘greatest lower bound’’, but not always.) However, $\mathrm{Sub}(X)$ is not structureless; one subobject of X may be *contained in* another, and this inclusion relation is seen to satisfy the conditions of *reflexivity*, *antisymmetry* and *transitivity*.

Again we abstract the situation, calling an arbitrary pair $P = (|P|, \leq)$, where $|P|$ is a set, and $\leq$ is a binary relation on $|P|$ satisfying the above three laws, a *partially ordered set*.

Examples of partial orderings arising in other ways than the above ‘‘prototypical’’ one are the relation ‘‘$\leq$’’ on the integers or the real numbers, and the logical relation ‘‘$\Rightarrow$’’ on a family of inequivalent propositions. Partially ordered sets are also natural models of various hierarchical and genealogical structures in nature, language, and human society.

Given an arbitrary partially ordered set P, will P be isomorphic to a ‘‘concrete’’ partially ordered set – a family of subsets of a set X, ordered by inclusion? Again, let us try to build such an X in as simple-minded a way as possible. We want to associate to every $p \in |P|$ a subset $\bar{p}$ of a set X, so as to duplicate the order relation among elements of P. To make sure all these sets are distinct, let us introduce for each $p \in |P|$ an element $x_p \in X$ belonging to $\bar{p}$, and hence necessarily to every $\bar{q}$ with $q \geq p$, but not to any of the other sets $\bar{q}$ $(q \ngeq p)$. It turns out that this works – if we define X to be the set of symbols $\{x_p \mid p \in |P|\}$, and if for $p \in |P|$ we set $\bar{p} = \{x_q \mid q \leq p\} \subseteq X$, we find that $\{\bar{p} \mid p \in |P|\}$, under the relation ‘‘$\subseteq$’’, forms a partially ordered set isomorphic to P. Again, the symbol ‘‘x’’ is really irrelevant, so we can get a simplified construction by taking $X = |P|$ and $\bar{p} = \{q \mid q \leq p\}$ $(p \in |P|)$. Thus we have ‘‘Cayley’s Theorem for partially ordered sets’’.

(d) ‘‘*Bimonoids.*’’ Let us go back to the ideas that led to the definition of a monoid, but make a small change. Suppose that X and Y are two mathematical objects of the same sort (two sets, two rings, etc.), and we consider the family of all homomorphisms among them. What structure does this system have?

First, it is a system of four sets:

$$\mathrm{Hom}(X, X), \qquad \mathrm{Hom}(X, Y), \qquad \mathrm{Hom}(Y, X), \qquad \mathrm{Hom}(Y, Y).$$

Elements of certain of these sets can be composed with elements of others, giving us *eight* composition maps:

$$\begin{aligned} \mu_{XXX}: &\ \mathrm{Hom}(X,X) \times \mathrm{Hom}(X,X) \to \mathrm{Hom}(X,X), \\ \mu_{XXY}: &\ \mathrm{Hom}(X,Y) \times \mathrm{Hom}(X,X) \to \mathrm{Hom}(X,Y), \\ &\quad \bullet \quad \bullet \quad \bullet \\ \mu_{YYY}: &\ \mathrm{Hom}(Y,Y) \times \mathrm{Hom}(Y,Y) \to \mathrm{Hom}(Y,Y). \end{aligned}$$

(There is no composition on the remaining eight pairs, e.g., $\mathrm{Hom}(X,Y) \times \mathrm{Hom}(X,Y)$.)

These composition operations are associative – we have *sixteen* associative laws; namely, for every 4-tuple (Z_0, Z_1, Z_2, Z_3) of objects from $\{X, Y\}$ (e.g., (Y, Y, X, Y)) we get the law

(6.1.1) $$(ab)c = a(bc)$$

for maps:

$$Z_0 \xrightarrow{c} Z_1 \xrightarrow{b} Z_2 \xrightarrow{a} Z_3.$$

(We could write (6.1.1) more precisely by specifying the four μ's involved.) We also have two neutral elements, $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ and $\mathrm{id}_Y \in \mathrm{Hom}(Y,Y)$, satisfying eight neutral element laws, which you can write down.

Cumbersome though this description is, it is clear that we have here a fairly natural mathematical structure, and we might abstract these conditions by defining a *bimonoid* to be any system of sets and operations

$$S = \big((|S|_{ij})_{i,j\in\{0,1\}},\ (\mu_{ijk})_{i,j,k\in\{0,1\}},\ (1_i)_{i\in\{0,1\}}\big)$$

such that the $|S|_{ij}$ are sets, the μ_{ijk} are maps

$$\mu_{ijk}\colon\ |S|_{jk} \times |S|_{ij} \ \to\ |S|_{ik},$$

satisfying associative laws $(ab)c = a(bc)$ on 3-tuples $(a, b, c) \in |S|_{jk} \times |S|_{ij} \times |S|_{hi}$ for all $h, i, j, k \in \{0, 1\}$, and such that the 1_i are elements of $|S|_{ii}$ $(i \in \{0,1\})$ satisfying

$$1_j\, a = a = a\, 1_i \quad (a \in |S|_{ij}).$$

Again, these objects can arise in ways other than the one just indicated:

We can get an analog of the ‘‘π_1’’ construction for groups: If X is a topological space and x_0, x_1 are two points of X, then the set of homotopy classes of paths in X whose initial and final points both lie in $\{x_0, x_1\}$ is easily seen to form a ‘‘bimonoid’’ which we might call $\pi_1(X;\, x_0, x_1)$.

Readers familiar with the ring-theoretic concept of a *Morita context* $(R, S;\ {}_RP_S,\ {}_SQ_R;\ \tau, \tau')$ will see that it also has this form: The underlying sets of the rings R and S play the roles of $|S|_{00}$ and $|S|_{11}$, the underlying sets of the bimodules P and Q give $|S|_{10}$ and $|S|_{01}$, and the required eight multiplication maps are given by the internal multiplication maps of R and S, the bimodule structures of P and Q, and the bilinear maps $\tau\colon P \times Q \to R$, and $\tau'\colon Q \times P \to S$.

Finally, if K is a field and for any two integers i and j we write $M_{ij}(K)$ for the set of $i \times j$ matrices over K, then for any m and n, the four systems of matrices $M_{mm}(K)$, $M_{nm}(K)$, $M_{mn}(K)$, $M_{nn}(K)$, form a ‘‘bimonoid’’ under matrix multiplication. (The astute reader will notice that this is really a disguised case of ‘‘two mathematical objects and maps among them’’, since matrix multiplication is designed precisely to encode composition of linear maps between vector spaces K^m and K^n. And the ring-theorist will note that this matrix example is also a Morita context.)

Is there a ‘‘Cayley’s Theorem for bimonoids’’, saying that any bimonoid S is isomorphic to a subbimonoid of the bimonoid of all maps between two sets X and Y? Following the models of the preceding cases, our approach should be to introduce a small number of elements in X and/or Y, and use them to ‘‘generate’’ the rest of X and Y under the action of elements of S. Will it suffice to introduce a single generator $x\in X$, and let X and Y consist of elements obtained from x by application of the elements of the $|S|_{0j}$? In particular, this would mean taking for Y the set $\{tx \mid t\in|S|_{01}\}$. For some bimonoids S this will work; but in general it will not. For example, one can define a bimonoid S by taking any two monoids for $|S|_{00}$ and $|S|_{11}$, and the empty set for both $|S|_{01}$ and $|S|_{10}$. For such an S, the above construction gives empty Y, though if $|S|_{11}$ is nontrivial it cannot be represented faithfully by an action on the empty set. In the same way, it will not suffice to take *only* a generator in Y.

Let us, therefore, introduce as generators one element $x\in X$ and one element $y\in Y$, and let X be the set of all symbols of either of the forms sx or ty with $s\in|S|_{00}$, $t\in|S|_{10}$, and Y the set of symbols ux or vy with $u\in|S|_{01}$, $v\in|S|_{11}$. If we let S ‘‘act on’’ this pair of sets by defining

$$a(bz) \;=\; (ab)z,$$

whenever $z\in\{x,y\}$, and a and b are members of sets $|S|_{ij}$ such that these symbolic combinations should be meaningful, then we find that this yields an embedding of S in the bimonoid of all maps between X and Y, as desired. The interested reader can work out the details.

(e) *Categories.* We could go on in the same vein, looking at maps among 3, 4, etc., mathematical objects, and define ‘‘trimonoids’’, ‘‘quadrimonoids’’ etc., with larger and larger collections of operations and identities.

But clearly it makes more sense to treat these as cases of one general concept! Let us now, therefore, try to abstract the algebraic structure we find when we look at an arbitrary *family* $\mathbf{X}$ of mathematical objects and the homomorphisms among them.

In the above development of ‘‘bimonoids’’, the index set $\{0,1\}$ that ran through our considerations was the same for all bimonoids. But in the general situation, the corresponding index set must be specified as part of the object. This is the first component of the 4-tuple described in the next definition.

Definition 6.1.2 (provisional). *A* category *will mean a 4-tuple*

$$\mathbf{C} \;=\; (\mathrm{Ob}(\mathbf{C}),\ \mathrm{Ar}(\mathbf{C}),\ \mu(\mathbf{C}),\ \mathrm{id}(\mathbf{C})),$$

where $\mathrm{Ob}(\mathbf{C})$ *is any collection of elements,* $\mathrm{Ar}(\mathbf{C})$ *is a family of sets indexed by the pairs of elements of* $\mathrm{Ob}(\mathbf{C})$:

$$\mathrm{Ar}(\mathbf{C}) \;=\; (\mathbf{C}(X,Y))_{X,\,Y\in\mathrm{Ob}(\mathbf{C})},$$

$\mu(\mathbf{C})$ *is a family of operations*

$$\mu(\mathbf{C}) \;=\; (\mu_{XYZ})_{X,\,Y,\,Z\in\mathrm{Ob}(\mathbf{C})}$$

$$\mu_{XYZ}:\ \mathbf{C}(Y,Z)\times\mathbf{C}(X,Y) \;\to\; \mathbf{C}(X,Z),$$

and $\mathrm{id}(\mathbf{C})$ *is a family of elements*

$$\mathrm{id}(\mathbf{C}) \;=\; (\mathrm{id}_X)_{X\in\mathrm{Ob}(\mathbf{C})}$$
$$\mathrm{id}_X \in \mathbf{C}(X,X),$$

such that, using multiplicative notation for the maps μ_{XYZ}, *the associative identity*

$$a(bc) \;=\; (ab)c$$

is satisfied for all elements $a\in\mathbf{C}(Y,Z)$, $b\in\mathbf{C}(X,Y)$, $c\in\mathbf{C}(W,X)$ $(W,X,Y,Z\in\mathrm{Ob}(\mathbf{C}))$; *and the identity laws*

$$a\,\mathrm{id}_X \;=\; a \;=\; \mathrm{id}_Y\, a$$

are satisfied for all $a\in\mathbf{C}(X,Y)$ $(X,Y\in\mathrm{Ob}(\mathbf{C}))$.

This definition is labeled ‘‘provisional’’ because it avoids the question of what we mean by a ‘‘*collection* of elements $\mathrm{Ob}(\mathbf{C})$’’. If we hope to be able to deal with categories within set theory, we should require the family $\mathrm{Ob}(\mathbf{C})$ to be a *set*. Yet we will find that the most useful applications of category theory are to cases where $\mathrm{Ob}(\mathbf{C})$ consists of *all* algebraic objects of a certain type (e.g., *all groups*), which calls for larger ‘‘collections’’. We will deal with this dilemma in §6.4. In the next section, where we give some examples of categories, we will interpret ‘‘collection’’ broadly or narrowly as the example requires.

I mentioned that the concept of an ‘‘abstract group’’ – a group given as a set of elements with certain operations on them, rather than as a concrete family of permutations of a set – was confusing to people when it was first introduced. The ‘‘abstract’’ concept of a category still causes many people problems – there is a great temptation for beginning students to imagine that the members of $\mathbf{C}(X,Y)$ must be actual *maps* between *sets* X and Y.

One reason for this confusion is that the terminology of category theory is set up to closely copy that of the situation which motivated the concept. The word ‘‘category’’ is suggestive to begin with; ‘‘$\mathrm{Ob}(\mathbf{C})$’’ stands for ‘‘objects of $\mathbf{C}$’’, and this is what elements of $\mathrm{Ob}(\mathbf{C})$ are called; elements of $f\in\mathbf{C}(X,Y)$ are called ‘‘morphisms’’ from X to Y, the objects X and Y are called the ‘‘domain’’ and ‘‘codomain’’ of f, these morphisms are often denoted diagrammatically by arrows, $X \xrightarrow{f} Y$; and objects and morphisms are shown together in the sort of diagrams that are used to represent objects and maps in other areas of mathematics. In place of $\mathbf{C}(X,Y)$, the notation $\mathrm{Hom}(X,Y)$ is very common. And $\mu_{XYZ}(f,g)$ is generally written fg or $f\cdot g$ or $f\circ g$, and so looks just like composition of functions.

So I urge you to note carefully the distinction between the situation that motivated our definition, and the definition itself. Within that definition, the collection $\mathrm{Ob}(\mathbf{C})$ is simply an ‘‘index set’’ for the families of elements on which the composition operation is defined. Hence in discussing an abstract category $\mathbf{C}$, one cannot give arguments based on considering ‘‘an *element* of the object X’’, ‘‘the *image* of the morphism a’’, etc.; any more than in considering an abstract group G one can refer to such concepts as ‘‘the set of points left fixed by G’’. (However, the latter concept is meaningful for concrete groups of permutations, and the former concepts are likewise meaningful for ‘‘concrete categories’’, a concept we will make precise later on.)

Of course, the motivating situation should not be forgotten, and a natural question is: Is every category isomorphic to a system of maps among some sets? We can give a qualified affirmative answer. The complete answer depends on the set-theoretic matters that we have postponed to §6.4.

but if $\mathrm{Ob}(\mathbf{C})$ is actually a *set*, then we can indeed construct sets $(\bar{X})_{X\in\mathrm{Ob}(\mathbf{C})}$, and set maps among these, including the identity map of each of these sets, which form under composition of maps a category isomorphic to $\mathbf{C}$. The proof is the analog of the one we sketched for ‘‘bimonoids’’.

Exercise 6.1:1. Write out the argument indicated above – ‘‘Cayley’s Theorem’’ for a category with only a *set* of objects.

Incidentally, we will now discard the term ‘‘bimonoid’’, since the structure it described was, up to notational adjustment, simply a category having for object-set the two-element set $\{0, 1\}$.

6.2. Examples of categories. To describe a category, one should, strictly, specify the class of *objects*, the *morphism-set* associated with any pair of objects, the *composition* operation on morphisms, and the *identity morphism* of each object. In practice, some of this structure is usually easy to guess from context. When one is dealing with the prototype situation – a family of mathematical objects and all homomorphisms among them – the whole structure is usually clear once the class of objects is named. In other cases the morphism-sets must be specified as well; once this is done the intended composition operation is usually (though not always) obvious. As to the identity elements, these are uniquely determined by the remaining structure (just as in groups or monoids), so the only problem is verifying that they exist, which is usually easy.

Categories consisting of families of mathematical objects and the homomorphisms among them are generally denoted by boldface or script names for the type of object (often abbreviated. The particular abbreviations may vary from author to author.) Some important examples are:

Set, the category of all sets and set maps among them. (Another symbol commonly used for this category is **Ens**, from the French word *ensemble*.)

Group, the category whose objects are all groups, and whose morphisms are the group homomorphisms; and similarly **Ab**, the category of abelian groups.

Monoid, **Semigroup**, **AbMonoid** and **AbSemigroup**, the categories of monoids, semigroups, abelian monoids and abelian semigroups.

$\mathbf{Ring}^1$, and $\mathbf{CommRing}^1$ the categories of associative, respectively associative commutative, rings with unity. (One could denote the corresponding categories of nonunital rings – i.e., 4-tuples $R = (|R|, +, \cdot, -, 0)$, where $|R|$ may or may not contain an element 1 satisfying the neutral law for multiplication, and where, even if rings do happen to possess such elements, morphisms are not required to respect them – by the same symbols with the superscripts ‘‘^{1}’’ deleted; but we will not refer to those categories often enough in these notes to want to fix names form them.)

If R is a unital associative ring, we will write the category of left R-modules R-**Mod**. (The category of right R-modules is often written as **Mod**-R; other notations for these two categories are ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ respectively.) Similarly, for G a group, the category of G-sets will be written G-**Set**; here the morphisms are the set maps respecting the actions of all elements of G.

Top denotes the category of all topological spaces and *continuous maps* among them. Topologists often find it useful to work with topological spaces with basepoint, (X, x_0), so we also define the category $\mathbf{Top}^{\mathrm{pt}}$ of *pointed* topological spaces, the objects of which are such pairs (X, x_0), and the morphisms of which are the continuous maps which send basepoint to basepoint. Much of topology is done under the assumption that the space is Hausdorff; thus one considers the subcategory **HausTop** of **Top** whose objects are the Hausdorff spaces.

We shall write **POSet** for the category of partially ordered sets, with isotone maps for morphisms. If we want to allow only strict isotone maps, i.e., maps respecting the relation ‘‘<’’, we can call the resulting category $\mathbf{POSet}_<$.

We have mentioned that our concept of ‘‘bimonoid’’ was a special case of the concept a category. Let us make this precise. The definition of a category requires specification of the object-set, whereas for bimonoids the implicit object-set was always $\{0,1\}$. So given a bimonoid $S = ((|S|_{ij}),\ (\mu_{ijk}),\ (1_i))$, to get a category $\mathbf{C}$, we throw in a formal first component $\mathrm{Ob}(\mathbf{C}) = \{0,1\}$. We can then define $\mathbf{C}(i,j) = |S|_{ij}$, and we have the category $\mathbf{C} = (\{0,1\},\ (|S|_{ij}),\ (\mu_{ijk}),\ (1_i))$, which we may denote $S_{\mathbf{cat}}$.

This works because the situation from which we abstracted the concept of a bimonoid was a special case of the situation from which we abstracted the concept of a category. Now in fact, the situations from which we abstracted the concepts of *group*, *monoid*, and *partially ordered set* were also special cases of that situation! Can objects of these types similarly be identified with certain kinds of categories?

The objects most similar to bimonoids are the monoids. Since they are modeled after the algebraic structure on the set of endomorphisms of a single algebraic object, let us associate to an arbitrary monoid S a *one*-object category $S_{\mathbf{cat}}$, with object-set $\{0\}$. The only morphism-set to define is $S_{\mathbf{cat}}(0,0)$, we take this to be $|S|$; for the composition map on pairs of elements of $S_{\mathbf{cat}}(0,0)$, we use the composition operation of S, and for the identity morphism, the neutral element of S.

Conversely, if $\mathbf{C}$ is any category with only one object, X, then the unique morphism set $\mathbf{C}(X,X)$ will form a monoid S under the composition operation of $\mathbf{C}$, such that the category $S_{\mathbf{cat}}$ formed as above is isomorphic to our original category $\mathbf{C}$, the only difference being the name of the one object. Thus, a category with exactly one object is ‘‘essentially’’ a monoid.

If we start with a group G, we can similarly form a category $G_{\mathbf{cat}}$ with just one object, 0, whose morphisms are the elements of G and whose composition operation is the composition of G. We cannot incorporate the inverse operation of G as an operation of the category; in fact, what we are doing is essentially forgetting the inverse operation, i.e., forming from G the monoid G_{md}, and then applying the previous construction; thus $G_{\mathbf{cat}} \cong (G_{\mathrm{md}})_{\mathbf{cat}}$. We see that via this construction, a *group* is equivalent to a category which has exactly one object, and in which every morphism is invertible.

Note that for G a group, the one member of $\mathrm{Ob}(G_{\mathbf{cat}})$ should not be thought of as the group G, but as a fictitious mathematical object on which G acts. Thus, morphisms in this category from that one object to itself do not correspond to endomorphisms of G, as students sometimes think, but to *elements* of G.

The case of partially ordered sets is a little different. In the motivating situation, though we started with a single object X, we considered a family of objects obtained from it, namely all its subobjects. Although there might exist many maps among these objects, the structure of partially ordered set only reveals a certain subfamily of these: the inclusion maps. (In fact, since a ‘‘homomorphism’’ means a map which respects the kind of structure being considered, and we are considering these objects as subobjects of X, one could say that a homomorphism *as subobjects* should mean a set map which respects the way the objects are embedded in X, i.e., an inclusion map; so from this point of view, these really are the ‘‘only’’ relevant maps.) A composite of inclusion maps is an inclusion map, and identity maps are (trivial) inclusions, so the subobjects of X and the inclusion maps among them form a category. In this category there is a morphism from A to B if and only if $A \subseteq B$, and the morphism is then unique, so the partial ordering of the subobjects determines the structure of the category.

If we start with an abstract partially ordered set P, we can construct from it an abstract category $P_{\mathbf{cat}}$ in the way suggested by this concrete prototype: Take $\mathrm{Ob}(P_{\mathbf{cat}}) = |P|$, and for all $A, B \in |P|$, define there to be one morphism from A to B if $A \leq B$ in P, none otherwise. What should we take this one morphism to be? This is like asking in our construction of $G_{\mathbf{cat}}$ what to call the one object. The choice doesn't really matter. Since we want to associate to each ordered pair (A, B) with $A \leq B$ in P some element, the easiest choice is to take for that element the pair (A, B) itself. Thus, we can define $P_{\mathbf{cat}}$ to have object-set $|P|$, and for $A, B \in |P|$, take $P_{\mathbf{cat}}(A, B)$ to be the singleton $\{(A, B)\}$ if $A \leq B$, the empty set otherwise. The reader can easily describe the composition operation and identity elements of $P_{\mathbf{cat}}$.

Exercise 6.2:1. Let $\mathbf{C}$ be a category. Show that $\mathbf{C}$ is isomorphic to $P_{\mathbf{cat}}$ for some partially ordered set P if and only if ''there is at most one morphism between any unordered pair of objects''; in the sense that each hom-set $\mathbf{C}(X, Y)$ has cardinality at most 1, and the hom-sets $\mathbf{C}(X, Y)$ and $\mathbf{C}(Y, X)$ do not *both* have cardinality 1 unless $X = Y$.

We mentioned that some groups, such as the cyclic groups $\mathbf{Z}_n$, are of interest as ''pieces'' in terms of which we look at general groups. Thus, to give an element of order n in a group G is equivalent to displaying an isomorphic copy of $\mathbf{Z}_n$ in G, and to give an element satisfying $x^n = e$ is equivalent to displaying a homomorphic image of $\mathbf{Z}_n$ in G. Various simple categories are of interest for essentially the same reason. For instance a *commutative square* $\begin{smallmatrix}\cdot\to\cdot\\ \downarrow\ \ \downarrow\\ \cdot\to\cdot\end{smallmatrix}$ of objects and morphisms in a category $\mathbf{C}$ corresponds to an image in $\mathbf{C}$ of a certain category with four objects, which we can name 0, 1, 2 and 3:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \\
2 \arrow[r] & 3
\end{tikzcd}$$

whose morphisms are the four identity morphisms and the five arrows shown, where the diagonal arrow is *both* the composite of the morphisms from 0 to 1 to 3 and the composite of the morphisms from 0 to 2 to 3. Indeed, this ''diagram category'' might be conveniently named '' $\begin{smallmatrix}\cdot\to\cdot\\ \downarrow\ \ \downarrow\\ \cdot\to\cdot\end{smallmatrix}$ ''.

A simpler example is the diagram category $\cdot\rightrightarrows\cdot$, with two objects and only two nonidentity morphisms, going in the same direction. Copies of this in a category $\mathbf{C}$ correspond to the type of data one starts with in the definitions of *difference kernels* and *difference cokernels*. Still simpler is $\cdot\to\cdot$, which is often called ''$\mathbf{2}$''; an image of this in a category corresponds to a choice of two objects and one morphism between them. (So the category $\mathbf{2}$ takes its place in our vocabulary beside the ordinal 2, the Boolean ring 2, the lattice 2, and the partially ordered set 2 !) A larger diagram category is

$$\cdot\to\cdot\to\cdot\to\cdot\to\ \dots$$

images of which in $\mathbf{C}$ correspond to infinite chains of morphisms. The morphisms of this diagram category are the identity morphisms, the arrows shown in the picture, *and* all composites of these arrows, of which we have exactly one from every object to every object to the right of it. Finally, one might denote by $\cdot\!\circlearrowleft$ a category having one object 0, and, aside from the identity morphism of 0, one other morphism x, and all its powers, x^2, x^3, etc.. An image of this in a category

C will correspond to a choice of an object and a morphism from this object to itself.

(In the above discussion I have been vague about what I meant by an ''image'' of one category in another. In §6.5 we shall introduce the category-theoretic concept analogous to *homomorphism*, in terms of which this can be made precise. At this point, for the sake of giving you some broad classes of examples to think about, I have spoken without having the formal definition at hand.)

The various types of examples we have discussed are by no means disjoint. Three of the above ''diagram categories'' can be recognized as having the form $P_{\mathbf{cat}}$, where P is respectively, a 4-element partially ordered set, the partially ordered set 2, and the partially ordered set of nonnegative integers, while the last example is $S_{\mathbf{cat}}$, for S the free monoid on one generator x.

Many of the other ''nonprototypical'' ways in which we saw that groups, etc., arise also have generalizations to categories:

If R is any ring, we see that multiplication of rectangular matrices over R satisfies precisely the laws for composition of morphisms in a category. Thus, we get a category $\mathbf{Mat}_R$ by defining the objects to be the nonnegative integers, the morphism-set $\mathbf{Mat}_R(m, n)$ to be the set of all $n \times m$ matrices over R, the composition μ to be matrix multiplication, and the morphisms id_n to be the identity matrices I_n. This is not very novel, since as we observed before, matrix multiplication is defined to encode composition of linear maps among free R-modules. But it is interesting to note that the abstract system of matrices over R is not limited to serving that function; if M is any left R-module, one can use $n \times m$ matrices over R to represent operations which carry m-tuples of elements of M to n-tuples formed from these using linear expressions with coefficients in R. This line of thought suggests similar constructions for other sorts of algebraic objects. For instance, we can define a category $\mathbf{C}$ whose objects are again the nonnegative integers, and such that $\mathbf{C}(m, n)$ represents all ways of getting an n-tuple of elements of an arbitrary *group* from an m-tuple using combinations of the *group* operations. Precisely, we can define $\mathbf{C}(m, n)$ to be the set of all n-tuples of *derived group-theoretic operations* in m variables. The composition maps

$$\mathbf{C}(n, p) \times \mathbf{C}(m, n) \rightarrow \mathbf{C}(m, p)$$

can be described in terms of substitution of derived operations.

Generalizing the construction of the fundamental group of a topological space X, one can define a category $\pi_1(X)$ whose objects are all points of the space X, and where a morphism from the point x_0 to the point x_1 is defined to mean a homotopy class of paths from x_0 to x_1.

We can also define categories which have familiar mathematical entities for their objects, but put unexpected twists into the definitions of the morphism-sets. Recall that in the category **Set**, the morphisms from the set X to the set Y are all functions from X to Y. Now formally, a function is a relation $f \subseteq X \times Y$ such that for every $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$. Suppose we drop this restriction, and consider arbitrary relations $R \subseteq X \times Y$. One can compose these using the same formula by which one composes functions: If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, one defines

$$S \circ R \;=\; \{(x, z) \in X \times Z \mid (\exists y \in Y)\ \ (x, y) \in R,\ \ (y, z) \in S\}.$$

This composition of relations is associative, and the identity relations satisfy the identity laws; hence one can define a category **RelSet**, whose objects are ordinary sets, but such that $\mathbf{RelSet}(X, Y)$ is the set of relations in $X \times Y$.

Algebraic topologists work with topological spaces, but instead of individual maps among them, they are most concerned with *homotopy classes* of maps. Thus, they use the category **HtpTop** whose objects are topological spaces, and whose morphisms are such homotopy classes. Composition of continuous maps respects homotopy, allowing one to define the composition operation of this category.

In complex variable theory, one often fixes a point z of the complex plane and considers all analytic functions defined in a neighborhood of z. Different functions in this set are defined on different neighborhoods of z, so these functions do not all have any domain of definition in common. Further, functions which are the same in a neighborhood of z may not agree on the full intersection of their domains, if this intersection is not connected. E.g., the natural logarithm function $\ln(z)$ with value zero at $z = 1$ extends to some connected regions of the plane so as to assume the value $+\pi i$ at the point -1, and to other such regions so as to assume the value $-\pi i$ at that point. To eliminate distinctions which are not relevant to the behavior of functions in the vicinity of the specified point z, one introduces the concept of a *germ of a function* at z. This is an equivalence class of functions defined on neighborhoods of z, under the relation making two functions equivalent if they agree on some common neighborhood of z.

An apparent inconvenience of this concept is that for germs of functions at z, one does not have a well-defined operation of composition. For instance, if f and g are germs of analytic functions at $z = 0$, one cannot generally attach a meaning to $g(f(z))$ unless $f(0) = 0$, because g does not have a well-defined ‘‘value’’ at $f(0)$. (This is the analog of the algebraic problem that given formal power series $f(z) = a_0 + a_1 z + \dots$ and $g(z) = b_0 + b_1 z + \dots$, one cannot in general ‘‘substitute f into g’’ to get another formal power series in z, unless $a_0 = 0$.) But this behavior ceases to be problematic if we define a category **GermAnal**, whose objects are the points of the complex plane, and where a morphism from z to w means a germ of an analytic function at z whose value at z is w. Then for any three points z_0, z_1, z_2, one sees that one does indeed have a well-defined composition operation

$$\mathbf{GermAnal}(z_1,\, z_2) \times \mathbf{GermAnal}(z_0,\, z_1) \;\to\; \mathbf{GermAnal}(z_0,\, z_2).$$

I.e., the partial operation of composition of germs of analytic functions is defined in exactly those cases where it should be, to make these germs the morphisms of a category.

These examples allow endless modification as needed. A topologist may impose the restriction that the topological spaces considered in a given context be Hausdorff, be locally compact, be given with basepoint, etc., and modify the category he or she uses accordingly. The definition of a *germ of a function* is not limited to complex variable theory, so analogs of **GermAnal** can be set up wherever needed. Here is an interesting case:

Exercise 6.2:2. If G and H are groups, let us define an *almost-homomorphism* from G to H to mean a homomorphism $f: G_f \to H$, whose domain G_f is a subgroup of *finite index* in G. Given two almost-homomorphisms f and g from G to H, with domains G_f and G_g, let us write $f \approx g$ if the subgroup $\{x \in |G_f| \cap |G_g| \mid f(x) = g(x)\}$ also has finite index in G.

(i) Show that $\approx$ is an equivalence relation on the set of almost-homomorphisms from G to H.

(ii) Construct a category $\mathbf{C}$ whose objects are all groups, and whose morphisms are the equivalence classes of almost-homomorphisms, under $\approx$.

(iii) Describe the endomorphism-monoid $\mathbf{C}(\mathbf{Z}, \mathbf{Z})$, where $\mathbf{C}$ is the category described above, and $\mathbf{Z}$ is the additive group of integers.

We noted earlier that isomorphism classes of abelian groups formed a monoid under $\otimes$. The reader with some ring-theoretic background might like the following generalization of this monoid to a category.

Exercise 6.2:3. Show that one can define a category $\mathbf{C}$ such that $\mathrm{Ob}(\mathbf{C})$ is the class of all rings, $\mathbf{C}(R,S)$ is, for each $R,S\in\mathrm{Ob}(\mathbf{C})$, the family of all isomorphism-classes $[P]$ of (S,R)-bimodules P, and the composite $[P][Q]$ is the isomorphism class of the tensor product, $[P\otimes_S Q]$, for $[P]\in\mathbf{C}(S,T)$, $[Q]\in\mathbf{C}(R,S)$. (Either ignore the problem that the classes involved in this definition are not sets, or modify the statement in some reasonable way to avoid this problem.)

If you are familiar with Morita equivalence, verify that two objects are isomorphic in this category if and only if they are Morita equivalent rings.

The following example shows that not every plausible definition works:

Exercise 6.2:4. Suppose one attempts to define a category $\mathbf{C}$ by taking all sets for the objects, and letting $\mathbf{C}(X,Y)$ consist of all equivalence classes of set maps $X\to Y$, under the relation that makes $f\approx g$ if $\{x\in X \mid f(x)\neq g(x)\}$ is finite. Show that this does not work, and give an example of the phenomenon that goes wrong (e.g., an example showing that something is not well-defined, or whatever). Instead of sets and finite subsets, you can alternatively use measure spaces and subsets of measure zero.

Here is an interesting variant on the construction $S_{\mathbf{cat}}$, for S a monoid. (For an application, see [**40**].)

Exercise 6.2:5. Let S be a monoid, and X an S-set. We may define a category whose objects are the elements of X, and such that a morphism $x\to y$ $(x,y\in|X|)$ is an element $s\in|S|$ such that $sx=y$. However, to help remind us of the intended domain and codomain of each morphism, let us, rather, take the morphisms $x\to y$ to be all 3-tuples (y,s,x) such that $s\in|S|$ and $sx=y$. We define composition by $(z,t,y)(y,s,x)=(z,ts,x)$; the definition of the identity morphisms should be clear.

(i) Show that the construction $S_{\mathbf{cat}}$ is a special case of this construction.

(ii) In general, can one reconstruct the monoid S and the S-set X from the category $X_{\mathbf{cat}}$?

I don't know the answer to

(iii) Is there a nice characterization of those categories expressible in the form $X_{\mathbf{cat}}$ for S a monoid and X an S-set? What about those which are so expressible with S in fact a group?

6.3. Other notations and viewpoints.

The language and notation of category theory are still far from uniform. Let me note some of the commonest variations on the conventions I have presented.

I have mentioned that what we are writing $\mathbf{C}(X,Y)$ is often written $\mathrm{Hom}(X,Y)$; this may at times be made more explicit as $\mathrm{Hom}_{\mathbf{C}}(X,Y)$; there is also the shorter notation (X,Y). Even though we shall not use the notation $\mathrm{Hom}(X,Y)$, we shall often call these sets ''hom-sets''.

More problematically, the order in which the objects are written may be reversed: Some authors write the set of morphisms from X to Y as $\mathbf{C}(Y,X)$, $\mathrm{Hom}(Y,X)$, etc.. There are advantages to each choice: The order we are using matches the conceptual order of going ''from X to Y'', and the use of arrows drawn from left to right, $X\xrightarrow{a}Y$, but has the disadvantage that composition of morphisms $X\to Y\to Z$ must be described as a map $\mathbf{C}(Y,Z)\times\mathbf{C}(X,Y)\to\mathbf{C}(X,Z)$. Under the reversed notation, composition goes more nicely, $\mathbf{C}(Z,Y)\times\mathbf{C}(Y,X)\to\mathbf{C}(Z,X)$. A different cure for the same problem is to continue to think of elements of $\mathbf{C}(X,Y)$ as morphisms from X to Y (as we are doing), but reverse the way composition is written, letting the composite of $a\in\mathbf{C}(X,Y)$ and $b\in\mathbf{C}(Y,Z)$ be denoted $ab\in\mathbf{C}(X,Z)$, rather than ba. However if one does this, then when writing functions on sets, one is more or less forced to abandon the conventional notation $f(x)$, which leads to the usual order of composition, and write xf instead.

Note that the above difficulties in category-theoretic notation simply mirror conflicts of notation already existing within mathematics!

The elements of $\mathbf{C}(X,Y)$, which we call ‘‘morphisms’’, are called ‘‘arrows’’ by some. (Our notation $\mathrm{Ar}(\mathbf{C})$ for the system of morphism-sets is based on that word; some authors write $\mathrm{Fl}(\mathbf{C})$, based on the French *flèche* (arrow).) Colloquially they are also called ‘‘maps’’ from X to Y, and I may allow myself to fall into this easy usage at times, hoping that you understand by now that they are *not* maps in the literal sense, i.e., functions.

The identity element in $\mathbf{C}(X,X)$ which we are writing id_X is also written I_X (like an identity matrix) or 1_X (just as the identity element of a group is often written 1).

The student has probably noticed at some point in his or her study of mathematics the petty but vexing question: If X is a subset of Y, is the inclusion map of X into Y the ‘‘same’’ as the identity map of X? If we follow the convenient formalization of a function as a set of ordered pairs $(x, f(x))$, then they are indeed the same. But this means that a question like ‘‘Is f surjective?’’ is meaningless; one can only ask whether f is surjective as a map from X to X, whether it is surjective as a map from X to Y, etc.. A formalization more in accord with the way we think about these things might be to define a function $f: X \to Y$ as a 3-tuple $(X, Y, |f|)$, where $|f|$ is the set of ordered pairs used in the usual definition. Then f is *surjective* if and only if the set of second components of members of $|f|$ equals the whole set Y. (Since X is determined by $|f|$, our making X a component of the 3-tuple is, strictly, unnecessary; but it seems worth doing for symmetry. Note that if one wants to use a similar notation for general *relations* $|R| \subseteq X \times Y$, then neither X nor Y will be determined by $|R|$, so one needs both of these in the tuple describing the relation. Having both in the tuple describing a function then allows one to consider the functions from X to Y as a subset of the relations between these sets.)

The same problem arises in abstract form in developing the concept of category: Can an element be a member of two different morphism-sets, $\mathbf{C}(X,Y)$ and $\mathbf{C}(X',Y')$, with $(X,Y) \neq (X',Y')$? Yes under our definitions; however some authors add to the definition of a category the condition that the sets of morphisms between distinct pairs of objects be disjoint.

Let us note what such a condition would entail. In the category **Group**, as an example, a group homomorphism $f: G \to H$ would have to specify not merely its set-theoretic domain and codomain $|G|$ and $|H|$, but the full group structures $G = (|G|, \mu_G, \iota_G, e_G)$ and $H = (|H|, \mu_H, \iota_H, e_H)$. When one thinks about it, this makes good sense, not only in category theory but in ordinary group theory; for without knowing the group structures on $|G|$ and $|H|$, one cannot say whether f is a homomorphism, let alone answer such group-theoretic questions as, say, whether its kernel contains all elements of order 2.

Observe that in set theory, even if one does not define a function so as to determine its codomain, certain things remain well-defined; for example, the composite fg of two composable maps can be defined knowing only the set of ordered pairs by which the maps are defined. But there is nothing in the axioms of a category that says that if g lies in both $\mathbf{C}(X,Y)$ and $\mathbf{C}(X',Y')$, while f lies in both $\mathbf{C}(Y,Z)$ and $\mathbf{C}(Y',Z')$, then the composites $\mu_{XYZ}(f,g)$ and $\mu_{X'\,Y'\,Z'}(f,g)$ need to be the same; so even the symbol ‘‘fg’’ is formally ambiguous.

On the whole, I think it desirable to include in the definition of a category the condition that morphism-sets be disjoint. However, we shall not do so in these notes, largely because it would increase the gap between our category theory and ordinary mathematical usage. So the difficulties mentioned above mean that we have to be careful, understanding for instance that in a given context, we are using fg as a shorthand for $\mu_{XYZ}(f,g)$, which is the only really unambiguous expression. Any structure which is a category $\mathbf{C}$ under our definition can be ‘‘translated’’ to a category $\mathbf{C}^{\mathrm{disj}}$ with disjoint morphism-sets, by using the same objects, and letting $\mathbf{C}^{\mathrm{disj}}(X,Y)$

consist of all 3-tuples $f = (X, Y, |f|)$ with $|f| \in \mathbf{C}(X, Y)$.

Those who do require morphism-sets to be disjoint can play some interesting variations on the definition of category. Instead of defining $\mathrm{Ar}(\mathbf{C})$ to be a family of sets, $\mathrm{Ar}(\mathbf{C}) = (\mathbf{C}(X, Y))_{X,\, Y \in \mathrm{Ob}(\mathbf{C})}$, they can take it to be a single set (or class), the union of all the $\mathbf{C}(X, Y)$'s. To recover *domains* and *codomains* of morphisms, one adds to the definition of a category two operations, dom, cod: $\mathrm{Ar}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$. One then makes composition of morphisms a single map

$$\mu\colon\ \{(f, g) \in \mathrm{Ar}(\mathbf{C})^2 \mid \mathrm{dom}(f) = \mathrm{cod}(g)\} \ \to\ \mathrm{Ar}(\mathbf{C}).$$

One can be even more radical and eliminate reference to objects, as sketched in the next exercise:

Exercise 6.3:1. (i) Let $\mathbf{C}$ be a category such that distinct ordered pairs of objects (X, Y) have disjoint morphism-sets. Let $A = \bigcup_{X,Y} \mathbf{C}(X, Y)$. Let μ denote the composition operation in A, considered now as a *partial map* from $A \times A$ to A, i.e., a function from a subset of $A \times A$ to A. Show that the pair (A, μ) determines $\mathbf{C}$ up to isomorphism.

(ii) Find conditions on a pair (A, μ), where A is a set and μ a partial binary operation on A, which are necessary and sufficient for it to arise, as above, from a category $\mathbf{C}$.

One gets a still nicer structure by combining the above approach with that of giving functions specifying the domain and codomain of each morphism. Namely, given a category $\mathbf{C}$ with disjoint morphism-sets, let A be defined as in (i), let dom: $A \to A$ be the map associating to each morphism f the *identity* morphism of its domain, and similarly let cod: $A \to A$ associate to each morphism the identity morphism of its codomain. Since the pair (A, μ) determines $\mathbf{C}$ up to isomorphism, the same will be true of the 4-tuple $(A, \mu, \mathrm{dom}, \mathrm{cod})$.

(iii) Find necessary and sufficient conditions on a 4-tuple $(A, \mu, \mathrm{dom}, \mathrm{cod})$ for it to arise as above from a category $\mathbf{C}$.

So one could redefine a category as an ordered pair (A, μ) or 4-tuple $(A, \mu, \mathrm{dom}, \mathrm{cod})$ satisfying appropriate conditions.

However, these differences in definition do not make a great difference in how one actually works with categories. If, for instance, one defines a category as a 5-tuple $\mathbf{C} = (\mathrm{Ob}(\mathbf{C}), \mathrm{Ar}(\mathbf{C}), \mathrm{dom}_{\mathbf{C}}, \mathrm{cod}_{\mathbf{C}}, \mathrm{id}_{\mathbf{C}})$, one then immediately makes the definition

$$\mathbf{C}(X, Y) \ = \ \{f \in \mathrm{Ar}(\mathbf{C}) \mid (\mathrm{dom}(f) = X) \wedge (\mathrm{cod}(f) = Y)\},$$

and works with these morphism sets as other category-theorists do. (But I will mention one notational consequence of the morphisms-only approach that can be confusing to the uninitiated: the use, by some categorists, of the name of an object as the name for its identity morphism as well.)

Changing the topic from technical details to attitudes, category theory has been seen by some as the new approach that would revolutionize, unify, and absorb all of mathematics; by others as a pointless abstraction whose content is trivial where it is not incomprehensible.

Neither of these characterizations is justified, but each has a grain of truth. The subject matter of essentially every branch of mathematics can be viewed as forming a category (or a family of categories); but this does not say how much value the category-theoretic viewpoint will have for workers in a given area. The actual role of category theory in mathematics is like that of group theory: Groups come up in all fields of mathematics, because for every mathematical object, we can look at its symmetries, and generally make use of them. In some situations the contribution of group theory is limited to a few trivial observations, and to providing a terminology consistent with that used for similar considerations in other fields. In others, deep group-theoretic results are applicable. Finally, group theory is a branch of algebra in its own right, with its own intrinsically interesting questions. The same is true of category theory.

As with the concept of ‘‘abstract group’’ for an earlier generation, many people are troubled by that of an ‘‘abstract category’’, whose ‘‘objects’’ are structureless primitives, not mathematical objects with ‘‘underlying sets’’, so that in particular, one cannot reason by ‘‘chasing elements’’ around diagrams. I think the difficulty is pedagogic. The problem comes from *expecting* to be able to ‘‘chase elements’’. As one learns category theory (or a given branch thereof), one learns the techniques one *can* use, which is, after all, what one needs to do before one can feel at home in any area of mathematics. These include some reasonable *approximations* of element-chasing, if one wants them.

And there is no objection to sometimes using a mental image in which objects are sets and morphisms are certain maps among them, since this is an important class of cases. One must merely bear in mind that, like all the mental images we use to understand mathematics, it is imperfect.

When one thinks of categories as *algebraic entities* themselves, one should note that the item in the definition of a category that corresponds to the *element* in the definition of a group, monoid etc., is the *morphism*. It is on these that the composition operation, analogous to the multiplication in a group or monoid, is defined. The *object*-set of $\mathbf{C}$, which has no analog in groups or monoids, is essentially an index set, used to classify these elements.

While on the subject of terminology, I will mention one distinction among words (relevant to, but not limited to category theory) which many mathematicians are sloppy about, but which I try to maintain: the difference between *composite* and *composition*. If f and g are maps of sets, or morphisms in a category, such that gf makes sense, it is their *composite*. The operation carrying the pair (f, g) to this element gf is *composition*. This is analogous to the distinction between the *sum* of two integers, $a+b$, and the operation of *addition*.

6.4. Universes. Let us now confront the problem we postponed, of how we can both handle category theory within set theory, and have category theory include concepts like ‘‘the category of sets’’.

A first approach is the following. Formulate the general definition of a category $\mathbf{C}$ so that $\mathrm{Ob}(\mathbf{C})$, and even the families $\mathbf{C}(X, Y)$, are assumed to be *classes*. Do as much as we can in that context – the resulting animals are called *large* categories. Then go on to consider those categories in which the families $\mathbf{C}(X, Y)$ are sets, and prove better results about these – they are called *legitimate* categories, and most of the examples of §6.2 are of this sort. Finally consider categories such that both $\mathrm{Ob}(\mathbf{C})$ and the families $\mathbf{C}(X, Y)$ are sets. These are called *small* categories, and in studying them one can use the full power of set theory.

Unfortunately, in conventional set theory one has one’s hands tied behind one’s back when trying to work with large or even legitimate categories, for there is no mechanism for dealing with *collections of classes*. To get around this, one might try extending set theory. One could remove the assumption that every member of a class must be a set, so as to allow certain classes of proper classes, and extend the axioms to apply to such classes as well as sets – and one would find essentially no difficulty – except that what one had been calling ‘‘classes’’ are now looking more and more like sets!

So let us change the names, and call our old sets *small sets* and the classes, collections of classes, etc., *large sets*. (The word ‘‘class’’ itself we then restore to the function of referring to arbitrary collections of the sets in our set theory, large and small. This includes the collection of all sets, which cannot itself be a member of that theory.) We shall not distinguish between large and small sets in the *axioms*; we will use the axioms of ZFC for arbitrary sets, large and small. To set up the distinction between large sets and small sets, we will use

Definition 6.4.1. *A* universe *is a set* U *satisfying*

(i) $X \in Y \in U \Rightarrow X \in U$.

(ii) $X, Y \in U \Rightarrow \{X, Y\} \in U$.

(iii) $X, Y \in U \Rightarrow X \times Y \in U$.

(iv) $X \in U \Rightarrow \mathbf{P}(X) \in U$.

(v) $X \in U \Rightarrow (\bigcup_{A \in X} A) \in U$.

(vi) $\omega \in U$.

(vii) *If* $X \in U$ *and* $f: X \to U$ *is a function, then* $\{f(x) \mid x \in X\} \in U$.

The axioms of ZFC introduced in §4.4 do not guarantee the existence of a set with the above properties (though if we replace ‘‘a set U’’ by ‘‘a class U’’ in the above definition, the class of all sets satisfies these conditions). Suppose, however, that a universe U exists, and suppose we look at the members of U, and the relation $\in$ among these. Then this subsystem of sets will itself satisfy the ZFC axioms (we have set up the definition of universe precisely to guarantee this) and the ‘‘operations’’ of power set, direct product, etc., will be the same for this ‘‘sub - set-theory’’ as for our given set theory. Hence we can call a member of U a ‘‘small set’’ and an arbitrary set a ‘‘large set’’, and use the indicated concepts of ‘‘small category’’, ‘‘legitimate category’’ and ‘‘large category’’, as defined above, within this context. We define ‘‘group’’, ‘‘ring’’, ‘‘lattice’’, ‘‘topological space’’, etc., as we always did; we further define one of these objects to be ‘‘small’’ if it is a member of U. Then, although *all* groups still do not form a set, all *small* groups do (though they do not form a small set). We can now define **Set**, **Group**, etc., to mean the categories of all small sets, small groups, etc., make the tacit assumption that small objects are all that ‘‘ordinary mathematics’’ cares about, and use large categories to study them! All that needs to be added to ZFC is an axiom saying that there exists a universe, and such an axiom is considered reasonable by set-theorists.

The above is the approach used by Mac Lane [**14**, pp.21-24]. However, we shall go a little further, and, following A. Grothendieck [**56**, §1.1], use ZFC plus an assumption that seems no less reasonable than the existence of one universe, and more elegant. Namely,

Axiom of Universes: *Every set is a member of a universe.*

We shall assume ZFC with the Axiom of Universes from now on.

Now we no longer have to imagine a 2-tiered set theory such that ordinary mathematicians work in the lower tier of ‘‘small’’ sets, and category theorists have access to the higher tier of ‘‘large’’ sets. Rather, categories, just like other mathematical objects, can exist ‘‘at any level’’. But when we want to use categories to study a given sort of mathematical object, we study the category of all objects of that sort belonging to some fixed universe U.

Let us now state things more formally.

Definition 6.4.2. Category *will be defined as in the provisional Definition 6.1.2, but with the ‘‘system’’ of objects* $\mathrm{Ob}(\mathbf{C})$ *explicitly meaning a* set.

Definition 6.4.3. *If U is a universe, a set X will be called U-*small *if $X \in U$. A mathematical object (e.g., a group, a ring, a topological space, etc. – or a category) will be called U-small if it is U-small as a set. In addition, a category $\mathbf{C}$ will be called U-*legitimate *if* $\mathrm{Ob}(\mathbf{C}) \subseteq U$ *and for all $X, Y \in \mathrm{Ob}(\mathbf{C})$, $\mathbf{C}(X, Y) \in U$. U-*large *will mean ''not necessarily U-small''.*

The categories of U-small sets, U-small groups, etc., will be denoted $\mathbf{Set}_{(U)}$, $\mathbf{Group}_{(U)}$, etc..

Thus, $\mathbf{Set}_{(U)}$, $\mathbf{Group}_{(U)}$ etc., are *U-legitimate* categories, and so for every universe U' having U as a member, they are *U'-small* categories.

But we don't want to encumber our notation with these subscripts $_{(U)}$, so we make a convention to suppress them:

Definition 6.4.4. *When the contrary is not indicated, some chosen universe U will be understood to be fixed, and the terms ''small'', ''legitimate'' and ''large'' will mean ''U-small'', ''U-legitimate'' and ''U-large''. When speaking of a collection of mathematical objects (sets, groups, rings, topological spaces etc.), its members will be assumed small. As an exception, ''category'' will mean* legitimate *category. In particular, the symbols* **Set**, **Group**, **Top** *etc., will denote the* legitimate *categories of* small *sets, groups, topological spaces, etc..*

(Note, incidentally, that our term ''U-large'' does not specify any conditions on a set; in particular, it does not mean that it is a subset of U. It simply removes any assumption of U-smallness.)

So things now look more or less as they did when we started, but we know what we are doing!

The distinctions between small and large objects will come into our considerations from time to time. For instance, when we generalize the construction of free groups and other universal objects as *subobjects of direct products*, we will see that the key condition we need is that we be able to choose an appropriate *small* set of objects over which to take the direct product.

Exercise 6.4:1. (i) Show that a group G is small if and only if $|G|$ is small.

(ii) Show that a category $\mathbf{C}$ is small if and only if $\mathrm{Ob}(\mathbf{C})$ is small, and for all $X, Y \in \mathrm{Ob}(\mathbf{C})$, the set $\mathbf{C}(X, Y)$ is small.

Although, as we have seen, one uses *non-small categories* to study *small* objects of other sorts, the tables can be turned. For instance, we may consider closure operators on classes of small (or legitimate) categories, and the lattice of closed sets of such an operator will then be a *large lattice*.

The next couple of exercises show some properties of the class of universes. (The Axiom of Universes is, of course, to be assumed if the contrary is not stated.)

Exercise 6.4:2. (i) Show that the class of universes is not a set.

(ii) Will this same result hold if we weaken the Axiom of Universes to the statement that there is at least one universe (as in Mac Lane)? What if we use the intermediate statement that there is a universe, and that every universe is a member of a larger universe?

Exercise 6.4:3. Let us recursively define the *depth* of a set by the condition that $\mathrm{depth}(X)$ is the least ordinal greater than all the ordinals $\mathrm{depth}(Y)$ for $Y \in X$, and the *width* of a set by the condition that $\mathrm{width}(X)$ is the least cardinal $\geq \mathrm{card}(X)$ and $\geq$ the cardinals $\mathrm{width}(Y)$ for all $Y \in X$.

(i) Say briefly why we can make these definitions.

(ii) Show that for every universe U there exists a cardinal α such that U consists of all sets of width $< \alpha$, and/or show that for every universe U there exists a cardinal α such that U consists of all sets of depth $< \alpha$.

(iii) Obtain bounds for the width of a set in terms of its depth, and vice versa, and if you only did one part of the preceding point, deduce the other part.

(iv) Characterize the cardinals α which determine universes in this manner.

Do the arguments you have used require the Axiom of Universes?

Exercise 6.4:4. Show that if $U \neq V$ are universes, then either $U \in V$ or $V \in U$. Deduce that the relation ‘‘$\in$ or $=$’’ is a well-ordering on the class of universes. (You may wish to use some results from the preceding exercise.)

Exercise 6.4:5. Suppose that we drop from our axioms for set theory the Axiom of Infinity, and in our definition of ‘‘universe’’ replace the condition that every universe contain ω by the condition that every universe contain $\varnothing$. Show that under the new axiom-system, one can recover the Axiom of Infinity using the Axiom of Universes. Show that all but one of the sets which are ‘‘universes’’ under the new definition will be universes under our existing definition, and characterize the one exception.

Exercise 6.4:6. In Exercise 4.5:13 we found that ‘‘most’’ infinite cardinals were regular; namely, that all singular cardinals were limit cardinals; but that among limit cardinals, regular cardinals were rare; we found no example but $\aleph_0$. Show now that the cardinality of any *universe* is a regular limit cardinal.

Remarks: Set-theorists call a regular limit cardinal a *weakly inaccessible cardinal*, because it cannot be ‘‘reached’’ from lower cardinals using either the cardinal successor operation or chains indexed by lower cardinals. The *inaccessible* cardinals, which are the cardinalities of universes, are the cardinals which cannot be reached from lower cardinals using *all* of the constructions of ZFC; i.e., the above two constructions together with the *power set* construction and the Axioms of the Empty Set and Infinity, which hand us 0 and $\aleph_0$. Whether every weakly inaccessible cardinal is inaccessible depends on the assumptions one makes on one’s set theory. The student familiar with the Generalized Continuum Hypothesis will see that this assumption implies that these two concepts do coincide. Discussions of inaccessible cardinals can be found in basic texts on set theory. (For their relation to universes, cf. [**74**]; for some alternative proposals for set-theoretic foundations of category theory, [**79**] and [**55**]; and for a proposal in the opposite direction, [**12**].)

Notice that introducing ‘‘large sets’’ has not eliminated the need for the concept of a ‘‘class’’ – in discussing set theory, one still needs to refer to the class of all sets; and one of the above exercises refers to the class of all universes. However, the need to refer to classes, and the difficulty of not being able to use set-theoretic techniques in such considerations, is greatly reduced, because for many purposes, references to large sets will do.

We cannot be sure that the axiomatization we have adopted will be satisfactory for all the needs of category theory. It is based on the assumption that ‘‘ordinary mathematics’’ can be done within any universe U, so that the set of all U-small objects is a reasonable substitute for what was previously treated as the class of *all* objects. If some area of mathematics studied using category theory should itself require the Axiom of Universes, then to get an adequate version of the set of ‘‘all’’ objects in that area, one might want to define a ‘‘second-order universe’’ to mean a universe V such that every set $X \in V$ is a member of a universe $U \in V$, and introduce a Second Axiom of Universes, saying that every set belongs to a second-order universe! However, the fact that for pre-category-theoretic mathematics, ZFC seemed an adequate set theory suggests that the set theory we have adopted here should be good for a while.

Concerning the basic idea of what we have done, namely to assume a set theory that contains ‘‘sub-set-theories’’ which look like traditional set theory, let us note that these are ‘‘sub-set-theories’’ in the best possible sense: They involve the same membership relation, the same power set operation, etc.. Set theorists often work with ‘‘sub-set-theories’’ in weaker senses; for example, allowing certain sets X to belong to the sub-set-theory without making all subsets of X members of the sub-set-theory. (E.g., they may allow only those that are ‘‘constructible’’ in some way.) The resulting model may still satisfy general axioms such as ZFC, but have other properties significantly different from those of the set theory one started with. This technique is used in proving results of the sort, ‘‘If a certain set of axioms is consistent, so is a modified set of axioms’’. The distinction between what we have done and this more general technique can be compared with the difference between considering a sublattice of a lattice, which by assumption has the same meet and join operations, and considering a subset which also has least upper bounds and greatest lower bounds, and hence can again be regarded as a lattice, but where these least upper and greatest lower bounds are not in general the same as in the original lattice, so that the object is not a sublattice.

We will find the following concept useful at times.

Definition 6.4.5. *A mathematical object will be called* quasi-small *if it is isomorphic to a small object.*

Here the meaning of ‘‘isomorphic’’ will be clear from the context. Thus, a quasi-small set will mean a set with the same cardinality as a small set. A quasi-small group is easily seen to be a group whose underlying set is a quasi-small set.

We shall now return to category theory proper. As we have indicated, our language will in general be, superficially, as before, but there is now a fixed arbitrary universe assumed in the background, and when the contrary is not stated, words such as ‘‘group’’ now mean ‘‘group that is small with respect to our fixed universe’’, etc., while ‘‘category’’ means ‘‘category legitimate with respect to that universe’’.

6.5. Functors. Since categories are themselves a sort of mathematical object, we should have a concept of ‘‘subcategory’’, and some sort of concept of ‘‘homomorphism’’ between categories. The first of these concepts is described in

Definition 6.5.1. *If* $\mathbf{C}$ *is a category, a* subcategory *of* $\mathbf{C}$ *means a category* $\mathbf{S}$ *such that* (i) $\mathrm{Ob}(\mathbf{S})$ *is a subset of* $\mathrm{Ob}(\mathbf{C})$, (ii) *for each* $X, Y \in \mathrm{Ob}(\mathbf{S})$, $\mathbf{S}(X,Y)$ *is a subset of* $\mathbf{C}(X,Y)$, *and* (iii) *the composition and identity operations of* $\mathbf{S}$ *are the restrictions of those of* $\mathbf{C}$.

Examples are clear: The category $\mathbf{Ab}$ of abelian groups is a subcategory of $\mathbf{Group}$. Within $\mathbf{Monoid}$, we can look at the subcategory whose objects are monoids all of whose elements are invertible (and whose morphisms are still all monoid-homomorphisms between these); this will be isomorphic to $\mathbf{Group}$. $\mathbf{Lattice}$ is likewise isomorphic to a subcategory of $\mathbf{POSet}$; here the lattice homomorphisms form a *proper* subset of the isotone maps. A subcategory of $\mathbf{POSet}$ with the same objects, but a smaller set of morphisms, is the one we called $\mathbf{POSet}_<$. Similarly, $\mathbf{Set}$ is a subcategory of $\mathbf{RelSet}$ with the same set of objects, but a more restricted set of morphisms. The *empty category* (no objects, and hence no morphisms) is a subcategory of every category.

The analog of homomorphisms between categories is given in

Definition 6.5.2. *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are categories, then a* functor $F\colon \mathbf{C} \to \mathbf{D}$ *will mean a pair* $(F_{\mathrm{Ob}}, F_{\mathrm{Ar}})$, *where* F_{Ob} *is a map* $\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$, *and* F_{Ar} *is a family* $F_{\mathrm{Ar}} = (F(X,Y))_{X,\,Y\in\mathrm{Ob}(\mathbf{C})}$ *of maps*

$$F(X,Y)\colon\ \mathbf{C}(X,Y) \to \mathbf{D}(F_{\mathrm{Ob}}(X),\, F_{\mathrm{Ob}}(Y)) \quad (X,Y\in\mathrm{Ob}(\mathbf{C})),$$

such that

(i) *for any two composable morphisms* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *in* $\mathbf{C}$, *one has*

$$F(X,Z)(fg) = F(Y,Z)(f)\, F(X,Y)(g).$$

and

(ii) *for every* $X\in\mathrm{Ob}(\mathbf{C})$,

$$F(X,X)(\mathrm{id}_X) = \mathrm{id}_{F_{\mathrm{Ob}}(X)}.$$

When there is no danger of ambiguity, F_{Ob}, F_{Ar}, *and* $F(X,Y)$ *are generally all abbreviated to* F. *Thus, in this notation, the last three displays become* (*more readably*)

$$F\colon\ \mathbf{C}(X,Y) \to \mathbf{D}(F(X),\, F(Y)) \quad (X,Y\in\mathrm{Ob}(\mathbf{C})),$$

$$F(fg) = F(f)\,F(g),$$

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

How do functors arise in the prototypical situation where $\mathbf{C}$ and $\mathbf{D}$ consist of mathematical objects and homomorphisms among them? Since we must first specify the object of $\mathbf{D}$ to which each object of $\mathbf{C}$ is carried, such a functor must be based on a *construction* which gives us for each object of $\mathbf{C}$ an object of $\mathbf{D}$. And indeed, most mathematical constructions, though often discussed as merely associating to each object of one sort an object of another, *also* have the property that to every morphism of objects of the first sort there corresponds naturally a morphism between the constructed objects, in a manner which satisfies just the conditions of the above definition.

Consider, for example the construction of the free group, with which we began this course. To every $X\in\mathrm{Ob}(\mathbf{Set})$ this associates a group $F(X)$, together with a map $u_X\colon X \to |F(X)|$ having a certain universal property. Now if $f\colon X \to Y$ is a set map, it is easy to see how to get a homomorphism $F(f)\colon F(X) \to F(Y)$. Intuitively, this homomorphism acts by ‘‘substituting $f(x)$ for x’’ in elements of $F(X)$ and evaluating the results in $F(Y)$. In terms of the universal property of $F(X)$, ‘‘substituting values in a group G for the generators of $F(X)$’’ means determining a group homomorphism $F(X) \to G$ by specifying its composite with the set map $u_X\colon X \to |F(X)|$. In this case, $F(f)$ is the unique group homomorphism $F(X) \to F(Y)$ such that $F(f)\circ u_X = u_Y\circ f$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u_X"] & Y \arrow[d, "u_Y"] \\
{|F(X)|} \arrow[r, "F(f)"] & {|F(Y)|.}
\end{tikzcd}$$

It is easy to check that when $F(f)$ is defined in this way, one has $F(fg) = F(f)\,F(g)$ and

$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, as required.

Looking in the same way at the construction of *abelianization*, associating to each group G the abelian group $G^{\mathrm{ab}} = G/[G, G]$, we see that every group homomorphism $f: G \to H$ yields a homomorphism of abelian groups $f^{\mathrm{ab}}: G^{\mathrm{ab}} \to H^{\mathrm{ab}}$, describable either concretely in terms of cosets, or using the universal property of the canonical homomorphism $G \to G^{\mathrm{ab}}$. The constructions of free semilattices, universal abelianizations of rings, etc., give similar examples.

Like most mathematical concepts, the concept of functor also has ‘‘trivial’’ examples, that by themselves would not justify the general definition, yet which turn out to have important roles in the theory. The ‘‘construction’’ associating to every group G its underlying set $|G|$ is a functor **Group** $\to$ **Set**, since homomorphisms of groups certainly give maps of underlying sets. One similarly has underlying-set functors from $\mathbf{Ring}^1$, **Lattice**, **Top**, **POSet**, etc., to **Set**. These all belong to the class of constructions called ‘‘forgetful functors’’. Those listed above ‘‘forget’’ all structure on the object, and so give functors to **Set**; other forgetful functors we have seen are the construction $G \mapsto G_{\mathrm{md}}$ of §3.11, taking a group $(|G|, \cdot, ^{-1}, e)$ to the monoid $(|G|, \cdot, e)$, which ‘‘forgets’’ the inverse operation, and the construction taking a ring to its underlying additive group, or to its underlying multiplicative monoid.

The term ‘‘forgetful functor’’ is not a technical one, so one cannot say precisely whether it should be applied to constructions like the one taking a lattice to its ‘‘underlying’’ partially ordered set (where the partial ordering is not part of the 3-tuple formally defining the lattice); but in any case, this is another example of a functor. I likewise don’t know whether one would apply the term ‘‘forgetful’’ to the inclusion of the subcategory **Ab** in the category **Group**, which might be said to ‘‘forget’’ that the groups are abelian, but this too, and indeed, the inclusion of any subcategory in any category, is clearly a functor. In particular, every category $\mathbf{C}$ has an identity functor, $\mathrm{Id}_{\mathbf{C}}$, taking each object and each morphism to itself.

If, instead of looking at the whole underlying set of a group, we consider the set of its elements of exponent 2, we get another example of a functor **Group** $\to$ **Set**; the reader should verify that every group homomorphism does indeed give a map between the corresponding sets.

If R is a ring, the *opposite* ring R^{op} is defined to have the same underlying set, and the same operations $+, -, 0, 1$ as R, but reversed multiplication: $x*y = yx$. A ring homomorphism $f: R \to S$ will also be a homomorphism $R^{\mathrm{op}} \to S^{\mathrm{op}}$, and we see that this makes $(\ )^{\mathrm{op}}$ a functor $\mathbf{Ring}^1 \to \mathbf{Ring}^1$; one which, composed with itself, gives the identity functor. One has similar opposite-multiplication constructions for monoids and groups. The definitions of the opposite (or dual) of a partially ordered set or lattice give functors with similar properties.

Recall that **HtpTop** is defined to have the same objects as **Top**, but has for morphisms *equivalence classes* of continuous maps under homotopy. Thus we have a functor **Top** $\to$ **HtpTop** which preserves objects, and sends every morphism to its homotopy class.

We have mentioned *diagram categories*, such as the ‘‘commuting square diagram’’ $$\begin{tikzcd}\cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot\end{tikzcd}$$ which is useful because ‘‘images’’ of it in any category $\mathbf{C}$ correspond to commuting squares of objects and arrows in $\mathbf{C}$. We can now say this more precisely: Commuting squares in $\mathbf{C}$ correspond to *functors* from this diagram-category into $\mathbf{C}$.

Let us note a few examples of mathematical constructions that are *not* functors. These tend to be of two sorts: those in which morphisms from one object to another can destroy some of the properties used by the construction, and those that involve arbitrary choices. We have noted that the construction associating to every group G the set of elements of exponent 2, $\{x \in |G| \mid x^2 = e\}$, is a functor **Group** $\to$ **Set**. However, if we define $T(G)$ to be the set of elements of *order* 2, $\{x\in|G| \mid x^2 = e,\ x \neq e\}$, we find that a group homomorphism f may take some of these elements to the identity element, so there is no natural way to define ‘‘$T(f)$’’. Similarly, the important group-theoretic construction of the *center* $Z(G)$ of a group G (the subgroup of elements $a\in|G|$ that commute with all elements of G) is not functorial, because even if a is in the center of G, when we apply a homomorphism $f: G \to H$, some elements of H which are not in the image of G may fail to commute with $f(a)$. The construction Aut, taking a group G to its automorphism group, is also not a functor, roughly because when we map G into another group H, there is no guarantee that H will have all the ‘‘symmetries’’ that G does.

Some constructions of these sorts can be ‘‘made into’’ functors by modifying the choice of domain category, so as to restrict the allowed morphisms to those that don’t ‘‘disturb’’ the structure involved. Thus, the construction associating to every group its set of elements of order 2 does give a functor $\mathbf{Group}_{\mathrm{inj}} \to \mathbf{Set}$, if we define $\mathbf{Group}_{\mathrm{inj}}$ to be the category whose objects are groups and whose morphisms are *injective* (one-to-one) group homomorphisms. The construction of the center likewise gives a functor $\mathbf{Group}_{\mathrm{surj}} \to \mathbf{Group}$, where the morphisms of $\mathbf{Group}_{\mathrm{surj}}$ are the *surjective* group homomorphisms. One may make Aut a functor by restricting morphisms to *isomorphisms* of groups.

An example of the other sort, where a construction is not a functor because it involves choices that cannot be made in a canonical way, is that of finding a basis for a vector space. Even limiting ourselves to finite-dimensional vector spaces, so that bases may be constructed without the Axiom of Choice, the finite sequence of choices made is still arbitrary, so that if one chooses a basis B_V for a vector space V, and a basis B_W for a vector space W, there is no natural way to associate to every linear map $V \to W$ a set map $B_V \to B_W$.

In the above discussion we have merely indicated where straightforward attempts to make these constructions into functors went wrong. In several of the following exercises you are asked to prove more precise negative results.

Exercise 6.5:1. (i) Show that there can be no functor F: **Group** → **Set** taking each group to the set of its elements of order 2, no matter how F is made to act on morphisms.

On the other hand,

(ii) Show how to define a functor **Group** → **RelSet** taking every group to its set of elements of order 2. (Since **RelSet** is an unfamiliar category, verify explicitly all parts of the definition of functor.)

Exercise 6.5:2. (i) Show that there can be no functor F: **Group** → **Group** taking each group to its center.

(ii) Can one construct a functor **Group** → **RelSet** taking every group to the set of its central elements?

Exercise 6.5:3. (i) Give an example of a group homomorphism $f: G \to H$ and an automorphism a of G such that there does not exist a unique automorphism a' of H such that $a' f = f a$. In fact, find such examples with f one-to-one (but not onto) and with f onto (but not one-to-one), and in each of these situations, if possible, an example where such a' does not exist, and an example where such a' exists but is not unique. (If you cannot get an example of one of the above combinations, can you show that it does not occur?)

(ii) Find similar examples involving partially ordered sets in place of groups.

(iii) Prove that there is no functor from **Group** (alternatively, from **POSet**) to **Set** (or even to **RelSet**) taking each object to its set of automorphisms.

Exercise 6.5:4. If K is a field, let $\bar{K}$ denote the algebraic closure of K. We recall that any field homomorphism $f: K \to L$ can be extended to a homomorphism of algebraic closures, $\bar{f}: \bar{K} \to \bar{L}$.

(i) Show, however, that there is no way to choose an extension $\bar{f}$ of each field homomorphism f so as to make the algebraic closure construction a functor.

(ii) If we remove the restriction that $\bar{f}$ be an extension of f, can we make algebraic closure a functor?

The next exercise is instructive and entertaining. A full solution to the second part is difficult, but one can get many interesting partial results.

Exercise 6.5:5. Let **FSet** denote the subcategory of **Set** having for objects the finite sets, and for morphisms all set maps among these.

(i) Show that every functor F from **FSet** to **FSet** determines a unique function f from the nonnegative integers to the nonnegative integers, such that for every finite set X, $\mathrm{card}(F(X)) = f(\mathrm{card}(X))$.

(ii) Investigate *which* integer-valued functions f can occur as the functions associated to such functors. If possible, determine necessary and sufficient conditions on f for such an F to exist.

Note that given functors $\mathbf{C} \xrightarrow{G} \mathbf{D} \xrightarrow{F} \mathbf{E}$ between any three categories, we can form the *composite* functor $\mathbf{C} \xrightarrow{FG} \mathbf{E}$ taking each object X to $F(G(X))$ and each morphism f to $F(G(f))$. Composition of functors is clearly associative, and identity functors satisfy the identity laws, so we have a ''category of categories''! This is named in

Definition 6.5.3. **Cat** *will denote the* (*legitimate*) *category whose objects are all* small *categories, and such that for two small categories* $\mathbf{C}$ *and* $\mathbf{D}$, $\mathbf{Cat}(\mathbf{C}, \mathbf{D})$ *is the set of all functors* $\mathbf{C} \to \mathbf{D}$, *with composition of functors defined as above.*

You might be disappointed with this definition, since only a few of the categories we have mentioned have been small (the diagram-categories, and the categories $S_{\mathbf{cat}}$ and $P_{\mathbf{cat}}$ constructed from monoids S and partially ordered sets P). Thus, **Cat** would appear to be of limited importance. But here the Axiom of Universes comes to our aid. The universe U relative to which we have defined ''small category'' is arbitrary. If we want to study the categories of all groups, rings, etc., belonging to a universe U, and functors among these categories, we may choose a universe U' having U as a member, and note that the abovementioned categories, and indeed all U-legitimate categories, are U'-small, hence are objects of $\mathbf{Cat}_{(U')}$. Thus we can apply general results about the construction **Cat** to this situation.

For some purposes, it might also be useful to give a name to the category of all U-legitimate categories, which lies strictly between $\mathbf{Cat}_{(U)}$ and $\mathbf{Cat}_{(U')}$, but we shall not do so here.

Considering functors as ''homomorphisms'' among categories, we should like to define properties of functors analogous to ''one-to-one-ness'' and ''onto-ness''. The complication is that a functor acts both on objects and on morphisms. We have observed that it is the *morphisms* in a category that are like the *elements* of a group or monoid; this leads to the pair of concepts named below. They are not the only analogs of one-one-ness and onto-ness that one ever uses, but they are the most important:

Definition 6.5.4. *Let* $F: \mathbf{C} \to \mathbf{D}$ *be a functor.*

F *is called* faithful *if for all* $X, Y \in \mathrm{Ob}(\mathbf{C}),$ *the map* $F(X,Y): \mathbf{C}(X,Y) \to \mathbf{D}(F(X), F(Y))$ *is one-to-one.*

F *is called* full *if for all* $X, Y \in \mathrm{Ob}(\mathbf{C}),$ *the map* $F(X,Y): \mathbf{C}(X,Y) \to \mathbf{D}(F(X), F(Y))$ *is onto.*

A subcategory of $\mathbf{C}$ *is said to be full if the corresponding* inclusion functor *is full.*

Thus, a full subcategory of $\mathbf{C}$ is determined by specifying a subset of the object-set, and using all the morphisms among the specified objects; the subcategory **Ab** of **Group** is an example. Some examples of nonfull subcategories are $\mathbf{Set} \subseteq \mathbf{RelSet}$ and $\mathbf{POSet}_{<} \subseteq \mathbf{POSet}$. The inclusion of a full subcategory in a category is a full and faithful functor, while the inclusion of a nonfull subcategory is a faithful functor, but is not full. The reader should verify that most of our examples of forgetful functors are faithful but not full, as is, also, the free-group functor $\mathbf{Set} \to \mathbf{Group}$. The functor $\mathbf{Top} \to \mathbf{HtpTop}$ which takes every object (topological space) to itself, and each morphism to its *homotopy class*, is an example of a functor that is full but not faithful. The functor associating to every group the set of its elements of exponent 2 is neither full nor faithful.

Exercise 6.5:6. Show that the abelianization construction, $\mathbf{Group} \to \mathbf{Ab}$ is neither full nor faithful.

Exercise 6.5:7. Is the functor $\mathbf{Monoid} \to \mathbf{Group}$ associating to every monoid its group of invertible elements full? Faithful?

Exercise 6.5:8. (i) Show that the construction associating to each partially ordered set P the category $P_{\mathbf{cat}}$ is a functor $F: \mathbf{POSet} \to \mathbf{Cat}$, and that this functor is full and faithful. Essentially, this says that the concept of functor, when restricted to the class of categories that correspond to partially ordered sets, just gives the concept of isotone map between these sets!

(ii) Which isotone maps between partially ordered sets correspond under F to full functors? To faithful functors?

(iii) Show similarly that the construction associating to each monoid S the category $S_{\mathbf{cat}}$ is a full and faithful functor $E: \mathbf{Monoid} \to \mathbf{Cat}$. Which monoid homomorphisms are sent by E to full, respectively faithful functors?

In §6.1 we sketched a way of ‘‘concretizing’’ any small category $\mathbf{C}$ (Exercise 6.1:1 and preceding discussion). Let us make the details precise now. (Below, we will use ‘‘U’’ for the concretization functor, based on the primary example of *underlying set* functors on categories of mathematical objects. Though we are still assuming a universe in the background, which we have from time to time called ‘‘U’’, we are not giving it any name here.)

Definition 6.5.5. *A* concrete category *means a category* $\mathbf{C}$ *given with a* faithful *functor* $U: \mathbf{C} \to \mathbf{Set}$ (*a ‘‘concretization functor’’*). (*More formally, one would say that the concrete category is the ordered pair* $(\mathbf{C}, U)$.)

So given any small category $\mathbf{C}$, we want to prove the existence of a faithful functor $U: \mathbf{C} \to \mathbf{Set}$. The idea we sketched was to let the family of representing sets – in our present language, the system of sets $U(X)$ – be ‘‘generated’’ by a family of elements $z_Y \in U(Y)$, one for each $Y \in \mathrm{Ob}(\mathbf{C})$, so that the general element of $U(X)$ would look like $U(a)(z_Y)$ $(Y \in \mathrm{Ob}(\mathbf{C}),$ $a \in \mathbf{C}(Y,X))$; and to impose no additional relations on these elements, so that they are all distinct.

Let us use the ordered pair (Y, a) for the element that is to become $U(a)(z_Y)$. Then we should define U to take $X \in \mathrm{Ob}(\mathbf{C})$ to $\{(Y, a) \mid Y \in \mathrm{Ob}(\mathbf{C}),\ a \in \mathbf{C}(Y,X)\}$. Given $b \in \mathbf{C}(X,W)$, we see that $U(b)$ should take $(Y, a) \in U(X)$ to $(Y, ba) \in U(W)$. It is easy to verify that this defines a faithful functor $U: \mathbf{C} \to \mathbf{Set}$, proving

Theorem 6.5.6 (Cayley's Theorem for small categories). *Every small category admits a concretization, i.e., a faithful functor to the category of small sets.* □

Exercise 6.5:9. Verify that the above construction U is a functor, and is faithful. Which elements correspond to the z_Y of our motivating discussion?

Incidentally, if we had required that categories have disjoint morphism-sets, we could have dropped the Y's from the pairs (Y, a), since each a would determine its domain. Then we could simply have taken $U(X) = \bigcup_{Y \in \mathrm{Ob}(\mathbf{C})} \mathbf{C}(Y, X)$.

It is natural to hope for stronger results, so you can try

Exercise 6.5:10. (i) Does every legitimate category admit a concretization – a faithful functor into the (legitimate) category of small sets? (Obviously, most of those we are familiar with do.)

Since this question involves ''big'' cardinalities, you might prefer to examine a mini-version of the same problem:

(ii) Suppose $\mathbf{C}$ is a category with countably many objects, and such that for all $X, Y \in \mathrm{Ob}(\mathbf{C})$, the set $\mathbf{C}(X, Y)$ is finite. Must $\mathbf{C}$ admit a faithful functor into the category of finite sets?

(iii) If the answer to either question is negative, can you find necessary and sufficient conditions for such concretizations to exist?

Of course, a given concretizable category will admit many concretizations, just as a given group has many representations by permutations.

In the construction used to prove Theorem 6.5.6, we introduced a generator in every $U(Y)$ to insure that our functor would be faithful. However the construction we get by taking some particular object Y and introducing just one generator $z_Y \in U(Y)$, again with no relations imposed among the elements $U(a)(z_Y)$, is also worth looking at. It will be the ''part'' of the above construction consisting of elements $U(a)(z_Y)$ for the given Y. Since Y is fixed, each such element is determined by $a \in \mathbf{C}(Y, X)$, so U may be described as taking each object X to the set $\mathbf{C}(Y, X)$. Although it is generally not faithful, this functor will play an important role in our subsequent work, so let us give its standard name (coming from the term ''hom-sets'' for the sets $\mathbf{C}(Y, X)$).

Definition 6.5.7. *For* $Y \in \mathrm{Ob}(\mathbf{C})$, *the* hom functor *induced by* Y, $h_Y : \mathbf{C} \to \mathbf{Set}$, *is defined on objects by*

$$h_Y(X) \;=\; \mathbf{C}(Y, X) \qquad (X \in \mathrm{Ob}(\mathbf{C})),$$

while for a morphism $b \in \mathbf{C}(X, W)$, $h_Y(b)$ *is defined to carry* $a \in \mathbf{C}(Y, X)$ *to* $ba \in \mathbf{C}(Y, W)$.

Examples: On the category $\mathbf{Group}$, the functor $h_{\mathbf{Z}}$ takes each group G to $\mathbf{Group}(\mathbf{Z}, G)$. But a homomorphism from $\mathbf{Z}$ to G is determined by what it does on the generator $1 \in |\mathbf{Z}|$, so the elements of $h_{\mathbf{Z}}(G)$ correspond to the elements of the underlying set of G; i.e., $h_{\mathbf{Z}}$ is essentially the underlying set functor. You should verify that its behavior on morphisms also agrees with that functor. Similarly, $h_{\mathbf{Z}_2}$ may be identified with the functor taking each group to the set of its elements of exponent 2.

Recalling that $2 \in \mathrm{Ob}(\mathbf{Set})$ is a 2-element set, we see that $h_2 : \mathbf{Set} \to \mathbf{Set}$ is essentially the construction $X \mapsto X^2$.

For a topological example, consider the category of topological spaces with basepoint, and homotopy classes of basepoint-preserving maps, and let $(S^1, 0)$ denote the circle with a basepoint chosen. Then $h_{(S^1,0)}(X, x_0) = |\pi_1(X, x_0)|$. (Of course, the most interesting thing about $\pi_1(X, x_0)$ is its group structure. How *this* can be described category-theoretically we shall discover in Chapter 9!)

In the last few paragraphs, we have said a couple of times that a certain functor is ‘‘essentially’’ a certain construction. What we mean should be intuitively clear; we will make these statements precise in §6.9.

6.6. Contravariant functors, and functors of several variables. Consider the construction associating to every set X the additive group $\mathbf{Z}^X$ of integer-valued functions on X, with pointwise operations. This takes objects of **Set** to objects of **Ab**, but given a set map $f: X \to Y$, there is not a natural map $\mathbf{Z}^X \to \mathbf{Z}^Y$ – rather, there is a homomorphism $\mathbf{Z}^Y \to \mathbf{Z}^X$ carrying each integer-valued function a on Y to the function af on X.

There are many similar examples – the construction associating to any set X the Boolean algebra $(\mathbf{P}(X), \cup, \cap, {}^{\mathrm{c}}, \varnothing, X)$ of its subsets, the construction associating to a set X the lower semilattice $(\mathbf{E}(X), \cap)$ of equivalence relations on X, the construction associating to a vector space V its dual V^*, the construction associating to a commutative ring the partially ordered set of its prime ideals. All have the property that from a map going one way among the given objects, one gets a map going the *other* way among constructed objects. It is clear that these constructions take identity maps to identity maps and composite maps to composite maps (though the order of composition must be reversed because of the reversal of the direction of the maps). These properties look like the definition of a functor turned backwards. Let us set up a definition to cover this:

Definition 6.6.1. *If* **C** *and* **D** *are categories, then a* contravariant functor $F: \mathbf{C} \to \mathbf{D}$ *will mean a pair* $(F_{\mathrm{Ob}}, F_{\mathrm{Ar}})$, *where* F_{Ob} (*written* F *when there is no danger of ambiguity*) *is a map* $\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$, *and* F_{Ar} *is a family of maps*

$$F(X, Y):\ \mathbf{C}(X, Y) \ \to\ \mathbf{D}(F(Y), F(X)) \qquad (X, Y \in \mathrm{Ob}(\mathbf{C})),$$

such that (*also abbreviating these maps* $F(X, Y)$ *to* F),

(i) *for any two composable morphisms* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *in* **C**, *one has*

$$F(fg) \ = \ F(g)\,F(f) \quad \text{in } \mathbf{D},$$

and

(ii) *for every* $X \in \mathrm{Ob}(\mathbf{C})$, *one has*

$$F(\mathrm{id}_X) \ = \ \mathrm{id}_{F(X)}.$$

Functors of the sort defined in the preceding section will be called covariant *functors when we want to contrast them with contravariant functors. When the contrary is not indicated, however, ‘‘functor’’ will still mean covariant functor.*

It is easy to see that a composite of two contravariant functors is a covariant functor, while a composite of a covariant and a contravariant functor, in either order, is a contravariant functor.

Contravariant functors can in fact be expressed in terms of covariant functors, thus eliminating the need to prove results separately for the two concepts. We shall do this with the help of

Definition 6.6.2. *If* $\mathbf{C}$ *is a category, then* $\mathbf{C}^{\mathrm{op}}$ *will denote the category defined by*

$$\mathrm{Ob}(\mathbf{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathbf{C}), \qquad \mathbf{C}^{\mathrm{op}}(X,Y) = \mathbf{C}(Y,X),$$
$$\mu(\mathbf{C}^{\mathrm{op}})(f,g) = \mu(\mathbf{C})(g,f), \qquad \mathrm{id}(\mathbf{C}^{\mathrm{op}})_X = \mathrm{id}(\mathbf{C})_X.$$

Thus, a *contravariant* functor $\mathbf{C} \to \mathbf{D}$ is equivalent to a covariant functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$. Of course, one could also describe it as equivalent to a covariant functor $\mathbf{C} \to \mathbf{D}^{\mathrm{op}}$, and at this point we have no way of deciding which reduction is preferable. However, we shall see soon that putting the ''$^{\mathrm{op}}$'' on the domain-category is more convenient.

As in the theory of partially ordered sets, the ''opposite'' construction introduced above allows us to dualize results. Whenever we have proved a result about a general category $\mathbf{C}$, the statement obtained by reversing the directions of all morphisms and the orders of all compositions is also a theorem, which may be proved by applying the original theorem to $\mathbf{C}^{\mathrm{op}}$.

There is a slight notational difficulty in dealing with a category $\mathbf{C}^{\mathrm{op}}$, while referring also to the original category $\mathbf{C}$, for though in the formal definition given above we could distinguish the two composition operations as $\mu(\mathbf{C})$ and $\mu(\mathbf{C}^{\mathrm{op}})$, the usual notation for composition, $f \circ g$ or fg, does not allow such a distinction. There are various ways of getting around this problem. One can use a modified symbol, such as $\circ^{\mathrm{op}}$ or $*$, for the composition of $\mathbf{C}^{\mathrm{op}}$. Or one can keep ''multiplicative notation'', but use different symbols for the same objects and morphisms when considered as elements of $\mathbf{C}$ and of $\mathbf{C}^{\mathrm{op}}$; e.g., let the morphism written $f \in \mathbf{C}(X,Y)$ also be written $\tilde{f} \in \mathbf{C}^{\mathrm{op}}(\tilde{Y},\tilde{X})$, so that one can write $\widetilde{fg} = \tilde{g}\tilde{f}$, relying on the convention that the operation denoted by juxtaposition is determined by context – specifically, by the structure to which the elements being juxtaposed belong. Still other solutions are possible. E.g., one could be daring, and denote the same composite by fg in both $\mathbf{C}$ and $\mathbf{C}^{\mathrm{op}}$, using different conventions, $fg = \mu(f,g)$ in $\mathbf{C}$ and $fg = \mu(g,f)$ in $\mathbf{C}^{\mathrm{op}}$; i.e., writing morphisms with domains ''on the right'' in one category and ''on the left'' in the other.

Most often, one avoids the problem by not writing equations in $\mathbf{C}^{\mathrm{op}}$. One uses this category as an auxiliary construct in discussing contravariant functors and in dualizing results, but avoids dealing explicitly with objects and morphisms inside it.

In these notes, we shall regularly write a contravariant functor from $\mathbf{C}$ to $\mathbf{D}$ as $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$, where F is a covariant functor on $\mathbf{C}^{\mathrm{op}}$, and shall take advantage of the principle of duality mentioned; these are the main uses we shall make of the $^{\mathrm{op}}$ construction. In the rare cases where we have to work explicitly inside $\mathbf{C}^{\mathrm{op}}$, we will generally use modified symbols such as $\tilde{X}$, $\tilde{f}$ (or X^{op}, f^{op}) for objects and morphisms in $\mathbf{C}^{\mathrm{op}}$.

Note that in the category of categories, $\mathbf{Cat}$, the morphisms are the *covariant* functors.

Exercise 6.6:1. (i) Show how to make $^{\mathrm{op}}$ a functor R from $\mathbf{Cat}$ to $\mathbf{Cat}$. Is R a covariant or a contravariant functor?

(ii) Let $R\colon \mathbf{Cat} \to \mathbf{Cat}$ be as in part (i), let $R'\colon \mathbf{POSet} \to \mathbf{POSet}$ be the functor taking every partially ordered set P to the opposite partially ordered set P^{op}, and let $C\colon \mathbf{POSet} \to \mathbf{Cat}$ denote the functor taking each partially ordered set P to the category $P_{\mathbf{cat}}$ (§6.2). Show that $RC = CR'$. This means that the construction of the opposite of a partially ordered set is essentially a case of the construction of the opposite of a category!

(iii) State the analogous result with *monoids* in place of partially ordered sets.

We noted in earlier chapters that given a set map $X \to Y$, there are ways of getting both a map $\mathbf{P}(X) \to \mathbf{P}(Y)$ and a map $\mathbf{P}(Y) \to \mathbf{P}(X)$ (where $\mathbf{P}$ denotes the power-set construction). The next few exercises look at situations of this sort.

Exercise 6.6:2. (i) Write down explicitly how to get from a set map $f\colon X \to Y$ a set map $P_1(f)\colon \mathbf{P}(X) \to \mathbf{P}(Y)$ and a set map $P_2(f)\colon \mathbf{P}(Y) \to \mathbf{P}(X)$. Show that these constructions make the power set construction a functor $P_1\colon \mathbf{Set} \to \mathbf{Set}$ and a functor $P_2\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ respectively. (These are called the *covariant* and *contravariant power set functors*.)

(ii) Examine what structure on $\mathbf{P}(X)$ is *respected* by maps of the form $P_1(f)$ and $P_2(f)$ defined as above. In particular, determine whether each sort of map always respects the operations of finite meets, finite joins, empty meet, empty join, unions of chains, intersections of chains, complements, and the relations ‘‘$\subseteq$’’ and ‘‘$\subset$’’ in power-sets $\mathbf{P}(X)$. (If you are familiar with the standard topologization of $\mathbf{P}(X)$, you can also investigate whether maps of the form $P_1(f)$ and $P_2(f)$ are continuous.) Accordingly, determine whether the constructions P_1 and P_2 which we referred to above as functors from $\mathbf{Set}$, respectively $\mathbf{Set}^{\mathrm{op}}$, to $\mathbf{Set}$, can in fact be made into functors from $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ to $\vee$-$\mathbf{Semilat}$, to $\mathbf{Bool}^1$, etc..

Exercise 6.6:3. Investigate similarly the construction associating to every set X the set $\mathbf{E}(X)$ of *equivalence relations* on X. I.e., for a set map $f\colon X \to Y$, look for functorial ways of inducing maps in one or both directions between the sets $\mathbf{E}(X)$, $\mathbf{E}(Y)$, and determine what structure on these sets is respected by each such construction.

Exercise 6.6:4. (i) Do the same for the construction associating to every group G the set of subgroups of G.

(ii) Do the same for the construction associating to every group G the set of *normal* subgroups of G.

As with covariant functors, there is an important class of contravariant functors which one can define on every category:

Definition 6.6.3. *For any category* $\mathbf{C}$ *and any object* $Y \in \mathrm{Ob}(\mathbf{C})$, *the* contravariant hom functor *induced by* Y, $h^Y\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, *is defined on objects by*

$$h^Y(X) = \mathbf{C}(X, Y) \qquad (X \in \mathrm{Ob}(\mathbf{C})),$$

while for a morphism $b \in \mathbf{C}(W, X)$ *the morphism* $h^Y(b)\colon \mathbf{C}(X, Y) \to \mathbf{C}(W, Y)$ *is defined to carry* $a \in \mathbf{C}(X, Y)$ *to* $ab \in \mathbf{C}(W, Y)$. (*The functor* h_Y *which we previously named ‘‘the hom functor induced by* Y*’’ will henceforth be called ‘‘the covariant hom functor induced by* Y*’’.*)

Examples: Let $\mathbf{C} = \mathbf{Set}$, and let Y be the set $2 = \{0, 1\}$. Recall that every map from a set X into 2 is the characteristic function of a unique subset $S \subseteq X$. Hence $\mathbf{Set}(X, 2)$ can be identified with $\mathbf{P}(X)$. The reader should verify that the behavior of $h^2\colon \mathbf{Set} \to \mathbf{Set}$ on morphisms is exactly that of the contravariant power-set functor.

Let k be a field, and in the category k-$\mathbf{Mod}$ of k-vector spaces, let k denote this field considered as a one-dimensional vector space. Then for any space V, $h^k(V)$ is the underlying set of the *dual* vector space.

Let $\mathbf{R} \in \mathrm{Ob}(\mathbf{Top})$ denote the real line. Then $h^{\mathbf{R}}$ is the construction associating to every topological space X the set of continuous real-valued functions on X. One can vary this example using categories of differentiable manifolds and differentiable maps, etc., in place of $\mathbf{Top}$.

Here are three examples for students familiar with more specialized topics:

In the category of commutative algebras over the rational numbers, if $\mathbf{C}$ denotes the algebra of complex numbers, then $h^{\mathbf{C}}$ is the functor associating to every algebra the set of its ‘‘complex-valued points’’, its *classical spectrum*. In particular, if R is presented by generators $x_0, \ldots, x_{n-1}$ and relations $p_0 = 0, \ldots, p_{m-1} = 0$, then $h^{\mathbf{C}}(R)$ can be identified with the solution-set of the system of polynomial equations $p_0 = 0, \ldots, p_{m-1} = 0$ in complex n-space.

If **LocCpAb** is the category of locally compact abelian groups, and $S = \mathbf{R}/\mathbf{Z}$ is the circle group, then $h^S(A)$ is the underlying set of the *Pontryagin dual* of the group A [**93**, §1.7]. (In the study of nontopological abelian groups, the object $S = \mathbf{Q}/\mathbf{Z}$ plays a somewhat similar role.)

Finally, in the category **HtpTop**, $h^{S^n}(X)$ (where S^n denotes the n-sphere) gives the underlying set of the nth *cohomotopy group* $\pi^n(X)$.

Exercise 6.6:5. Let $2 \in \mathrm{Ob}(\mathbf{POSet})$ denote the set $2 = \{0, 1\}$, ordered in the usual way.

(i) Show that $h^2 : \mathbf{POSet}^{\mathrm{op}} \to \mathbf{Set}$ is faithful.

(ii) Show that for $P \in \mathrm{Ob}(\mathbf{POSet})$, the set $h^2(P)$ can be made a *lattice* with a greatest and a least element, under pointwise operations. Show that h^2 induces a functor $A : \mathbf{POSet}^{\mathrm{op}} \to \mathbf{Lattice}^{0,1}$, where $\mathbf{Lattice}^{0,1}$ denotes the category of lattices with greatest and least elements, and lattice homomorphisms respecting these elements.

(iii) Let us also write $2 \in \mathrm{Ob}(\mathbf{Lattice}^{0,1})$ for the 2-element lattice! Thus we get a functor $h^2 : (\mathbf{Lattice}^{0,1})^{\mathrm{op}} \to \mathbf{Set}$. Show that this functor is *not* faithful.

(iv) Show that for $L \in \mathrm{Ob}(\mathbf{Lattice}^{0,1})$, the set $h^2(L)$ is not in general closed under meet or join, and may not contain a greatest or least element, but that if we partially order lattice homomorphisms by pointwise comparison, h^2 yields a functor $B : (\mathbf{Lattice}^{0,1})^{\mathrm{op}} \to \mathbf{POSet}$.

(v) Show that for P a *finite* partially ordered set, $B(A(P)) \cong P$.

This is just a part of the story of this pair of functors – the student can discover more for him or herself now, or wait till we resume this investigation in §9.11 with more powerful tools at our disposal.

Exercise 6.6:6. Following up on the idea of Exercise 6.5:5, observe that every *contravariant* functor from the category **FSet** of finite sets into itself also determines a nonnegative integer-valued function on the nonnegative integers. Investigate which functions on the nonnegative integers arise as functions associated with contravariant functors.

Exercise 6.6:7. Let **RelFSet** denote the full subcategory of **RelSet** whose objects are finite sets. Investigate similarly the integer-valued functions associated with functors $\mathbf{RelFSet} \to \mathbf{FSet}$, $\mathbf{FSet} \to \mathbf{RelFSet}$, and $\mathbf{RelFSet} \to \mathbf{RelFSet}$. In these cases, it does not matter whether we look at covariant or contravariant functors – why not?

Exercise 6.6:8. We have noted that a composite of two contravariant functors is a covariant functor, etc.. But in terms of the description of contravariant functors as covariant functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$, it is not clear how to formally describe the composite of two contravariant functors (or a composite of the form (contravariant functor)$\circ$(covariant functor)). Show how to reduce these cases to composition of covariant functors, with the help of Exercise 6.6:1(i).

There are still some types of well-behaved constructions which we have not yet fitted into our functorial scheme: (a) Given a *pair* of sets (A, B), we can form the *product* set $A \times B$. We likewise have product constructions for groups, rings, topological spaces, etc., *coproducts* for most of the same types of objects, and the *tensor product* construction on abelian groups. (b) From two objects A and B of any category $\mathbf{C}$, one gets the object $\mathbf{C}(A, B) \in \mathrm{Ob}(\mathbf{Set})$. (c) There are also constructions that combine objects of different categories. For instance, from a commutative ring R and a set X, one can form the *polynomial ring* over R in an X-tuple of indeterminates, $R[X]$.

In each of these cases, maps on the given objects yield maps on the constructed objects. In cases (a) and (c), the maps of constructed objects go the same way as the maps of the given objects, while in case (b) the direction depends on which argument one varies: A morphism $Y \to Y'$ yields a map $\mathbf{C}(X, Y) \to \mathbf{C}(X, Y')$, but a morphism $X \to X'$ yields a map $\mathbf{C}(X', Y) \to \mathbf{C}(X, Y)$.

It is natural to call such constructions *functors of two variables*, and like the concept of contravariant functor, that of a functor of more than one variable can be reduced to our original definition of functor via an appropriate construction on categories.

Definition 6.6.4. *Let* $(\mathbf{C}_i)_{i\in I}$ *be a family of categories. Then the* product *category* $\prod_{i\in I}\mathbf{C}_i$ *will mean the category* $\mathbf{C}$ *defined by*

$$\mathrm{Ob}(\mathbf{C}) \;=\; \prod_I \mathrm{Ob}(\mathbf{C}_i) \qquad \mathbf{C}((X_i)_I, (Y_i)_I) \;=\; \prod_I \mathbf{C}(X_i, Y_i),$$

$$\mu((f_i)_I, (g_i)_I) \;=\; (\mu(f_i, g_i))_I, \qquad \mathrm{id}_{(X_i)_I} \;=\; (\mathrm{id}_{X_i})_I\,.$$

The product of a finite family of categories may be written $\mathbf{C}\times\ldots\times\mathbf{E}$.

A functor F *on a product category is called a* functor of several variables*; a functor of two variables is often called a* bifunctor.

Thus, a functor on a category of the form $\mathbf{C}\times\mathbf{D}^{\mathrm{op}}$ may be described as a ‘‘bifunctor covariant in a $\mathbf{C}$-valued variable and contravariant in a $\mathbf{D}$-valued variable’’. Note that if we tried to express contravariance by putting ‘‘$^{\mathrm{op}}$’’ onto the *codomains* instead of the domains of functors, we would not be able to express this mixed type of functor; hence the preference for putting $^{\mathrm{op}}$ on domains.

A product category $\prod_{i\in I}\mathbf{C}_i$ has a *projection functor* onto each of the categories $\mathbf{C}_i$ $(i\in I)$, taking each object and each morphism to its ith component, and as we might expect from our experience with products of other sorts of mathematical objects, this is characterizable by a universal property:

Theorem 6.6.5. *Let* $(\mathbf{C}_i)_{i\in I}$ *be a family of categories,* $\mathbf{C} = \prod\mathbf{C}_i$ *their product, and* $P_i\colon \mathbf{C}\to\mathbf{C}_i$ *the projection functors. Then for every category* $\mathbf{D}$ *and family of functors* $F_i\colon \mathbf{D}\to\mathbf{C}_i$, *there exists a unique functor* $F\colon \mathbf{D}\to\mathbf{C}$ *such that for each* $i\in I$, $F_i = P_i F$. □

Exercise 6.6:9. Prove the above theorem.

Exercise 6.6:10. Show that a family of categories also has a *coproduct*. (First state the universal property desired.)

Let us note that the two sorts of hom-functors, h_X and h^Y, are in fact pieces of a single bifunctor. In the definition of this functor below, we use ‘‘$\tilde{X}$’’-notation for objects and morphisms in opposite categories, though in presentations elsewhere, you are likely to see no distinctions made.

Definition 6.6.6. *The* bivariant hom-functor *of a category* $\mathbf{C}$ *means the functor*

$$h\colon\ \mathbf{C}^{\mathrm{op}}\times\mathbf{C}\ \to\ \mathbf{Set}$$

which is defined on objects by

$$h(\tilde{X}, Y) \;=\; \mathbf{C}(X, Y) \qquad (X, Y\in\mathrm{Ob}(\mathbf{C})),$$

while for a morphism $(\tilde{p}, q)\in\mathbf{C}^{\mathrm{op}}(\tilde{X},\tilde{W})\times\mathbf{C}(Y, Z)$ *(formed from morphisms* $p\in\mathbf{C}(W, X)$, $q\in\mathbf{C}(Y, Z)$*) we define* $h(\tilde{p}, q)$ *to carry* $a\in\mathbf{C}(X, Y)$ *to* $qap\in\mathbf{C}(W, Z)$.

Thus, each covariant hom-functor h_X can be described as taking objects Y to the objects $h(\tilde{X}, Y)$, and morphisms q to the morphisms $h(\widetilde{\mathrm{id}_X}, q)$, and the contravariant hom-functors h^Y are similarly obtained by putting Y and id_Y in the right-hand slot of the bifunctor h.

Exercise 6.6:11. Extend further the ideas of Exercises 6.5:5, 6.6:6 and 6.6:7, by investigating functions in two nonnegative-integer-valued variables induced by bifunctors $\mathbf{FSet}\times\mathbf{FSet}\to\mathbf{FSet}$, $\mathbf{FSet}^{\mathrm{op}}\times\mathbf{FSet}\to\mathbf{FSet}$, etc..

6.7. Category-theoretic versions of some common mathematical notions: properties of morphisms. We have mentioned that in an abstract category, one cannot speak of ''elements'' of an object, hence one cannot meaningfully ask whether a given morphism is one-to-one or onto. However, we have occasionally spoken of two objects of a category $\mathbf{C}$ being ''isomorphic''. What we meant was, I hope, clear: An *isomorphism* between X and Y means an element $f\in\mathbf{C}(X,Y)$ for which there exists a 2-sided inverse, that is, a morphism $g\in\mathbf{C}(Y,X)$ such that $fg=\mathrm{id}_Y$, $gf=\mathrm{id}_X$. It is clear that in virtually any naturally occurring category, the invertible morphisms are the things one wants to think of as the isomorphisms. (However, in some cases other words are commonly used: In set theory the term is *bijection*, an invertible morphism in **Top** is called a *homeomorphism*, and differential geometers call their invertible maps *diffeomorphisms*.) If X and Y are isomorphic, we will as usual write $X\cong Y$. An isomorphism of an object X with itself is called an *automorphism* of X; these together comprise the *automorphism group* of X.

Exercise 6.7:1. Let $\mathbf{C}$ be a category.

(i) Show that if a morphism $f\in\mathbf{C}(X,Y)$ has both a right inverse g and a left inverse g', then these are equal. (Hence if h and h' are both two-sided inverses of f, then $h=h'$.)

(ii) Show that the relation $X\cong Y$ is an equivalence relation on $\mathrm{Ob}(\mathbf{C})$.

(iii) Show that isomorphic objects in a category have isomorphic automorphism groups.

Our aim in this and the next section will be to look at various other concepts occurring in ''concrete mathematics'' and ask, in each case, whether we can define a concept for abstract categories which will yield the given concept in *many* concrete cases. We cannot expect that there will always be as perfect a fit as there was for the concept of isomorphism! But lack of perfect fit with existing concepts will not necessarily detract from the usefulness of the concepts we find.

Let us start with the concepts of ''one-to-one map'' and ''onto map''. The next exercise shows that we will not be able to get a ''perfect fit'' in either of these cases.

Exercise 6.7:2. Show that a category $\mathbf{C}$ can have two different concretizations $U, V\colon \mathbf{C}\to\mathbf{Set}$ such that for a particular morphism f in $\mathbf{C}$, $U(f)$ is one-to-one but not onto, and $V(f)$ is onto but not one-to-one. (Suggestion: Take $\mathbf{C}=S_{\mathbf{cat}}$, where S is the free monoid on one generator, or $\mathbf{C}=2_{\mathbf{cat}}$, where 2 is the 2-element totally ordered set.)

Nevertheless, there is a category-theoretic property which in the vast majority of naturally occurring concrete categories does correspond to one-one-ness.

Definition 6.7.1. *A morphism $f\colon X\to Y$ in a category $\mathbf{C}$ is called a* monomorphism *if for all $W\in\mathrm{Ob}(\mathbf{C})$ and all pairs of morphisms $g,h\in\mathbf{C}(W,X)$, one has $fg=fh\Rightarrow g=h$; equivalently, if every covariant hom-functor $h_W\colon\mathbf{C}\to\mathbf{Set}$ $(W\in\mathrm{Ob}(\mathbf{C}))$ carries f to a one-to-one set map.*

Exercise 6.7:3. (i) Show that if $(\mathbf{C},U)$ is a concrete category (i.e., $\mathbf{C}$ is a category and $U\colon\mathbf{C}\to\mathbf{Set}$ a faithful functor) and f is a morphism in $\mathbf{C}$ such that $U(f)$ is one-to-one, then f is a monomorphism in $\mathbf{C}$.

(ii) If $\mathbf{C}$ is a small category, show that a morphism f in $\mathbf{C}$ is a monomorphism if and only if for at least one faithful functor $U\colon\mathbf{C}\to\mathbf{Set}$, $U(f)$ is one-to-one.

Exercise 6.7:4. Show that in the categories **Set**, **Group**, **Monoid**, $\mathbf{Ring}^1$, **POSet** and **Lattice**, a morphism is one-to-one if and only if it is a monomorphism. (Suggestion: look for some general criterion, that you can quickly verify in all these cases.) If you are familiar with the basic definitions of general topology, also verify this for **Top**.

Naturally occurring concrete categories where monomorphisms are not the one-to-one maps are rare, but here is an example:

Exercise 6.7:5. A group G is called *divisible* if for every $x\in|G|$ and every positive integer n, there exists $y\in|G|$ such that $x = y^n$.

(i) Show that in the category of divisible groups (a full subcategory of **Group**), the quotient map $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ (where $\mathbf{Q}$ is the additive group of rational numbers and $\mathbf{Z}$ the subgroup of integers) is a monomorphism, though it is not a one-to-one map.

(ii) Can you characterize group-theoretically the homomorphisms that are monomorphisms in this category?

(iii) Can you find a category-theoretic property equivalent in this category to being one-to-one? (You may, if you prefer, examine this last question in the category of divisible *abelian* groups.)

If you are familiar with topological group theory, you may consider the category of connected Lie groups and the canonical map $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$ instead of, or in addition to, divisible groups and $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$.

It is natural to look at the dual to the concept of monomorphism.

Definition 6.7.2. *A morphism* $f\colon X \to Y$ *in a category* $\mathbf{C}$ *is called an* epimorphism *if for all* $Z\in\mathrm{Ob}(\mathbf{C})$ *and all pairs of morphisms* $g,h\in\mathbf{C}(Y,Z)$ *one has* $gf = hf \Rightarrow g = h$; *equivalently, if all the contravariant hom-functors* $h^Z\colon \mathbf{C} \to \mathbf{Set}$ $(Z\in\mathrm{Ob}(\mathbf{C}))$ *carry* f *to one-to-one set maps; equivalently, if the morphism* $\tilde{f}$ *in* $\mathbf{C}^{\mathrm{op}}$ *is a monomorphism.*

This concept *sometimes* coincides with that of surjective homomorphism in naturally occurring concrete categories, but equally often it does not:

Exercise 6.7:6. (i) Show that if $(\mathbf{C}, U)$ is a concrete category, and f a morphism in $\mathbf{C}$ such that $U(f)$ is surjective, then f is an epimorphism in $\mathbf{C}$.

(ii) Show that in the categories **Set** and **Ab**, the epimorphisms are precisely the surjective morphisms.

(iii) Show that in the category **Monoid**, the inclusion of the free monoid on one generator in the free group on one generator is an epimorphism, though not surjective with respect to the underlying-set concretization. (Hint: uniqueness of inverses.) Show similarly that in $\mathbf{Ring}^1$, the inclusion of any integral domain in its field of fractions is an epimorphism.

(iv) If you are familiar with elementary point-set topology, show that in the category **HausTop** of Hausdorff topological spaces, the epimorphisms are precisely the continuous maps with *dense* image.

Exercise 6.7:7. (i) Determine the epimorphisms in **Group**.

(ii) Show the relation between this problem and Exercise 3.10:9.

(iii) Does the method you used in (i) also yield a description of the epimorphisms in the category of *finite* groups? If not, can you nevertheless determine these?

Exercise 6.7:8. (i) Show that for an object A of $\mathbf{Ring}^1$ (or if you prefer, $\mathbf{CommRing}^1$), the following conditions are equivalent: (a) The unique morphism $\mathbf{Z} \to A$ is an epimorphism. (b) For each object R, there is at most one morphism $A \to R$ in $\mathbf{C}$.

(ii) Investigate the class of rings A with the above property. (Cf. Exercise 3.12:7, and last sentence of Exercise 6.7:6(iii).)

Exercise 6.7:9. Show that if R is a commutative ring, and $f: R \to S$ is an epimorphism in $\mathbf{Ring}^1$, then S is also commutative.

Though the property of being an epimorphism is not a reliable equivalent of surjectivity, we see that it is an interesting concept in its own right. In concrete categories, the statement that $f: A \to B$ is an epimorphism means intuitively that the image $f(A)$ ‘‘controls’’ all of B, in terms of behavior under morphisms.

There is an unfortunate tendency for some categorical enthusiasts to consider ‘‘epimorphism’’ to be the ‘‘category-theoretically correct’’ translation of ‘‘surjective map’’, even in cases when the concepts do not agree. For instance, a standard definition in module theory calls a module P *projective* if for every surjective module homomorphism $f: M \to N$, every homomorphism $P \to N$ factors through f. (If you haven’t seen this concept, draw a diagram, and verify that every *free* module is projective.) I have heard it claimed that one should therefore define an object P of a general category $\mathbf{C}$ to be projective if and only if for every *epimorphism* $f: M \to N$ of $\mathbf{C}$, every morphism $P \to N$ factors through f. This property *is* of interest, but there is no reason to consider it to the exclusion of others, in particular to reject the concept of projective object defined in terms of factorization through *surjective* maps $M \to N$, if the category is a concrete one. The fact that a property can be defined purely category-theoretically does not make it automatically superior to another property.

(The right context for developing a theory of ‘‘projective objects’’ is probably that of a category $\mathbf{C}$ given with a subfamily of morphisms S, which we wish to put in the role of surjections. To make things behave nicely, one will presumably want to put certain restrictions on S; for instance that it be *contained* in the class of epimorphisms, as the surjective maps in concrete categories always are by Exercise 6.7:6(i); probably also that it contain all invertible morphisms, and be closed under composition. We would then say that an object P is ‘‘projective with respect to the class S’’ if for every morphism $f: M \to N$ belonging to S, every morphism $P \to N$ factors through f. Such an approach is taken in [**72**], where a large number of properties are defined relative to a *pair* of classes of morphisms, one in the role of the surjections and the other in the role of the injections.)

The use of the words ‘‘monomorphism’’ and ‘‘epimorphism’’ is itself unsettled. In the days before category theory, the words were introduced by Bourbaki with the meanings ‘‘injective (i.e., one-to-one) homomorphism’’ and ‘‘surjective (i.e., onto) homomorphism’’. The early category-theorists brazenly gave these words the abstract category-theoretic meanings we have been discussing. This, of course, made the terms ambiguous in situations where the category-theoretic definition did not agree with the old meaning. Mac Lane [**14**] has tried to remedy the situation by restoring ‘‘epimorphism’’ and ‘‘monomorphism’’ to their old meanings (applicable in concrete categories only) and calling the category-theoretic concepts that we have been discussing ‘‘monic’’ and ‘‘epic’’ morphisms, or ‘‘monos’’ and ‘‘epis’’ for short. However, the category-theoretic meanings are already well-established in many areas; e.g., there have been many published papers dealing with epimorphisms in categories of rings. (A concept which includes the construction of the field of fractions of a commutative domain is bound to be of interest!) My feeling is that the words ‘‘epimorphism’’ and ‘‘epic morphism’’ sound too similar to usefully carry Mac Lane’s distinction; and that we should now stick with the category-theoretic meanings of ‘‘epimorphism’’ and ‘‘monomorphism’’. The phrases ‘‘surjective (or onto) homomorphism’’ and ‘‘injective (or one-to-one) homomorphism’’ give us more than enough ways of referring to the concrete concepts.

In any case, when you see these words used by other authors, you should make sure which meaning they are giving them.

Exercise 6.7:10. Suppose $f\in\mathbf{C}(Y,Z)$, $g\in\mathbf{C}(X,Y)$. Investigate implications holding among the conditions ''f is a monomorphism'', ''g is a monomorphism'', ''fg is a monomorphism'' ''f is an epimorphism'', ''g is an epimorphism'' and ''fg is an epimorphism''.

A full answer would be an exact determination of which among the 64 possible combinations of truth-values for these 6 statements can hold for a pair of morphisms, and which cannot! As a partial answer, you might determine which of the 8 possible combinations of truth-values of the first 3 conditions can hold. Then see whether duality allows you to deduce which combinations of the last 3 can hold, and whether, by examining when morphisms in a *product* of categories are monomorphisms or epimorphisms, you can use the results you have found to get a complete or nearly complete answer to the full 64-case question.

Exercise 6.7:11. Although for most natural categories of mathematical objects, the two obvious questions about a morphism are whether it is one-to-one and whether it is onto, in the category **RelSet** we can ask additional questions such as whether a given relation is a *function*.

(i) Can you find an abstract category-theoretic condition on a morphism which, when applied to morphisms in this category, is equivalent to being a function?

(ii) Examine other properties of relations, and whether they can be characterized by category-theoretic properties in **RelSet**. For instance, which members of $\mathbf{RelSet}(X,X)$ represent partial orderings on X? Given $f,g\in\mathbf{RelSet}(X,Y)$, how can one determine whether $f\subseteq g$ as relations? Can one construct from the category-structure of **RelSet** the contravariant functor $R\colon \mathbf{RelSet}^{\mathrm{op}}\to\mathbf{RelSet}$ taking each relation $f\in\mathbf{RelSet}(X,Y)$ to the opposite relation, $R(f)\in\mathbf{RelSet}(Y,X)$?

Because of the way we used duality in getting from the concept of monomorphism to that of epimorphism, both of them refer to *one-one-ness* of the images of a morphism under certain hom-functors. Let us look at the conditions that these same images be *onto*:

Exercise 6.7:12. (i) Given $f\in\mathbf{C}(X,Y)$, show that the following conditions are equivalent:

(a) For all $Z\in\mathrm{Ob}(\mathbf{C})$, $h_Z(f)$ is surjective.

(b) f is right invertible; i.e., there exists $g\in\mathbf{C}(Y,X)$ such that $fg=\mathrm{id}_Y$.

(c) For every covariant functor $F\colon\mathbf{C}\to\mathbf{Set}$, $F(f)$ is surjective.

(d) For every contravariant functor $F\colon\mathbf{C}\to\mathbf{Set}$, $F(f)$ is injective.

(e) For every category $\mathbf{D}$ and covariant functor $F\colon\mathbf{C}\to\mathbf{D}$, $F(f)$ is an epimorphism.

(f) For every category $\mathbf{D}$ and contravariant functor $F\colon\mathbf{C}\to\mathbf{D}$, $F(f)$ is a monomorphism.

(For partial credit, simply establish the equivalence of (a) and (b). Hint: $\mathrm{id}_Y\in h_Y(Y)$.)

(ii) State the *dual* of the result you get in part (i).

Let us look at what condition (b) of the above exercise means in familiar categories; in other words, what it means to have two morphisms satisfying a one-sided inverse relation,

$$(6.7.3)\qquad fg=\mathrm{id}_Y\qquad (f\in\mathbf{C}(X,Y),\ g\in\mathbf{C}(Y,X)).$$

Let us first take $\mathbf{C}=\mathbf{Set}$. Then we see that if (6.7.3) holds, g must be one-to-one (if two elements of Y fell together under g, there would be no way for f to ''separate'' them); so let us think of g as embedding a copy of Y in X. The map f sends X to Y so as to take each element $g(y)$ back to y, while acting in an unspecified way on elements of X that are not in the image of g. Thus the composite $gf\in\mathbf{C}(X,X)$ leaves elements of the image of g fixed, and carries all elements not in that image into that image; i.e., it ''retracts'' X onto the embedded copy of Y. Hence in an arbitrary category, a pair of morphisms satisfying (6.7.3) is called a *retraction* of the object X onto the object Y. In this situation Y is said to be a *retract* of X (via the morphisms f and g).

Exercise 6.7:13. (i) Show that a morphism in **Set** is left invertible if and only if it is one-to-one, with the exception of certain cases involving $\varnothing$, and right invertible if and only it is onto (without exceptions).

(ii) Show that X is a retract of Y in the category **Ab** of abelian groups (or more generally, the category R-**Mod** of left R-modules) if and only if X is isomorphic to a direct summand in Y.

(iii) Give examples of a morphism in **Ab** that is surjective, but not right invertible, and a morphism that is one-to-one, but not left invertible.

(iv) Characterize retractions in **Group** in terms of familiar group-theoretic constructions. Do they all arise from direct-product decompositions, as in **Ab**?

Combining part (i) of the above exercise with parts of two earlier exercises, we see that in a concrete category, one has

$$\text{left invertible} \Rightarrow \text{one-to-one} \Rightarrow \text{monomorphism},$$
$$\text{right invertible} \Rightarrow \text{onto} \Rightarrow \text{epimorphism}.$$

On the other hand, part (iii) of the above exercise and parts of earlier exercises show that none of these implications are reversible.

Exercise 6.7:14. Investigate what combinations of the properties ‘‘epimorphism’’, ‘‘monomorphism’’, ‘‘left invertible’’ and ‘‘right invertible’’ force a morphism in a category to be an isomorphism.

(Warning in connection with the results of the above exercises: The meanings of the terms ‘‘left’’ and ‘‘right’’ invertible become reversed when category-theorists – or other mathematicians – compose their maps in the opposite sense to the one we are using!)

Note that in the situation of (6.7.3), the other composite, $e = gf$, is an idempotent endomorphism of the object X, whose image in concrete situations is a copy of the retract Y. The next exercise establishes two category-theoretic versions of the idea that this idempotent morphism ‘‘determines’’ the structure of the retract Y of X.

Exercise 6.7:15. (i) Let $X, Y, Y' \in \mathrm{Ob}(\mathbf{C})$, and suppose that $f \in \mathbf{C}(X,Y)$, $f' \in \mathbf{C}(X,Y')$ have right inverses g, g' respectively. Show that $gf = g'f' \Rightarrow Y \cong Y'$.

(ii) Let $\mathbf{C}$ be a category, and $e \in \mathbf{C}(X,X)$ be an idempotent morphism: $e^2 = e$. Show that $\mathbf{C}$ may be embedded as a full subcategory in a category $\mathbf{D}$, *unique up to isomorphism*, with one additional object Y (i.e., $\mathrm{Ob}(\mathbf{D}) = \mathrm{Ob}(\mathbf{C}) \cup \{Y\}$) and such that there exist morphisms $f \in \mathbf{D}(X,Y)$, $g \in \mathbf{D}(Y,X)$ satisfying

$$fg = \mathrm{id}_Y \ (\text{in } \mathbf{D}(Y,Y)), \qquad gf = e \ (\text{in } \mathbf{D}(X,X) = \mathbf{C}(X,X)).$$

Returning to our search for conditions which correspond to familiar mathematical concepts in many cases, let us ask whether we can define a concept of a *subobject* of an object X in a category $\mathbf{C}$.

If by this we mean a criterion for *which* objects of a category such as **Set** or **Group** actually *lie in* which other objects, the answer is ‘‘certainly not’’: There can be no way to distinguish an object that *is* a subobject of another from one that is simply *isomorphic* to such a subobject. However, in particular categories of mathematical objects, we may well be able to say when a given morphism is an *embedding*, i.e., corresponds to an isomorphism of its domain object with a subobject of its codomain. For instance, in the categories **Set**, **Group**, **Monoid**, **Ring**^{1}, **Lattice** and similar categories, the embeddings are the monomorphisms. In these cases, and more generally, whenever we know which morphisms we want to regard as embeddings, we can recover the partially ordered set of subobjects of X as equivalence classes of these morphisms:

Exercise 6.7:16. Let $\mathbf{C}$ be a category, and suppose we are given a subcategory $\mathbf{C}_{\mathrm{emb}}$ of $\mathbf{C}$ which includes all the objects of $\mathbf{C}$, and whose set of morphisms is contained in the set of *monomorphisms* of $\mathbf{C}$. The morphisms of $\mathbf{C}_{\mathrm{emb}}$ are the morphisms of $\mathbf{C}$ that we intend to *think of* as embeddings. (Note, however, that you may not assume anything about this class except what we have stated.) For any object X of $\mathbf{C}$, let $\mathbf{Emb}_X$ denote the category whose objects are pairs (Y, f), where $Y \in \mathrm{Ob}(\mathbf{C})$ and $f \in \mathbf{C}_{\mathrm{emb}}(Y, X)$, and where a morphism from (Y, f) to (Z, g) means a morphism $a\colon Y \to Z$ of $\mathbf{C}$ such that $f = ga$.

(i) Show that each hom-set $\mathbf{Emb}_X(U, V)$ has at most one element. Deduce that $\mathbf{Emb}_X$ is of the form $\mathrm{Emb}(X)_{\mathbf{cat}}$ for some (possibly large) preorder $\mathrm{Emb}(X)$. (We defined $P_{\mathbf{cat}}$ for partial orders in §6.2; the definition for preorders is the same.)

(ii) Let us call the partially ordered set constructed from the preorder $\mathrm{Emb}(X)$ as in Proposition 4.2.2 ‘‘Sub(X)’’. Show that if $\mathbf{C}$ is one of **Set**, **Group**, **Ring** or **Lattice**, and we take $\mathbf{C}_{\mathrm{emb}}$ to have for its morphisms precisely the *monomorphisms* of $\mathbf{C}$, then $\mathrm{Sub}(X)$ is isomorphic to the partially ordered set of subsets, subgroups, etc., of X.

(iii) Let X be a set, in general infinite, and S the monoid of set maps of X into itself. Form the category $S_{\mathbf{cat}}$, and take $(S_{\mathbf{cat}})_{\mathrm{emb}}$ to have the monomorphisms of $S_{\mathbf{cat}}$ for its morphisms. Calling the one object of $S_{\mathbf{cat}}$ ‘‘0’’, describe the partially ordered set $\mathrm{Sub}(0)$.

The categories of algebraic objects mentioned so far in discussing one-one-ness have the property that every one-to-one morphism gives an isomorphism of its domain with a subobject of its codomain. An example of a category for which this is not true is **POSet**. For instance if P and Q are finite partially ordered sets having the same underlying set, but the order-relation on Q is stronger than that of P, then the identity map of the underlying set is a one-to-one isotone map from P to Q, but some elements of Q satisfy order-relations that they don’t satisfy in P, so we cannot regard P as a subobject of Q with the induced ordering. This leads to the questions

Exercise 6.7:17. (i) If the construction of the preceding exercise is applied with $\mathbf{C}$ the category **POSet**, and $\mathbf{C}_{\mathrm{emb}}$ taken to consist of all the monomorphisms of $\mathbf{C}$, how can the partially ordered sets $\mathrm{Sub}(X)$ be described?

(ii) Can you find a category-theoretic property characterizing those morphisms of **POSet** which are ‘‘genuine’’ embeddings, i.e., correspond to isomorphisms of their domain with subsets of their codomain, partially ordered under the induced ordering?

6.8. More categorical versions of common mathematical notions: special objects. Let us start this section with some ‘‘trivialities’’. In many of the classes of structures we have dealt with, there were one, or sometimes two objects that one would call the ‘‘trivial’’ objects: the one-element group; the one-element set, and also the empty set; the one-element lattice and likewise the empty lattice. The following definition abstracts the common properties of these objects.

Definition 6.8.1. *An* initial object *in a category* $\mathbf{C}$ *means an object* I *such that for every* $X \in \mathrm{Ob}(\mathbf{C})$, $\mathbf{C}(I, X)$ *has exactly one element.*

A terminal object *in a category* $\mathbf{C}$ *means an object* T *such that for every* $X \in \mathrm{Ob}(\mathbf{C})$, $\mathbf{C}(X, T)$ *has exactly one element.*

An object that is both initial and terminal is often called a zero object.

Thus, in **Set**, the empty set is the initial object, while any one-element set is a terminal object; in **Group**, a one-element group is both initial and terminal, hence is a zero object. The categories **Lattice**, **POSet**, **Top** and **Semigroup** are like **Set** in this respect, while $\mathbf{Top}^{\mathrm{pt}}$ and **Monoid** are like **Group**. In $\mathbf{Ring}^1$, the initial object is $\mathbf{Z}$, which we might not have thought to call ‘‘trivial’’; the terminal object is the one-element ring with $1=0$ (which some people do not call a ring).

A category need not have an initial or terminal object: The category of nonempty sets, or nonempty partially ordered sets, or nonempty lattices, or finite rings, has no initial object; $\mathbf{POSet}_<$ has no terminal object, nor does the category of nonzero rings (rings in which $1\neq 0$). If P is the partially ordered set of the integers, then $P_{\mathbf{cat}}$ has neither an initial nor a terminal object. Terminal objects are also called ‘‘final’’ objects (and I may sometimes slip and use that word in class).

Lemma 6.8.2. *If I, I' are two initial objects in a category $\mathbf{C}$, then they are isomorphic, via a unique isomorphism. Similarly, any two terminal objects are isomorphic via a unique isomorphism.* □

Exercise 6.8:1. Write out the proof of Lemma 6.8.2.

Exercise 6.8:2. Consider the following conditions on a category $\mathbf{C}$:

(a) $\mathbf{C}$ has a zero object (an object that is both initial and terminal).

(b) It is possible to choose in each hom-set $\mathbf{C}(X,Y)$ a morphism $0_{X,Y}$ in such a way that for all $X,Y,Z\in\mathrm{Ob}(\mathbf{C})$ and $f\in\mathbf{C}(X,Y)$, $g\in\mathbf{C}(Y,Z)$ one has $0_{Y,Z}\,f = 0_{X,Z} = g\,0_{X,Y}$.

(c) It is possible to choose in each hom-set $\mathbf{C}(X,Y)$ a morphism $0_{X,Y}$ such that for all $X,Y,Z\in\mathrm{Ob}(\mathbf{C})$ one has $0_{Y,Z}\,0_{X,Y} = 0_{X,Z}$.

(d) For all X, $Y\in\mathrm{Ob}(\mathbf{C})$, $\mathbf{C}(X,Y)\neq\varnothing$.

(i) Show that (a)⇒(b)⇒(c)⇒(d), but that none of these implications is reversible.

(ii) Show that if $\mathbf{C}$ has either an initial or a terminal object, then the first implication is reversible, but not, in general, the second or third.

(iii) Show that if $\mathbf{C}$ has an initial object *and* a terminal object, then (d)⇒(a), so that all four conditions are equivalent.

Exercise 6.8:3. If $\mathbf{C}$ is a category with a terminal object T, let $\mathbf{C}^{\mathrm{pt}}$ denote the category whose objects are pairs (X,p), where $X\in\mathrm{Ob}(\mathbf{C})$ $p\in\mathbf{C}(T,X)$, and where $\mathbf{C}^{\mathrm{pt}}((X,p),(Y,q)) = \{f\in\mathbf{C}(X,Y)\mid fp = q\}$. Verify that this indeed defines a category, that $\mathbf{C}^{\mathrm{pt}}$ will have a *zero* object, and that if we start with $\mathbf{C} = \mathbf{Top}$ we get precisely the category we earlier named $\mathbf{Top}^{\mathrm{pt}}$.

Exercise 6.8:4. If $\mathbf{C}$ is a category, call an object A of $\mathbf{C}$ *quasi-initial* if it satisfies condition (b) of Exercise 6.7:8(i). Generalize the result of that exercise to a characterization of quasi-initial objects in categories with initial objects.

What about the concept of *free* object? The definition of a free group F on a set X refers to *elements* of groups, hence the generalization should apply to a *concrete* category $(\mathbf{C},U)$. You should verify that when $\mathbf{C} = \mathbf{Group}$ and U is the underlying set functor, the following definition reduces to the usual definition of free group.

Definition 6.8.3. *If $\mathbf{C}$ is a category, $U\colon\mathbf{C}\to\mathbf{Set}$ a faithful functor, and X a set, then a* free object *of $\mathbf{C}$ on X with respect to the concretization U will mean a pair (F_X, u), where $F_X\in\mathrm{Ob}(\mathbf{C})$, $u\in\mathbf{Set}(X,U(F_X))$, and this pair has the universal property that for any pair (G, v) with $G\in\mathrm{Ob}(\mathbf{C})$, $v\in\mathbf{Set}(X,U(G))$, there is a unique morphism $h\in\mathbf{C}(F_X,G)$ such that $v = U(h)\,u$.*

Loosely, we often call the object F_X the free object, and u the associated universal map.

Exercise 6.8:5. Let V denote the functor associating to every group G the set $|G|^2$ of ordered pairs (x,y) of elements of G, and W the functor associating to G the set of ''unordered pairs'' $\{x,y\}$ of elements of G (where $x=y$ is allowed).

(i) State how these functors should be defined on morphisms. Show that they are both faithful.

(ii) Show that for any set X, there exists a free group with respect to the functor V, and describe this group.

(iii) Show that there do not in general exist free groups with respect to W.

Exercise 6.8:6. Let $U\colon \mathbf{Ring}^1 \to \mathbf{Set}$ be the functor associating to every ring R the set of 2×2 invertible matrices over R. Show that U is faithful. Does there exist for every set X a free ring R_X on X with respect to U?

The next exercise shows why the concept of monomorphism characterizes the one-to-one maps in most of the categories we know – or at least, shows that this follows from another property of these categories.

Exercise 6.8:7. Let $(\mathbf{C}, U)$ be a concrete category. Show that if there exists a free object on a one-element set with respect to U, then a morphism f of $\mathbf{C}$ is a monomorphism if and only if $U(f)$ is one-to-one.

We could go further into the study of free objects, proving, for instance, that they are unique up to isomorphism when they exist, and that when $\mathbf{C}$ has free objects on all sets, the free-object construction gives a functor $\mathbf{Set} \to \mathbf{C}$. Some of this will be done in Exercise 6.9:7, later in this chapter, but for the most part, we shall get such results in the next chapter, as part of a theory embracing more general universal constructions.

Let us turn to another pair of constructions that we have seen in many categories (including **Cat** itself), those of *product* and *coproduct*. No concretization or other additional structure is needed to translate these concepts into category-theoretic terms.

Definition 6.8.4. *Let $\mathbf{C}$ be a category, I a set, and $(X_i)_{i\in I}$ a family of objects of $\mathbf{C}$.*

A product *of this family in $\mathbf{C}$ means a pair $(P, (p_i)_{i\in I})$, where $P\in\mathrm{Ob}(\mathbf{C})$ and for each $i\in I$, $p_i\in\mathbf{C}(P, X_i)$, having the universal property that for any pair $(Y, (y_i)_{i\in I})$ $(Y\in\mathrm{Ob}(\mathbf{C})$, $y_i\in\mathbf{C}(Y, X_i))$ there exists a unique morphism $r\in\mathbf{C}(Y, P)$ such that $y_i = p_i r$ $(i\in I)$.*

Likewise, a coproduct *of the family $(X_i)_{i\in I}$ means a pair $(Q, (q_i)_{i\in I})$, where $Q\in\mathrm{Ob}(\mathbf{C})$ and for each $i\in I$, $q_i\in\mathbf{C}(X_i, Q)$, having the universal property that for any pair $(Y, (y_i)_{i\in I})$ $(Y\in\mathrm{Ob}(\mathbf{C})$, $y_i\in\mathbf{C}(X_i, Y))$ there exists a unique morphism $r\in\mathbf{C}(Q, Y)$ such that $y_i = r q_i$ $(i\in I)$.*

Loosely, we call P and Q the product and coproduct of the objects X_i, the $p_i\colon P\to X_i$ the projection *maps, and the $q_i\colon X_i \to Q$ the* coprojection *maps. (The term* injection *is used by some authors instead of* coprojection.*)*

The category $\mathbf{C}$ is said to have finite products if every finite family of objects of $\mathbf{C}$ has a product in $\mathbf{C}$, and to have small products (often simply ''to have products'') if every family of objects of $\mathbf{C}$ indexed by a small *set has a product; and similarly for finite and small coproducts.*

Standard notations for product and coproduct objects are $P = \prod_{i\in I} X_i$ and $Q = \coprod_{i\in I} X_i$. For a product of finitely many objects one also writes $X_0 \times \ldots \times X_{n-1}$. There is no analogous standard notation for coproducts of finitely many objects; we used ''$*$'' as the operation-symbol in Chapter 3, following group-theorists' notation for ''free products''; one sometimes sees $+$ or $\oplus$, based on module-theoretic notation. In these notes we shall from now on write $X_0 \amalg \dots \amalg X_{n-1}$, which also occurs in the literature.

Observe that a product of the empty family is equivalent to a terminal object, while a coproduct of the empty family is equivalent to an initial object.

Exercise 6.8:8. If P is a partially ordered set, what does it mean for a family of objects of $P_{\mathbf{cat}}$ to have a product? A coproduct?

Exercise 6.8:9. (i) Suppose we are given a *family of families of objects* in a category $\mathbf{C}$, $((X_{ij})_{i\in I_j})_{j\in J}$, such that for each j, $\prod_{I_j} X_{ij}$ exists, and such that we can also find a product of these product objects, $P = \prod_J (\prod_{I_j} X_{ij})$. Show that P will be a product of the family $(X_{ij})_{i\in I_j,\, j\in J}$.

(ii) Deduce that if a category has products of pairs of objects, it has products of all finite nonempty families of objects.

When a concrete category has enough coproducts, we get an interesting relation between two concepts introduced earlier (equivalence of (a) and (b′) below):

Exercise 6.8:10. Let $(\mathbf{C}, U)$ be a concrete category. Show that the following conditions are equivalent. (a) The concretization functor U is *representable*. (b) $\mathbf{C}$ has a free object on one generator. Moreover, show that if $\mathbf{C}$ has small coproducts, then these are also equivalent to (b′) $\mathbf{C}$ has free objects on all sets.

Exercise 6.8:11. (i) Let X be a set (in general infinite) and S the monoid of maps of X into itself. When, if ever, does the category $S_{\mathbf{cat}}$ have products of pairs of objects? (Of course, there is only one ordered pair of objects, and only one object to serve as their product, so the question comes down to whether two morphisms p_1 and p_2 can be found having appropriate properties.)

(ii) Is there, in some sense, a ‘‘universal’’ example of a monoid S such that $S_{\mathbf{cat}}$ has products of pairs of objects?

We saw in Exercise 6.7:13(ii) that in **Ab** and R-**Mod**, any retraction of an object arises from a decomposition as a direct sum, which in those categories is both a product and coproduct. The next exercise examines the relation between retractions, products and coproducts in general.

Exercise 6.8:12. (i) Show that if $\mathbf{C}$ is a category with a zero object, then for any objects A and B of $\mathbf{C}$, if the product $A\times B$ exists, then A can be identified with a retract of this product, and if the coproduct $A \amalg B$ exists, then A can be identified with a retract of this coproduct.

(ii) Can you find a condition more general than the existence of a zero object under which these conclusions hold?

One does not have the converse to either part of (i). Indeed, let A and B be nontrivial objects of **Group**, so that by (i) above, A can be identified with a retract of $A\times B$ and also with a retract of $A \amalg B$. Now

(iii) Show that the subgroup $A \subseteq A \amalg B$, though a retract, is not a factor in any *product* decomposition of that group, and that $A \subseteq A\times B$, though a retract, is not a factor in any *coproduct* decomposition of that group.

The next exercise shows that when one requires more than products of *small* families, one's categories tend to become degenerate.

Exercise 6.8:13. Let $\mathbf{C}$ be a category and α a cardinal (e.g., the cardinality of some universe) such that $\mathrm{Ob}(\mathbf{C})$ and all morphism sets $\mathbf{C}(X, Y)$ have cardinality $\leq \alpha$.

(i) Show that if every family of objects of $\mathbf{C}$ indexed by a set of cardinality $\leq \alpha$ has a product in $\mathbf{C}$, then $\mathbf{C}$ has the form $P_{\mathbf{cat}}$, where P is a preorder whose associated partially ordered set $P/\approx$ is a complete lattice.

(ii) Deduce that in this case *every* family of objects of $\mathbf{C}$ (indexed by any set whatsoever) has a product and a coproduct.

It is an easy fallacy to say ‘‘since product is a category-theoretic notion, functors must respect products’’. Rather

Exercise 6.8:14. Find an example of categories $\mathbf{C}$ and $\mathbf{D}$ having finite products, and a functor $\mathbf{C} \to \mathbf{D}$ which does not respect such products.

On the other hand:

Exercise 6.8:15. Show that if $(\mathbf{C}, U)$ is a concrete category, and there exists a free object on one generator with respect to U, then U respects all products which exist in $\mathbf{C}$. (Cf. Exercise 6.8:7.)

Thus, in most of the concrete categories we have been interested in, the underlying set of a product object is the direct product of the underlying sets of the given objects. However, there is a well-known example for which this fails:

Exercise 6.8:16. A *torsion* group (also called a ‘‘periodic’’ group) is a group all of whose elements are of finite order. Let $\mathbf{TorAb}$ be the category of all torsion abelian groups.

(i) Show that a product in $\mathbf{Ab}$ of an infinite family of torsion abelian groups is not in general a torsion group.

(ii) Show, however, that the category $\mathbf{TorAb}$ has small products.

(iii) Deduce that the underlying set functor $\mathbf{TorAb} \to \mathbf{Set}$ does not respect products.

Exercise 6.8:17. Does the category $\mathbf{TorGroup}$ of *all* torsion groups have small products?

What about category-theoretic versions of the constructions of *kernel* and *cokernel*? We saw that these constructions were specific to fairly limited kinds of mathematical objects, such as groups and rings, but that a pair of concepts which embrace them but are much more versatile are those of *difference kernel* and *difference cokernel*. These concepts are abstracted in

Definition 6.8.5. *Let* $\mathbf{C}$ *be a category,* $X, Y \in \mathrm{Ob}(\mathbf{C})$, *and* $f, g \in \mathbf{C}(X, Y)$.

Then a difference kernel (*also called an* equalizer) *of* f *and* g *means a pair* (K, k), *where* K *is an object, and* $k: K \to X$ *a morphism which satisfies* $fk = gk$, *and is universal for this property, in the sense that for any pair* (W, w) *with* W *an object and* $w: W \to X$ *a morphism such that* $fw = gw$, *there exists a unique morphism* $h: W \to K$ *such that* $w = kh$.

Likewise, a difference cokernel (*also called a* coequalizer) *of* f *and* g *means a pair* (C, c) *where* C *is an object, and* $c: Y \to C$ *a morphism which satisfies* $cf = cg$, *and is universal for this property, in the sense that for any pair* (Z, z) *with* Z *an object and* $z: Y \to Z$ *a morphism such that* $zf = zg$, *there exists a unique morphism* $h: C \to Z$ *such that* $z = hc$.

Loosely, K *and* C *are called the difference kernel and cokernel objects, and* k, c *the difference kernel and cokernel morphisms, or the canonical morphisms associated with the difference* (*co*)*kernel construction. We say that* $\mathbf{C}$ *has difference kernels* (*respectively difference cokernels*) *if every pair of morphisms between every pair of objects of* $\mathbf{C}$ *has a difference kernel* (*cokernel*).

It turns out that the definition of difference cokernel actually leads to a better approximation to the concept of a *surjective* morphism in familiar categories than those we investigated earlier:

Exercise 6.8:18. (i) Show that in each of the categories **Group**, **Ring**[1], **Set**, **Monoid**, a morphism out of an object Y is surjective on underlying sets if and only if it is the difference cokernel morphism for some pair of morphisms from an object X into Y.

(ii) Is the same true in **POSet**? In the category of *finite* groups?

(iii) In the categories considered in (i) (and optionally, those considered in (ii)) investigate whether, likewise, the condition of being a difference *kernel* is equivalent to one-one-ness.

(iv) Investigate what implications hold in a general category between the conditions of being an epimorphism, being right invertible, and being a difference cokernel map.

Exercise 6.8:19. Let $f, g \in \mathbf{Set}(X, Y)$ be morphisms, and (C, c) their difference cokernel.

(i) Show that $\mathrm{card}(X) + \mathrm{card}(C) \geq \mathrm{card}(Y)$. If you wish, assume X and Y are finite.

(ii) Can one establish some similar relation between the cardinalities of X, of Y, and of the difference *kernel* of f and g in **Set**?

(iii) What can be said of the corresponding questions in **Ab**? In **Group**?

In categories (such as **Group** and **Ab**) which have a zero object, the concepts of the *kernel* and *cokernel* of a morphism $f\colon X \to Y$ may be defined as the difference kernel and difference cokernel of f with the zero morphism $X \to Y$ (see Exercise 6.8:2).

We turn next to a pair of constructions which we have not discussed before, but which are related both to products and coproducts and to difference kernels and cokernels.

Definition 6.8.6. *Given objects X_1, X_2, X_3 of a category* $\mathbf{C}$, *and morphisms $f_1\colon X_1 \to X_3$, $f_2\colon X_2 \to X_3$ (diagram below), a* pullback *of the pair of morphisms f_1, f_2 means a* 3*-tuple (P, p_1, p_2), where P is an object, and $p_1\colon P \to X_1$, $p_2\colon P \to X_2$ are morphisms satisfying $f_1 p_1 = f_2 p_2$, and which is universal for this property, in the sense that any* 3*-tuple (Y, y_1, y_2), satisfying $f_1 y_1 = f_2 y_2$ is induced by a unique morphism $h\colon Y \to P$.*

(6.8.7)

$$\begin{tikzcd}
Y \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& P \arrow[r, "p_1"'] \arrow[d, "p_2"] & X_1 \arrow[d, "f_1"'] \\
& X_2 \arrow[r, "f_2"] & X_3 .
\end{tikzcd}$$

Dually, for objects X_0, X_1, X_2 and morphisms $g_1\colon X_0 \to X_1$, $g_2\colon X_0 \to X_2$, a pushout *of g_1 and g_2 means a* 3*-tuple (Q, q_1, q_2), where $q_1\colon X_1 \to Q$, $q_2\colon X_2 \to Q$ satisfy $q_1 g_1 = q_2 g_2$, and which is universal for this property in the sense shown below:*

(6.8.8)

$$\begin{tikzcd}
X_0 \arrow[r, "g_1"'] \arrow[d, "g_2"] & X_1 \arrow[d, "q_1"'] \arrow[ddr] & \\
X_2 \arrow[r, "q_2"] \arrow[drr] & Q \arrow[dr] & \\
 & & Y.
\end{tikzcd}$$

A commuting square in **C** *is called a pullback diagram* (*respectively, a pushout diagram*) *if the upper left-hand* (*lower right-hand*) *object is a pullback* (*pushout*) *of the remainder of the diagram. As in the case of products and coproducts, the universal morphisms* p_1, p_2 *from a pullback object* P *are called its* projection morphisms *to the* X_i, *and the universal morphisms* q_1, q_2 *to a pushout object* Q *are called its* coprojection morphisms.

We say that a category **C** *has pullbacks if every diagram of objects and morphisms* X_1, X_2, X_3, f_1, f_2 *as in* (6.8.7) *has a pullback* P, *and that* **C** *has pushouts if every diagram of objects and morphisms* X_0, X_1, X_2, g_1, g_2 *as in* (6.8.8) *has a pushout* Q.

The next exercise shows how to construct these creatures.

Exercise 6.8:20. (i) Show that if a category **C** has finite products and has difference kernels, then it has pullbacks. Namely, for every system of objects and morphisms, X_1, X_2, X_3, f_1, f_2 as in the first part of the above definition, construct a pullback as the difference kernel of a certain pair of morphisms $X_1 \times X_2 \to X_3$.

(ii) State the dual result for pushouts.

To get a picture of pullbacks in **Set**, note that any set map $f: X \to Y$ can be regarded as a decomposition of the set X into subsets $f^{-1}(y)$, indexed by the elements $y \in Y$. When looking at f this way, one calls X a set *fibered* by Y, and calls $f^{-1}(y)$ the *fiber* of X at $y \in Y$. Now in a pullback situation (6.8.7) in **Set**, we see that from two sets X_1 and X_2, each fibered by X_3, we obtain a third set P fibered by X_3, with maps into the first two. From the preceding exercise one can verify that the fiber of P at each $y \in X_3$ is the direct product of the fibers of X_1 and of X_2 at y. Consequently, pullbacks are sometimes called *fibered products*, whether or not one is working in a concrete category. The next exercise shows that such ‘‘fibered products’’ can be regarded as products in an appropriate category of ‘‘fibered objects’’.

Exercise 6.8:21. Given a category **C** and any $Z \in \mathrm{Ob}(\mathbf{C})$, let $\mathbf{C}_Z$ denote the category of ‘‘objects of **C** fibered by Z’’, that is, the category having for objects all pairs (X, f) where $X \in \mathrm{Ob}(\mathbf{C})$ and $f \in \mathbf{C}(X, Z)$, and having for morphisms $(X, f) \to (Y, g)$ all members of $\mathbf{C}(X, Y)$ making commuting triangles with the morphisms f and g into Z.

Show that a *pullback* (6.8.7) in **C** is equivalent to a *product* of the objects (X_1, f_1), (X_2, f_2) in $\mathbf{C}_{X_3}$.

Exercise 6.8:22. Let **C** be a category having pullbacks and pushouts, and let X_1, X_2, X_3, f_1, f_2 be as in (6.8.7). Suppose we form their pullback P, then form the pushout of the system P, X_2, X_3, p_1, p_2, and so on, going back and forth between pullbacks and pushouts. Will this process ever ‘‘stabilize’’?

(Suggestion: Given the two objects X_1 and X_2, consider the set A of all objects W given with morphisms into X_1 and X_2, and the set B of all objects Y given with morphisms into them from X_1 and X_2, and let $R \subseteq A \times B$ denote the relation ‘‘the four morphisms form a commuting square’’. Examine the resulting Galois connection between A and B.)

We note

Lemma 6.8.9. *A morphism $f\colon X \to Y$ of a category $\mathbf{C}$ is a monomorphism if and only if the diagram*

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "1_X"] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

is a pullback diagram. Similarly f is an epimorphism if and only if

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "1_X"] \\
Y \arrow[r, "1_Y"] & Y
\end{tikzcd}$$

is a pushout diagram. □

Exercise 6.8:23. Prove Lemma 6.8.9.

The category of ''objects of $\mathbf{C}$ fibered over Z'' used in Exercise 6.8:21 has a far-reaching generalization:

Exercise 6.8:24. (i) Given three categories and two functors, $\mathbf{D} \xrightarrow{S} \mathbf{C} \xleftarrow{T} \mathbf{E}$, show that we can define a category $(S\downarrow T)$ having for objects all 3-tuples (D, f, E), where $D\in\mathrm{Ob}(\mathbf{D})$, $E\in\mathrm{Ob}(\mathbf{E})$, and $f\in\mathbf{C}(S(D), T(E))$, and where a morphism $(D, f, E) \to (D', f', E')$ means a pair consisting of morphisms $d\colon D \to D'$, $e\colon E \to E'$, such that $S(d)$ and $T(e)$ make a commuting square with f and f'. Specifically, write out the required commutativity condition, indicate how composition should be defined in $(S\downarrow T)$, and verify that the result is a category.

(ii) Given a category $\mathbf{C}$ and an object Z of $\mathbf{C}$, suppose we take for $\mathbf{E}$ the trivial category, with only one object and its identity morphism, let $T\colon \mathbf{E} \to \mathbf{C}$ be the functor taking the object of $\mathbf{E}$ to Z, and let $\mathbf{D} = \mathbf{C}$, with $S\colon \mathbf{C} \to \mathbf{C}$ the identity functor. Show that the category $(S\downarrow T)$ can then be identified with the category we called $\mathbf{C}_Z$ in Exercise 6.8:21.

(iii) For S and T as in (ii) above, also describe the category $(T\downarrow S)$.

The construction described in part (i) of the above exercise is sometimes written (S, T). We follow Mac Lane [**14**] in writing it $(S\downarrow T)$, because, as he observes, ''the comma is already overworked''. However, the older notation is the source of its name, the *comma category* construction. The most frequently used cases of this construction are those noted in (ii) and (iii) above, often written $(\mathbf{C}\downarrow Z)$ and $(Z\downarrow \mathbf{C})$.

If $\mathbf{C}$ is a category with a terminal object T, the construction $\mathbf{C}^{\mathrm{pt}}$ of Exercise 6.8:3 can clearly be described as $(T\downarrow \mathbf{C})$. However, there is a related comma category construction that is also sometimes called the category of ''pointed objects'' of $\mathbf{C}$:

Exercise 6.8:25. Suppose $(\mathbf{C}, U)$ is a concrete category having a free object F_1 on the one-element set 1. Show that the following categories are isomorphic:

(i) $(F_1 \downarrow \mathbf{C})$.

(ii) The category of objects X of $\mathbf{C}$ given with a distinguished element of $U(X)$, and having for morphisms the morphisms of $\mathbf{C}$ that respect distinguished elements.

(iii) $(1\downarrow U)$, where ''1'' denotes the functor from the one-object one-morphism category to **Set** taking the unique object to the one-element set 1.

Since the one-point topological space is both the terminal object T of **Top** and the free object F_1 on one generator in that category (under the concretization by underlying sets), the constructions $(T\downarrow\mathbf{C})$ and $(F_1\downarrow\mathbf{C})$ on a general category $\mathbf{C}$ each generalize the construction of $\mathbf{Top}^{\mathrm{pt}}$. In fact, each of these comma categories is sometimes called the category of ''pointed objects of $\mathbf{C}$'', though they may be quite different from one another. For instance, since the terminal object of **Group** is also initial (i.e., is a zero object, as defined in Definition 6.8.1), we see that every object of **Group** admits a unique homomorphism of this terminal object into it, so such a homomorphism contains no new information, and $\mathbf{Group}^{\mathrm{pt}}$ is isomorphic to **Group**. Hence an author who speaks of the category of ''pointed groups'' probably does not mean $\mathbf{Group}^{\mathrm{pt}}$, but the category $(F_1\downarrow\mathbf{Group})$, of groups with a distinguished element.

While the pullback P in (6.8.7) is often called the ''fibered product of X_1 and X_2 over X_3'', the pushout Q in (6.8.8) is often called the ''coproduct of X_1 and X_2 with *amalgamation* of X_0'', especially in concrete situations where the morphisms f_1 and f_2 are embeddings. In the spirit of Chapter 3, you might do

Exercise 6.8:26. (i) Show by general arguments that the category **Group** has pushouts.

(ii) Obtain an explicit description of pushouts of groups in the case where the given maps $f_1\colon G_0\to G_1$ and $f_2\colon G_0\to G_2$ are one-to-one, assuming for notational convenience that these maps are inclusions, and that the underlying sets of G_1 and G_2 are disjoint except for the common subgroup G_0. (This is a classical construction, called by group theorists ''the free product of G_1 and G_2 with amalgamation of the common subgroup G_0''. If you are already familiar with this construction, and the proof of its normal form by van der Waerden's trick, skip to the next part.)

(iii) Describe how to reduce the construction of an arbitrary pushout of groups to the case where the given maps f_1 and f_2 are one-to-one, as above.

We end this section with one more example of a category-theoretic translation of a familiar concept. Let G be a group, and recall that a G-set is a set with an *action* of G on it by permutations. More generally, one can consider an action of G by automorphisms on any mathematical object, that is, on any object X of a category $\mathbf{C}$; one defines such an action as a homomorphism f of G into the monoid $\mathbf{C}(X,X)$. Now observe that the pair consisting of such an object X and such a homomorphism $f\colon G\to\mathbf{C}(X,X)$ is equivalent to a *functor* $G_{\mathbf{cat}}\to\mathbf{C}$; the object X gives the image of the one object of $G_{\mathbf{cat}}$, and f determines the images of the morphisms. Thus, group actions are examples of functors!

6.9. Morphisms of functors (or ''natural transformations''). We have seen that various sorts of mathematical structures can be regarded as functors from ''diagram'' categories to categories of simpler objects: As just noted, G-sets are equivalent to functors from $G_{\mathbf{cat}}$ to **Set**; another example is the type of structure which is the input of the difference kernel and difference cokernel constructions, consisting of two objects of a category $\mathbf{C}$ and a pair of morphisms from the first object to the second, (X,Y,f,g). If we call such a 4-tuple a ''parallel pair'' of morphisms in $\mathbf{C}$, then as observed in §6.2, parallel pairs correspond to functors from the 2-object diagram category $\cdot\rightrightarrows\cdot$ to $\mathbf{C}$.

Now if we regard such functors as new sorts of mathematical ‘‘objects’’, it is natural to ask whether we can define *morphisms* among these objects.

There is a standard concept of a morphism of G-sets – a set map which ‘‘respects’’ the action of G. Is there a similar concept of ‘‘morphism of parallel pairs’’? Given two parallel pairs $S = (X, Y, f, g)$ and $S' = (X', Y', f', g')$, it seems reasonable to define a morphism $S \to S'$ to be a pair of morphisms $x \in \mathbf{C}(X, X')$, $y \in \mathbf{C}(Y, Y')$ which respects the structure of parallel pairs, in the sense that $yf = f'x$ and $yg = g'x$:

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & X' \arrow[d, shift right, "f'"'] \arrow[d, shift left, "g'"] \\
Y \arrow[r, "y"] & Y'
\end{tikzcd}$$

It is clear how to compose such morphisms, and immediate to verify that this composition makes the class of parallel pairs in $\mathbf{C}$ into a category.

We find that with this definition, difference kernels and difference cokernels join the ranks of constructions which, though originally thought of only as operating on objects, can also be applied to morphisms. Indeed, if the two parallel pairs of the above diagram each have a difference kernel, then it is not hard to check that the morphism (x, y) induces a morphism z of these difference kernels, and if *every* parallel pair in $\mathbf{C}$ has a difference kernel, then this way of associating to every morphism of parallel pairs a morphism of their difference kernels makes the difference kernel construction a functor. Likewise, if each parallel pair has a difference cokernel, the difference cokernel construction becomes a functor.

Exercise 6.9:1. Prove the assertions about difference kernels in the above paragraph.

Exactly similar considerations apply to the configurations in a category $\mathbf{C}$ for which we defined the concepts of *pullbacks* and *pushouts*. Such configurations can be regarded as functors from diagram categories $\begin{smallmatrix} & \cdot \\ & \downarrow \\ \cdot \to & \cdot \end{smallmatrix}$, respectively $\begin{smallmatrix} \cdot & \to \cdot \\ \downarrow & \\ \cdot & \end{smallmatrix}$ into $\mathbf{C}$, and the set of all configurations of one or the other of these kinds can be made into a category, by letting a morphism from one such configuration to another mean a system of maps between corresponding objects, which respect the given morphisms among these. One can verify that this makes the pullback and pushout constructions, when they exist, into functors on these categories of configurations.

In each of these cases, we have had a diagram category $\mathbf{D}$ and a general category $\mathbf{C}$, and we have discovered a concept of ‘‘morphism’’ between functors from $\mathbf{D}$ to $\mathbf{C}$. So, although we have so far regarded functors as the morphisms of $\mathbf{Cat}$, it seems that there is also a concept of morphisms among functors! We formalize this as

Definition 6.9.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories and* $F, G: \mathbf{D} \to \mathbf{C}$ *functors. Then a* morphism of functors $a: F \to G$ *means a family* $(a(X))_{X \in \mathrm{Ob}(\mathbf{D})}$ *such that for each* $X \in \mathrm{Ob}(\mathbf{D})$, $a(X) \in \mathbf{C}(F(X), G(X))$, *and for each* $f \in \mathbf{D}(X, Y)$,

$$a(Y)\,F(f) \;=\; G(f)\,a(X). \tag{6.9.2}$$

Pictorially, for each arrow f *as at left below, we have a commuting square as at right:*

$$\begin{tikzcd}
X \arrow[d, "f"] & & F(X) \arrow[r, "a(X)"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
Y & & F(Y) \arrow[r, "a(Y)"] & G(Y).
\end{tikzcd}$$

Given functors F, G, H: $\mathbf{D} \to \mathbf{C}$ *and morphisms* $F \xrightarrow{a} G \xrightarrow{b} H$, *the composite morphism* ba: $F \to H$ *is defined by*

$$ba(X) \ = \ b(X)\,a(X) \qquad (X \in \mathrm{Ob}(\mathbf{D})),$$

while the identity morphism of a functor F *is defined by*

$$\mathrm{id}_F(X) \ = \ \mathrm{id}_{F(X)} \qquad (X \in \mathrm{Ob}(\mathbf{D})).$$

The category whose objects are all the functors from $\mathbf{D}$ *to* $\mathbf{C}$, *with morphisms, composition, and identity defined in this way, will be denoted* $\mathbf{C}^{\mathbf{D}}$.

Note that if $\mathbf{D}$ is small, then $\mathbf{C}^{\mathbf{D}}$ will be small or legitimate according as $\mathbf{C}$ is, but that if $\mathbf{D}$ is legitimate, then $\mathbf{C}^{\mathbf{D}}$ will in general be large! But again, the Axiom of Universes shows us that we may consider these large functor categories as small categories with respect to a larger universe.

We see that if G is a group, the above definition of a morphism between functors $G_{\mathbf{cat}} \to \mathbf{Set}$ indeed agrees with the concept of a morphism between G-sets, hence the category G-$\mathbf{Set}$ can be identified with $\mathbf{Set}^{(G_{\mathbf{cat}})}$. Since $G_{\mathbf{cat}}$ is a small category, $\mathbf{Set}^{(G_{\mathbf{cat}})}$ is a legitimate category.

Let us note some examples where $\mathbf{D}$ is not a small category, using functors we have seen before. Let F, A: $\mathbf{Set} \to \mathbf{Group}$ be the functors taking a set X to the free group and the free abelian group on X respectively. For every set X there is a homomorphism $a(X)$: $F(X) \to A(X)$ taking each generator of $F(X)$ to the corresponding generator of $A(X)$. It is easy to see that these form commuting squares with group homomorphisms induced by set maps, hence they constitute a morphism of functors a: $F \to A$.

Let F again be the free group construction, and let U: $\mathbf{Group} \to \mathbf{Set}$ be the underlying set functor. Recall that for each $X \in \mathrm{Ob}(\mathbf{Set})$, the universal property of $F(X)$ involves a set map $u(X)$: $X \to U(F(X))$. It is easy to check that these maps $u(X)$, taken together, give a morphism u: $\mathrm{Id}_{\mathbf{Set}} \to U \circ F$ of functors $\mathbf{Set} \to \mathbf{Set}$, where $\mathrm{Id}_{\mathbf{Set}}$ denotes the identity functor of the category $\mathbf{Set}$.

Exercise 6.9:2. Verify the above claim that u is a morphism of functors.

Statements that two different constructions are ‘‘essentially the same’’ can usually be formulated precisely as saying that they are isomorphic as functors. For instance

Exercise 6.9:3. (i) Let F: $\mathbf{Set} \to \mathbf{Group}$ denote the free group construction, A: $\mathbf{Set} \to \mathbf{Group}$ the free abelian group construction, and C: $\mathbf{Group} \to \mathbf{Group}$ the abelianization construction. Show that $C \circ F \cong A$. (In what functor category?)

(ii) When we gave examples of covariant hom-functors h_X: $\mathbf{C} \to \mathbf{Set}$ at the end of §6.5, we observed that for $\mathbf{C} = \mathbf{Group}$, the functor $h_{\mathbf{Z}}$ was ‘‘essentially’’ the underlying set functor, and that for $\mathbf{C} = \mathbf{Set}$ and $2 = \{0, 1\} \in \mathrm{Ob}(\mathbf{Set})$, h_2 was ‘‘essentially’’ the construction $X \mapsto X \times X$. Similarly, in §6.6 we noted that, the contravariant hom-functor h^2 on $\mathbf{Set}$ ‘‘could be identified with’’ the contravariant power-set functor. Verify that in each of these cases, we have an *isomorphism* of functors.

(iii) Let T: $\mathbf{Ab}\times\mathbf{Ab} \to \mathbf{Ab}$ be the tensor product construction, and R: $\mathbf{Ab}\times\mathbf{Ab} \to \mathbf{Ab}\times\mathbf{Ab}$ the construction taking each pair of abelian groups (A,B) to the pair (B,A), and acting similarly on morphisms. Show that $T \cong T\circ R$.

(iv) Show that the isomorphisms of Exercise 6.6:5(v) give an isomorphism of functors $\mathrm{Id}_{\mathbf{POSet}} \cong BA$.

A venerable example of the concepts we have been discussing arises in considering duality of finite-dimensional vector spaces. We know that a finite-dimensional vector space V, its dual V^*, and its double dual V^{**} are all isomorphic. Now the isomorphism $V \cong V^*$ is not ''natural'' – these spaces are isomorphic simply because they have the same dimension. But there *is* a natural way to construct an isomorphism $V \cong V^{**}$, by taking each vector v to the operator $\overline{v}$ defined by $\overline{v}(f) = f(v)$ $(f\in V^*)$. What this natural construction shows is that for $\mathbf{C}$ the category of finite-dimensional k-vector spaces, the functors $\mathrm{Id}_{\mathbf{C}}$ and ** are isomorphic. (One cannot even attempt to construct an isomorphism between $\mathrm{Id}_{\mathbf{C}}$ and *, since one functor is covariant and the other contravariant.)

Examples such as this had long been referred to as ''natural'' isomorphisms, and people had gradually noticed that ''natural'' constructions respected maps among objects. When Eilenberg and Mac Lane introduced category theory in [**6**], they therefore gave the name *natural transformation* to what we are calling a *morphism of functors*; that term is still widely used, though we shall not use it here. One can also call such an entity a *functorial map*, to emphasize that it is not merely a system of maps between individual objects $F(X)$ and $G(X)$, but that these respect the morphisms $F(f)$ and $G(f)$ that make the constructions F and G functors.

In fact, we used the term ''functorial'' – deferring the explanation – in Exercises 2.3:6 and 2.3:7. What we called there a ''functorial group-theoretic operation in n variables'' is in our new language a morphism $U^n \to U$, where U is the underlying-set functor $\mathbf{Group} \to \mathbf{Set}$, and U^n the functor associating to every group G the direct product of n copies of $U(G)$ – the set of n-tuples of elements of $U(G)$. Some cases of those exercises reappear, along with other problems, in the following exercises, which should give you practice thinking about morphisms of functors.

Exercise 6.9:4. In each part below, attempt to describe *all* morphisms among the functors listed, including morphisms from functors to themselves. (I describe functors below in terms of their behavior on objects. The definitions of their behavior on morphisms should be clear. If you are at all in doubt, begin your answer by saying how you think these functors should act on morphisms.)

(i) The functors Id, A and B: $\mathbf{Set} \to \mathbf{Set}$ given by $\mathrm{Id}(S) = S$, $A(S) = S\times S$, $B(S) = \{\{x,y\} \mid x,y\in S\}$. (Note that a member of $B(S)$ may have either one or two elements.)

(ii) The functors U, V and W: $\mathbf{Group} \to \mathbf{Set}$ given by $U(G) = |G|$, $V(G) = |G|\times|G|$, $W(G) = \{x\in|G| \mid x^2 = e\}$.

(iii) The underlying set functor U: $\mathbf{FGroup} \to \mathbf{Set}$, where $\mathbf{FGroup}$ is the category of finite groups.

Exercise 6.9:5. (i) Show that for any category $\mathbf{C}$, the monoid $\mathbf{C}^{\mathbf{C}}(\mathrm{Id}_{\mathbf{C}}, \mathrm{Id}_{\mathbf{C}})$ of endomorphisms of the identity functor of $\mathbf{C}$ is commutative.

(ii) Attempt to determine this monoid for the following categories $\mathbf{C}$: $\mathbf{Set}$, $\mathbf{Group}$, $\mathbf{Ab}$, $\mathbf{FAb}$, the last being the category of finite abelian groups.

(iii) Do the same for $\mathbf{C} = S_{\mathbf{cat}}$ where S is an arbitrary monoid.

(iv) Is the endomorphism monoid of a full and faithful functor F: $\mathbf{C} \to \mathbf{D}$ in general isomorphic to the endomorphism monoid of the full subcategory of $\mathbf{D}$ that is its image? If not, is it at least abelian? If you get such a result, can either ''full'' or ''faithful'' be deleted from the hypothesis?

Exercise 6.9:6. (i) Let F: **Set** → **Set** be the functor associating to every set S the set S^ω of all sequences $(s_0, s_1, \ldots)$ of elements of S. Determine all morphisms from F to the identity functor of **Set**.

(ii) Let G: **FSet** → **Set** be the restriction of the above functor to the category of finite sets; i.e., the functor taking every finite set S to the (generally infinite) set of all sequences of members of S. Determine all morphisms from F to the inclusion functor **FSet** → **Set**.

We have mentioned that constructions such as that of free groups, product objects, etc., could be made into functors by using the universal properties to get the required morphisms between the constructed objects. Since then, we have talked about *the* free group functor, *the* product functor on a category, etc.. Part (ii) of the next exercise justifies this use of the definite article.

Exercise 6.9:7. (i) Let $(\mathbf{C}, U)$ be a concrete category having free objects, and let Φ be a function associating to every $X \in \mathrm{Ob}(\mathbf{Set})$ a free object on X in $\mathbf{C}$, $\Phi(X) = (F(X), u(X))$. Show that there is a unique way of extending F (the first component of Φ) to a functor (i.e., defining $F(f)$ for each morphism f of **Set** in a functorial manner) so that u becomes a morphism of functors $\mathrm{Id}_{\mathbf{C}} \to UF$.

(ii) Suppose Φ: $X \mapsto (F(X), u(X))$ and Ψ: $X \mapsto (G(X), v(X))$ are two constructions *each* assigning to every set X a free object in $\mathbf{C}$ with respect to U, as in part (i). Show that the functors F and G obtained from Φ and Ψ as above are isomorphic; in fact, that there is a *unique* isomorphism making an appropriate diagram commute.

(iii) Write up the analogs of (i) and (ii) for one other functor associated with a universal construction, e.g., products, difference kernels, tensor products of abelian groups, etc.. You may abbreviate steps that parallel the free-object case closely.

Exercise 6.9:8. Consider a category $\mathbf{C}$ having finite products. When we spoke of making the product construction into a functor (in motivating the concept of a functor of two variables), the domain category was to be the set of *pairs* of objects of $\mathbf{C}$. Clearly we can do the same using I-tuples for any *fixed* finite set I. But what if we look at the product construction as simultaneously applying to I-tuples of objects as I ranges over *all* finite index sets?

To make this question precise, let $\mathrm{Ob}(\mathbf{C})^+$ denote the class of all families $(X_i)_{i\in I}$ such that I is a finite set (varying from family to family) and the X_i are objects of $\mathbf{C}$. Can you make this the object-set of a category $\mathbf{C}^+$ in a natural way, so that the product construction becomes a functor $\mathbf{C}^+ \to \mathbf{C}$? If so, will the same category $\mathbf{C}^+$ serve as domain for the *coproduct* construction, assuming $\mathbf{C}$ has finite coproducts?

Exercise 6.9:9. (i) Suppose F, G: $\mathbf{C} \to \mathbf{D}$ are functors, and a: $F \to G$ a morphism of functors. What is the relation between the conditions: (a) for all $X \in \mathrm{Ob}(\mathbf{C})$, $a(X)$ is a monomorphism in $\mathbf{D}(F(X), G(X))$, and (b) a is a monomorphism in $\mathbf{D}^{\mathbf{C}}(F, G)$?

(ii) Suppose F_1, F_2, P: $\mathbf{C} \to \mathbf{D}$ are functors, and p_1: $P \to F_1$, p_2: $P \to F_2$ are morphisms. What is the relation between the conditions (a) for all $X \in \mathrm{Ob}(\mathbf{C})$, $P(X)$ is a product of $F_1(X)$ and $F_2(X)$ in $\mathbf{D}$, with projection morphisms $p_1(X)$ and $p_2(X)$, and (b) P is a product of F_1 and F_2 in $\mathbf{D}^{\mathbf{C}}$, with projection morphisms p_1 and p_2?

To motivate what comes next, let us consider the following three pairs of constructions: (a) To every group G, we may associate the set of its elements of exponent 2, and also its set of elements of exponent 4; this gives two functors V_2 and V_4 from **Group** to **Set** such that for every G, $V_2(G) \subseteq V_4(G)$. (b) To every set X we can associate the set $\mathbf{P}(X)$ of its subsets, and also the set $\mathbf{P}_f(X)$ of its finite subsets. If we regard the power-set construction as a covariant functor P: **Set** → **Set**, this gives a second functor P_f: **Set** → **Set** such that for all X, $P_f(X) \subseteq P(X)$. (We used the covariant power-set functor here because the inverse image of a finite set under a set map may not be finite, so there is no natural way to make a *contravariant* functor out of

$\mathbf{P}_f$.) (c) If Inv: **Monoid** → **Monoid** denotes the functor associating to every monoid its submonoid of invertible elements, then for each monoid S, $\mathrm{Inv}(S)$ is a submonoid of $S = \mathrm{Id}_{\mathbf{Monoid}}(S)$.

These examples suggest that we want a concept of ''subfunctor'' of a functor. Of course, the examples were based on having the concept of a ''subobject'' of an object, and as we have observed, there is no way to define this in an arbitrary category. However, if we assume a concept of subobject *given*, we can define the concept of subfunctor relative to it:

Lemma 6.9.3. *Let* $\mathbf{C}$ *be a category, and* $\mathbf{C}_{\mathrm{incl}}$ *be a subcategory having for objects all the objects of* $\mathbf{C}$, *and having for morphisms a subclass of the monomorphisms of* $\mathbf{C}$, *called the* inclusions, *such that there is at most one inclusion morphism between any unordered pair of objects* (*i.e., such that* $\mathbf{C}_{\mathrm{incl}}$ *is a* (*possibly large*) *partially ordered set*). *For* $X_0, X \in \mathrm{Ob}(\mathbf{C})$, *let us call* X_0 *a* subobject *of* X (*or when there is a possibility of ambiguity, a ''subobject with respect to the distinguished subcategory* $\mathbf{C}_{\mathrm{incl}}$'') *if there exists an inclusion morphism* $X_0 \to X$. *If* X_0 *and* Y_0 *are subobjects of* X *and* Y *respectively, and* $f \in \mathbf{C}(X, Y)$, *let us say* f carries X_0 into Y_0 *if there exists a* (*necessarily unique!*) *morphism* $f_0 \in \mathbf{C}(X_0, Y_0)$ *making a commuting square with* f *and the inclusions of* X_0 *and* Y_0 *in* X *and* Y.

Then for $\mathbf{C}$ *as above, and* F *any functor from another category* $\mathbf{D}$ *into* $\mathbf{C}$, *the following data are equivalent:*

(a) *A choice for each* $X \in \mathrm{Ob}(\mathbf{D})$ *of a subobject* $F_0(X)$ *of* $F(X)$ *such that for each* $f \in \mathbf{D}(X, Y)$, $F(f)$ *carries* $F_0(X)$ *into* $F_0(Y)$.

(b) *A functor* $F_0 : \mathbf{D} \to \mathbf{C}$ *such that each* $F_0(X)$ *is a subobject of* $F(X)$, *and the inclusion maps give a morphism of functors* $F_0 \to F$.

(c) *A subobject* F_0 *of* F *with respect to the subcategory of* $\mathbf{C}^{\mathbf{D}}$ *having for objects all the objects of that category* (*all functors* $\mathbf{C} \to \mathbf{D}$), *and for morphisms those morphisms of functors whose values at all objects of* $\mathbf{D}$ *are inclusion morphisms* (*relative to* $\mathbf{C}_{\mathrm{incl}}$). *We may call such an* F_0 *a* subfunctor *of* F. □

Exercise 6.9:10. (i) Prove the above lemma, including the assertion of unicity noted parenthetically in the first paragraph.

(ii) Can the subcategory of $\mathbf{C}^{\mathbf{D}}$ referred to in point (c) of that lemma be described as $(\mathbf{C}_{\mathrm{incl}})^{\mathbf{D}}$?

In considering categories $\mathbf{C}$ of familiar algebraic objects, when we speak of subobjects and subfunctors, the distinguished subcategory $\mathbf{C}_{\mathrm{incl}}$ will be understood to have for morphisms the ''ordinary'' inclusions, unless the contrary is stated.

Exercise 6.9:11. Let G be a group.

(i) Show that if S a subfunctor of the identity functor of **Group**, then $S(G)$ will be a subgroup of G which is carried into itself by every endomorphism of G. (Group theorists call such a subgroup *completely invariant*.)

(ii) Is it true, conversely, that if H is any completely invariant subgroup of G, then there exists a subfunctor S of $\mathrm{Id}_{\mathbf{Group}}$ such that $H = S(G)$?

(iii) Given a subgroup H of G such that *some* subfunctor S of $\mathrm{Id}_{\mathbf{Group}}$ exists for which $H = S(G)$, will there exist a *least* S with this property? A greatest?

(iv) Generalize your answers to (i)-(iii), in one way or another.

Exercise 6.9:12. Let k be a field of characteristic 0, and k-**Mod** the category of k-vector-spaces. For each positive integer n let $\otimes^n: k\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ denote the n-fold tensor product functor, $V \mapsto V^{\otimes n} =_{\text{def}} V \otimes \ldots \otimes V$ (n factors).

(i) Determine all subfunctors of the functors $\otimes^1$ and $\otimes^2$.

(ii) Investigate subfunctors of higher $\otimes^n$'s.

(iii) Are the results you obtained in (i) and/or (ii) valid over fields k of arbitrary characteristic?

We have observed that the idea that two constructions of some sort of mathematical object are ''equivalent'' can often be made precise as a statement that two functors are isomorphic. A different type of statement is that two *sorts* of mathematical object are ''equivalent''. In some cases, this can be formalized by giving an *isomorphism* (invertible functor) between the categories of the two sorts of objects. E.g., the category of Boolean rings is isomorphic to the category of Boolean algebras, and **Group** is isomorphic to the category of those monoids all of whose elements are invertible. But there are times when this does not work, because the two sorts of objects differ in certain ''irrelevant'' structure which makes it impossible, or unnatural, to set up such an isomorphism. For instance, groups with underlying set contained in ω are ''essentially'' the same as arbitrary countable groups, although there cannot be an isomorphism between the categories of such groups, because one category is small and the other has the cardinality of the universe in which we are working. Monoids are ''essentially the same'' as categories with just one object, but the natural construction taking one-object categories to monoids is not one-to-one, because it forgets what element was the one object; and the way we found to go in the other direction (inserting ''1'' as the object) is likewise not onto. For these purposes, a concept weaker than isomorphism is useful.

Definition 6.9.4. *A functor* $F: \mathbf{C} \to \mathbf{D}$ *is called an* equivalence *between the categories* $\mathbf{C}$ *and* $\mathbf{D}$ *if there exists a functor* $G: \mathbf{D} \to \mathbf{C}$ *such that* $GF \cong \mathrm{Id}_{\mathbf{C}}$ *and* $FG \cong \mathrm{Id}_{\mathbf{D}}$ *(isomorphisms of functors). If such an equivalence exists, one says ''*$\mathbf{C}$ *is equivalent to* $\mathbf{D}$*'', often written* $\mathbf{C} \approx \mathbf{D}$.

Lemma 6.9.5. *A functor* $F: \mathbf{C} \to \mathbf{D}$ *is an equivalence if and only if it is full and faithful, and every object of* $\mathbf{D}$ *is isomorphic to* $F(X)$ *for some* $X \in \mathrm{Ob}(\mathbf{C})$.

Idea of Proof. ''$\Rightarrow$'' is straightforward. To show ''$\Leftarrow$'', choose for each object Y of $\mathbf{D}$ an object $G(Y)$ of $\mathbf{C}$ and an isomorphism $i(Y): Y \to FG(Y)$. One finds that there is a unique way to make G a functor so that i becomes an isomorphism $\mathrm{Id}_{\mathbf{D}} \cong FG$, and a straightforward way to construct an isomorphism $\mathrm{Id}_{\mathbf{C}} \cong GF$. □

Exercise 6.9:13. Give the details of the above proof.

Exercise 6.9:14. Let k be a field and k-**FMod** the category of finite-dimensional vector spaces over k. Let $\mathbf{Mat}_k$ denote the category whose objects are the nonnegative integers, and such that a morphism from m to n is an $n \times m$ matrix over k, with composition of morphisms given by matrix multiplication. Show that $\mathbf{Mat}_k \approx k\text{-}\mathbf{FMod}$.

Exercise 6.9:15. Let k and k-**FMod** be as in the preceding exercise. Show that duality of vector spaces gives a *contravariant equivalence* of k-**FMod** with itself, i.e., an equivalence between $k\text{-}\mathbf{FMod}^{\mathrm{op}}$ and k-**FMod**.

Exercise 6.9:16. Show that **Set** is not equivalent to $\mathbf{Set}^{\mathrm{op}}$. For additional credit, demonstrate the non-equivalence of a few other pairs of familiar categories, e.g., show that **Set** is not equivalent to **Group**.

Exercise 6.9:17. Let $\mathbf{FBool}^1$ denote the category of finite Boolean rings, and **FSet** the category of finite sets. In $\mathbf{FBool}^1$, 2 will denote the 2-element Boolean ring with underlying set $\{0,1\}$, while in **FSet**, 2 will as usual denote the set $\{0,1\}$. Note that for any $B\in\mathrm{Ob}(\mathbf{FBool}^1)$, the hom-set $\mathbf{FBool}^1(B,2)$, a finite set of homomorphisms $B\to 2$, induces a homomorphism $m_B\colon B\to\prod_{i\in\mathbf{FBool}^1(B,2)}2$.

(i) Show that m_B is always an isomorphism. In particular, this says that every finite Boolean ring is a finite product of copies of the ring 2.

(ii) Show with the help of the preceding result that the category $\mathbf{FBool}^1$ is equivalent to $\mathbf{FSet}^{\mathrm{op}}$.

Exercise 6.9:18. Let R be a ring, n a positive integer, and $M_n(R)$ the ring of $n\times n$ matrices over R. For any left R-module M, let $\mathrm{Col}_n(M)$ denote the set of column vectors of height n of elements of M, and let this be made a left $M_n(R)$-module in the obvious way. This gives a functor $\mathrm{Col}_n\colon R\text{-}\mathbf{Mod}\to M_n(R)\text{-}\mathbf{Mod}$.

Show that Col_n is an equivalence of categories.

(Rings such as R and $M_n(R)$ which have equivalent module categories are said to be *Morita equivalent*. Morita equivalence was mentioned in Exercise 6.2:3, in terms of isomorphisms in a peculiar category having rings as objects and bimodules as morphisms. I hope in the future to add to Chapter 9 an introduction to Morita theory, from which we will be able to see why the ‘‘invertible bimodule’’ property and the above condition are equivalent.)

The following definition and lemma reduce the question of whether two categories are *equivalent* to the question of whether two other categories are *isomorphic*.

Definition 6.9.6. *If* **C** *is a category, then a* skeleton *of* **C** *means a full subcategory having exactly one representative of each isomorphism class of objects of* **C***; i.e., by Lemma 6.9.5, a minimal full subcategory* $\mathbf{C}_0$ *such that the inclusion of* $\mathbf{C}_0$ *in* **C** *is an equivalence.*

The Axiom of Choice allows us to construct a skeleton for every category.

Lemma 6.9.7. *Let* **C** *and* **D** *be categories, with skeleta* $\mathbf{C}_0$ *and* $\mathbf{D}_0$. *Then* **C** *and* **D** *are equivalent if and only if* $\mathbf{C}_0$ *and* $\mathbf{D}_0$ *are isomorphic.* □

Exercise 6.9:19. Write out the proof of Lemma 6.9.7.

Exercise 6.9:20. Let X be a pathwise connected topological space. Recall that one can define a category $\pi_1(X)$ whose objects are the points of X, and in which a morphism from x to y means a homotopy class of paths from x to y. What does a skeleton of this category look like?

For future reference, let us make

Definition 6.9.8. *Let* I *be a set* (*for instance, a natural number or other cardinal*), *and* **C** *a category having I-fold products. If* X *is an object of* **C**, *then when the contrary is not stated,* X^I *will denote the I-fold product of* X *with itself, which we may call the ‘‘Ith power of* X*’’. Likewise, if* F *is a functor from another category* **D** *to* **C**, *then when the contrary is not stated,* F^I *will denote the functor taking each object* Y *of* **D** *to the object* $F(Y)^I$ *of* **C**, *and behaving in the obvious way on morphisms.*

(Note that if $F\colon \mathbf{C} \to \mathbf{C}$ is an endofunctor of some category $\mathbf{C}$, we might want to write F^n for the n-fold composite of F with itself. In that case we would have to make an explicit exception to the above convention.)

6.10. Properties of functor categories. In the preceding section we defined morphisms of functors, and saw some applications of the resulting category structure on $\mathbf{C}^{\mathbf{D}}$. Let us now set down a few basic properties of these constructions.

First, consider any bifunctor

$$F\colon\ \mathbf{C} \times \mathbf{D} \ \to\ \mathbf{E},$$

in other words, any object of $\mathbf{E}^{\mathbf{C} \times \mathbf{D}}$. If we fix an object $Y \in \mathrm{Ob}(\mathbf{D})$, it is easy to verify that F induces a functor $F(-, Y)\colon \mathbf{C} \to \mathbf{E}$, i.e., an object of $\mathbf{E}^{\mathbf{C}}$, sending each object X of $\mathbf{C}$ to $F(X, Y)$ and each morphism f of $\mathbf{C}$ to $F(f, \mathrm{id}_Y)$.

Having made this observation for each *object* of $\mathbf{D}$, let us now note that for each *morphism* between such objects, $g \in \mathbf{D}(Y, Y')$, the morphisms $F(\mathrm{id}_X, g)$ $(X \in \mathrm{Ob}(\mathbf{C}))$ yield a morphism of functors $F(-, g)\colon F(-, Y) \to F(-, Y')$. Thus our system of objects $F(-, Y)$ of $\mathbf{E}^{\mathbf{C}}$ has become a functor $F'\colon \mathbf{D} \to \mathbf{E}^{\mathbf{C}}$. That is, from our object F of $\mathbf{E}^{\mathbf{C} \times \mathbf{D}}$ we have gotten an object F' of $(\mathbf{E}^{\mathbf{C}})^{\mathbf{D}}$.

In constructing F', we have not used the values of F at all the morphisms of $\mathbf{C} \times \mathbf{D}$, but only at morphisms of the forms (id_X, g) and (f, id_Y); so we might wonder whether F' embodies all the information contained in F. But in fact, an arbitrary morphism of $\mathbf{C} \times \mathbf{D}$, $(f, g)\colon (X, Y) \to (X', Y')$, can be written $(f, \mathrm{id}_{Y'})(\mathrm{id}_X, g)$, so the images of morphisms of those two sorts do indeed determine the images of all morphisms of $\mathbf{C} \times \mathbf{D}$. In fact, we have

Lemma 6.10.1 (Law of exponents for categories)**.** *For any categories* $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ *one has* $\mathbf{E}^{\mathbf{C} \times \mathbf{D}} \cong (\mathbf{E}^{\mathbf{C}})^{\mathbf{D}}$, *via the construction sketched above.* □

Exercise 6.10:1. Prove the above lemma. In particular, describe how to map morphisms of $\mathbf{E}^{\mathbf{C} \times \mathbf{D}}$ to morphisms of $(\mathbf{E}^{\mathbf{C}})^{\mathbf{D}}$.

Exercise 6.10:2. Does one have other laws of exponents for functor categories? In particular, is $(\mathbf{D} \times \mathbf{E})^{\mathbf{C}} \cong (\mathbf{D}^{\mathbf{C}}) \times (\mathbf{E}^{\mathbf{C}})$, and is $\mathbf{E}^{\mathbf{C} \amalg \mathbf{D}} \cong (\mathbf{E}^{\mathbf{C}}) \times (\mathbf{E}^{\mathbf{D}})$? (For the meaning of $\mathbf{C} \amalg \mathbf{D}$, cf. Exercise 6.6:10.)

Next, suppose that $G_1, G_2\colon \mathbf{C} \to \mathbf{D}$ are functors, and $a\colon G_1 \to G_2$ is a morphism between them. If we compose G_1, respectively G_2 with a functor H from any other category into $\mathbf{C}$, we get functors $G_1 H$, respectively $G_2 H$, and not surprisingly, the morphism $a\colon G_1 \to G_2$ induces a morphism $G_1 H \to G_2 H$. Likewise, given a functor F out of $\mathbf{D}$, a induces a morphism $F G_1 \to F G_2$. These induced morphisms of functors are written $a \circ H\colon G_1 H \to G_2 H$ and $F \circ a\colon F G_1 \to F G_2$ respectively.

Example: Let a be the canonical morphism from the free group functor F to the free abelian group functor A. If we compose on the right with, say, the functor U taking every lattice to its underlying set,

$$\begin{tikzcd}
\mathbf{Lattice} \arrow[r, "U"] & \mathbf{Set} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "A"'{name=A}] & \mathbf{Group}, \arrow[from=F, to=A, "a"]
\end{tikzcd}$$

we get a morphism of functors $a \circ U$ taking free groups on the underlying sets of lattices L to the free abelian groups on the same underlying sets. If instead we compose on the left with the underlying-set functor V out of the category of groups,

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F"] \arrow[r, bend right, "A"'] \arrow[r, phantom, "\downarrow a"] & \mathbf{Group} \arrow[r, "V"] & \mathbf{Set}, \end{tikzcd}$$

we get a morphism of functors $V \circ a$ from the construction of the underlying set of the free group on each set X to that of the underlying set of the free abelian group on X.

We record below the above constructions and note the laws that they satisfy. The reader is advised to draw (or visualize) pictures like those above for the various situations described.

Lemma 6.10.2. *Let* $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{E}$ *be categories.*

(i) *Given a morphism* $a\colon G_1 \to G_2$ *of functors* $\mathbf{C} \to \mathbf{D}$, *and any functor* $F\colon \mathbf{D} \to \mathbf{E}$, *a morphism* $F \circ a\colon FG_1 \to FG_2$ *is defined by setting* $(F \circ a)(X) = F(a(X))$ $(X \in \mathrm{Ob}(\mathbf{C}))$.

(ii) *Given any functor* $G\colon \mathbf{C} \to \mathbf{D}$, *and a morphism* $b\colon F_1 \to F_2$ *of functors* $\mathbf{D} \to \mathbf{E}$, *a morphism* $b \circ G\colon F_1 G \to F_2 G$ *is defined by setting* $(b \circ G)(X) = b(G(X))$ $(X \in \mathrm{Ob}(\mathbf{C}))$.

(iii) *Given morphisms* $G_1 \xrightarrow{a_1} G_2 \xrightarrow{a_2} G_3$ *of functors* $\mathbf{C} \to \mathbf{D}$, *and any functor* $F\colon \mathbf{D} \to \mathbf{E}$, *one has*

$$(F \circ a_2 a_1) \;=\; (F \circ a_2)(F \circ a_1).$$

(iv) *Given any functor* $G\colon \mathbf{C} \to \mathbf{D}$, *and morphisms* $F_1 \xrightarrow{b_1} F_2 \xrightarrow{b_2} F_3$ *of functors* $\mathbf{D} \to \mathbf{E}$, *one has*

$$(b_2 b_1 \circ G) \;=\; (b_2 \circ G)(b_1 \circ G).$$

(v) *Given both a morphism* $a\colon G_1 \to G_2$ *of functors* $\mathbf{C} \to \mathbf{D}$, *and a morphism* $b\colon F_1 \to F_2$ *of functors* $\mathbf{D} \to \mathbf{E}$, *one has*

$$(b \circ G_2)(F_1 \circ a) \;=\; (F_2 \circ a)(b \circ G_1)$$

as morphisms $F_1 G_1 \to F_2 G_2$.

(vi) *Given functors* $G\colon \mathbf{C} \to \mathbf{D}$ *and* $F\colon \mathbf{D} \to \mathbf{E}$, *one has*

$$\mathrm{id}_F \circ G \;=\; \mathrm{id}_{FG} \;=\; F \circ \mathrm{id}_G .$$

(vii) *In summary, composition of functors* $\mathbf{C} \to \mathbf{D} \to \mathbf{E}$ *induces a functor* $\mathbf{E}^{\mathbf{D}} \times \mathbf{D}^{\mathbf{C}} \to \mathbf{E}^{\mathbf{C}}$. □

Exercise 6.10:3. (i) Prove statements (i)-(vi) of the above lemma.

(ii) Show that statement (vii) summarizes all of statements (i)-(vi), except for the explicit descriptions of how $F \circ a$ and $b \circ G$ are defined.

The above lemma shows that the operation of composing functors, which, to begin with, was defined as a set map

$$\mathbf{Cat}(\mathbf{D}, \mathbf{E}) \times \mathbf{Cat}(\mathbf{C}, \mathbf{D}) \;\to\; \mathbf{Cat}(\mathbf{C}, \mathbf{E}), \tag{6.10.3}$$

actually gives a functor

(6.10.4) $$\mathbf{E}^{\mathbf{D}} \times \mathbf{D}^{\mathbf{C}} \to \mathbf{E}^{\mathbf{C}}.$$

In making **Cat** a category, we had to verify that the set map (6.10.3) satisfied the associativity and identity laws; we now ought to check that these laws hold, not merely as equalities of set maps, but as equalities of functors! The case of the identity laws is easy, but as part (ii) of the next exercise shows, is still useful:

Exercise 6.10:4. (i) Given a morphism $a\colon G_1 \to G_2$ of functors $G_1, G_2\colon \mathbf{C} \to \mathbf{D}$, show that
$$a \circ \mathrm{Id}_{\mathbf{C}} = a = \mathrm{Id}_{\mathbf{D}} \circ a.$$

(ii) Show that the above result, together with Lemma 6.10.2(v), immediately gives the result of Exercise 6.9:5(i).

It is more work to write out the details of

Exercise 6.10:5. For categories $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ establish identities (like those of Lemma 6.10.2) showing that the two iterated-composition functors $\mathbf{E}^{\mathbf{D}} \times \mathbf{D}^{\mathbf{C}} \times \mathbf{C}^{\mathbf{B}} \to \mathbf{E}^{\mathbf{B}}$ are equal as functors.

In doing the above exercises, you may wish to use the notation which represents the common value of the two sides of the equation of (v) above as $b \circ a$. Note, however, point (i) of

Exercise 6.10:6. (i) Show that if the above notation is adopted, there are situations where $b \circ a$ and ba are both defined, but are unequal.

(ii) Can you find any important class of cases where they must be equal?

Exercise 6.10:7. Suppose we have an equivalence of categories, given by functors $F\colon \mathbf{C} \to \mathbf{D}$, $G\colon \mathbf{D} \to \mathbf{C}$ with $\mathrm{Id}_{\mathbf{C}} \cong GF$, $\mathrm{Id}_{\mathbf{D}} \cong FG$. Given a particular isomorphism of functors $i\colon \mathrm{Id}_{\mathbf{C}} \to GF$, can one in general choose an isomorphism $j\colon \mathrm{Id}_{\mathbf{D}} \to FG$ such that the two isomorphisms of functors, $i \circ G,\ G \circ j\colon G \to GFG$ are equal, and likewise the two isomorphisms $j \circ F,\ F \circ i\colon F \to FGF$?

How are we to look at a functor category $\mathbf{C}^{\mathbf{D}}$? Should we think of its objects as ‘‘maps’’ or as ‘‘things’’? As a category, is it ‘‘like’’ $\mathbf{C}$, ‘‘like’’ $\mathbf{D}$, or like neither?

My general advice is to think of its objects as ‘‘things’’ and its morphisms as ‘‘maps’’; more precisely, its objects are ‘‘things’’ composed of *systems* of objects of $\mathbf{C}$, linked together by morphisms in a way parametrized by $\mathbf{D}$. With respect to basic properties, such a functor category usually behaves more like $\mathbf{C}$ than like $\mathbf{D}$. For example, if $\mathbf{C}$ has finite products, so does $\mathbf{C}^{\mathbf{D}}$: One can construct the product $F \times G$ of two functors $F, G \in \mathbf{C}^{\mathbf{D}}$ ‘‘objectwise’’, by taking $(F \times G)(X)$ to be the product $F(X) \times G(X)$ for each $X \in \mathrm{Ob}(\mathbf{D})$ (cf. Exercise 6.9:9(ii)). On the other hand, existence of products in $\mathbf{D}$ tells us nothing about $\mathbf{C}^{\mathbf{D}}$.

6.11. Enriched categories. A recurring trick in category theory is to characterize some type of mathematical entity as a certain sort of structure in a particular category, such as **Set**, analyze what properties of **Set** are needed for the concept to make sense, and then create a generalized definition, like the original one, except that **Set** is replaced by a general category having the required properties.

There is in fact an important application of this idea to the concept of *category* itself! We shall sketch this briefly below. We will begin with a few examples to motivate the idea, and then discuss what is involved in the general case, though we shall not give formal definitions.

Recall that a category, as we have defined it, is given by a *set* of objects, and a *set* of morphisms between any two objects, with composition operations given by *set maps*, $\mu: \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \to \mathbf{C}(X,Z)$. But now consider the category **Cat**. Though we still have a *set* of objects, for each pair of objects $\mathbf{C}$, $\mathbf{D}$ we have seen that we can speak of a *category* of morphisms $\mathbf{C}^{\mathbf{D}}$, and composition in that category is given by *functors* $\mu: \mathbf{E}^{\mathbf{D}} \times \mathbf{D}^{\mathbf{C}} \to \mathbf{E}^{\mathbf{C}}$.

Likewise, for any ring R, it is well known that the homomorphisms from one R-module to another form an additive group, so that R-**Mod** can be developed as having, for each pair of objects, an *abelian group* of morphisms. Here composition is given by *bilinear maps* among these abelian groups.

One expresses these facts by saying that **Cat** can be regarded as a **Cat**-*based category*, or a **Cat**-category for short, and R-**Mod** as an **Ab**-*category*. Similarly, in various situations where one has a natural topological structure on sets of morphisms, such that composition is bicontinuous, one can speak of having a **Top**-based category.

These generalized categories are called *enriched* categories.

Note that when we referred to R-**Mod** as being an **Ab**-category, this included the observation that the composition maps are *bilinear*. Thus, they correspond to abelian group homomorphisms

$$\mu_{XYZ}: \mathbf{C}(Y,Z) \otimes \mathbf{C}(X,Y) \to \mathbf{C}(X,Z).$$

The general definition of enriched category requires that the base category (the category in which the hom-objects are taken to lie; i.e., **Set**, **Ab**, etc.) be given with a bifunctor into itself having certain properties, which is to be used, as above, in describing the composition maps. In the case of **Ab** this is the *tensor-product* functor, while in the cases of **Set**, **Cat**, and **Top**, the corresponding role is filled by the *product*. See [**14**, §VII.7] for more details.

One should, strictly, distinguish between R-**Mod** as an ordinary (i.e., **Set**-based) category and as an **Ab**-category, writing these two entities as, say, R-**Mod** and R-$\mathbf{Mod}_{(\mathbf{Ab})}$, and similarly distinguish **Cat** and $\mathbf{Cat}_{(\mathbf{Cat})}$ – just as one ought to distinguish between the set of integers, the additive group of integers, the lattice of integers, the ring of integers, etc.. This notational problem will not concern us, however, since we will not formalize the concept of enriched category in these notes. Outside this section, if we have occasion to refer to the special properties of categories such as R-**Mod** or **Cat**, we shall not assume familiarity with the theory of enriched categories, but simply use in an ad hoc fashion what we know about the extra structure.

We remark that **Ab**-based categories, and more generally, k-**Mod**-based categories for k a commutative ring (called ‘‘k-linear categories’’), are probably more widely used than all the other sorts of enriched categories together; see [**7**] for a lively development of the subject.

The **Cat**-based category **Cat** contains a vast number of interesting sub-**Cat**-categories. Here is one:

Exercise 6.11:1. Consider the full subcategory of **Cat** whose objects are the categories $G_{\mathbf{cat}}$, for groups G. Characterize in group-theoretic terms the morphisms, and morphisms of morphisms, in this **Cat**-category.

Translate your answer into a description of a **Cat**-category structure one can put on the category **Group**.

The student interested in ring theory might note that the category of Exercise 6.2:3 (with rings as objects, and bimodules ${}_RB_S$ as morphisms) can be made a **Cat**-category, by using bimodule homomorphisms as the morphisms-among-morphisms; moreover, each morphism-category $\mathbf{C}(R,S)$ (for R and S rings) is in fact an **Ab**-category! What this says is that this category is an **AbCat**-category, where **AbCat** is the category of **Ab**-categories. There is an explanation: This category is equivalent to the subcategory of **Cat** whose objects are the **Ab**-categories R-**Mod**, and whose morphisms are the functors ${}_R B \otimes_S -: S$-**Mod** → R-**Mod** induced by bimodules ${}_R B_S$. So this observation is really a special case of the fact that the category of **Ab**-categories, which we may write **AbCat**, is an **AbCat**-category, just as **Ab** is an **Ab**-category and **Cat** is a **Cat**-category.

We have mentioned (in Exercise 6.3:1 and the preceding discussion) that there is a version of the definition of category which eliminates reference to objects, and assumes only one kind of element, the morphism. (The objects are hidden under the guise of their identity morphisms.) If we apply this idea twice to the concept of a **Cat**-category, we likewise get a structure with only one type of element – what we have been calling the morphisms of morphisms – but with two partial composition operations on these elements, ab and $a \circ b$ (Exercise 6.10:6). Described in this way, **Cat**-categories have been called ‘‘2-categories’’ [**14**, p. 44]. (The relation between the two types of composition is slightly asymmetric. If one drops the asymmetric condition – that every identity element with respect to the *first* composition is also an identity element with respect to the *second* – one gets a slightly more general concept, also defined in [**14**], and called a ‘‘double category’’.)

Having begun by considering **Cat** as an ordinary, i.e., **Set**-based category, with objects and morphisms (i.e., functors), and then having found that there was an important concept of *morphisms between morphisms* (morphisms of functors), we may ask whether one can define, further, *morphisms between morphisms between morphisms* in this category. The answer is ‘‘yes and no’’. On the yes side, let us observe that in fact, one can set up a concept of ‘‘morphisms between morphisms’’ in any category **C**! For a morphism in **C** is the same as an object of $\mathbf{C}^{\mathbf{2}}$, where **2** denotes the diagram category $\cdot\!\to\!\cdot$, and we know how to make $\mathbf{C}^{\mathbf{2}}$ a category. So in particular, given categories **C** and **D** we can define a ‘‘morphism of morphisms in $\mathbf{C}^{\mathbf{D}}$’’, which is thus a ‘‘morphism of morphisms of morphisms’’ in **Cat**.

However, this construction does not constitute a nontrivial enrichment of structure, since the concept of morphism of morphisms we have just described in an arbitrary category **C** is defined in terms of its existing category structure. (Indeed, when applied to **Cat**, it does not give the concept of ‘‘morphism of functors’’, but that of ‘‘commuting square of functors’’.) So we come to the ‘‘no’’ side of the answer – so far as I know, the category **Cat** has no enriched structure beyond that of a **Cat**-category.

However, if one turns from the category **Cat** of all **Set**-based categories, to the category **CatCat** of all **Cat**-based categories, one finds that here one has a natural and nontrivial concept of morphisms between morphisms between morphisms – in other words, **CatCat** is a **CatCat**-based category. And this process can be iterated ad infinitum.

But it is time to return from these vertiginous heights to the main stream of our subject.

Chapter 7. Universal constructions in category-theoretic terms.

The language of category theory has enabled us to give general definitions of ‘‘free object’’, ‘‘product’’, ‘‘coproduct’’, ‘‘difference kernel’’ and various other universal constructions. It is clear that these different constructions have many properties in common. Let us now look for ways to unify them, so that we will be able to prove results about them by general arguments, rather than piecemeal.

7.1. Universality in terms of initial and terminal objects. In all the above constructions, we deal with mathematical entities with certain ‘‘extra’’ structure, and seek one such entity, F, that is ‘‘universal’’. This suggests that we make the class of entities with such extra structure into a category, and examine the universal property of F there.

For instance, the free group on three generators is universal among systems (G, a, b, c) where G is a group, and $a, b, c \in |G|$. If we define a category whose objects are these systems (G, a, b, c), and where a morphism $(G, a, b, c) \to (G', a', b', c')$ means a group homomorphism $f: G \to G'$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, we see that the universal property of the free group (F, x, y, z) says that it has a unique morphism into every object of the category – in other words, that it is an initial object.

Similarly, given a group G, the *abelianization* of G is universal among pairs (A, f) where A is an abelian group, and f a group homomorphism $G \to A$. If we define a morphism from one such pair (A, f) to another such pair (B, g) to mean a group homomorphism $m: A \to B$ such that $mf = g$, we see that the definition of the abelianization of G says that it is initial in *this* category.

Finally, a group, a ring, a lattice, etc., with a presentation $<X \mid R>$ clearly means an initial object in the category whose objects are groups, etc., with specified X-tuples of elements satisfying the equations R, and whose morphisms are group homomorphisms respecting these distinguished X-tuples.

The above were examples of what we named ‘‘left universal’’ properties in §3.8. Let us look at one ‘‘right universal’’ property, that of a *product* of two objects A and B in a category $\mathbf{C}$. We see that the relevant auxiliary category should have for objects all 3-tuples (X, a, b), where $X \in \mathrm{Ob}(\mathbf{C})$, $a \in \mathbf{C}(X, A)$ $b \in \mathbf{C}(X, B)$, and for morphisms $(X, a, b) \to (Y, a', b')$ all morphisms $X \to Y$ in $\mathbf{C}$ making commuting triangles with the maps into A and B. A direct product of A and B in $\mathbf{C}$ is seen to be a *terminal* object (P, p_1, p_2) in this category.

You can likewise easily translate the universal properties of *pushouts, pullbacks* and *coproducts* in arbitrary categories to those of initial or terminal objects in appropriately defined auxiliary categories.

So all the universal properties we have considered reduce to those of being an initial or a terminal object in an appropriate category. This approach is followed by Lang [**28**, p. 57 et seq.], who gives these the more poetic designations ‘‘universally repelling’’ and ‘‘universally attracting’’ objects. Moreover, since a terminal object in $\mathbf{C}$ is an initial object in $\mathbf{C}^{\mathrm{op}}$, all universal properties reduce to that of initial objects!

Lemma 6.8.2 tells us that initial (and hence terminal) objects are *unique* up to unique isomorphism. This gives us, in one fell swoop, uniqueness up to canonical isomorphism for free groups, abelianizations, products, coproducts, pushouts, pullbacks, objects presented by generators and relations, and all the other universal constructions we have considered. (The canonical isomorphisms that these constructions are ‘‘unique up to’’ correspond to the unique morphisms between any two initial objects of a category. I.e., given two realizations of one of our universal constructions, these isomorphisms will be the unique morphisms from each to the other that preserve the extra structure.)

We will look at questions of *existence* in §7.10.

7.2. Representable functors, and Yoneda’s Lemma. The above approach to universal constructions is impressive for its simplicity; but we would also like to relate these universal objects to the original categories in question: Though the free group on an S-tuple of generators is initial in the category of groups given with S-tuples of elements, and the kernel of a group homomorphism $f: G \to H$ is terminal in the category of groups L given with homomorphisms $L \to G$ having trivial composite with f, we also want to understand these constructions in relation to the category $\mathbf{Group}$.

Note that the objects of the auxiliary categories we have used are pairs (X, a), where X is an object of the original category $\mathbf{C}$, and a is some additional structure on X. If we write $F(X)$ for the set of *all possible values* of this additional structure (e.g., in the case that leads to the free group on a set S, the set of all S-tuples of elements of X), we find that F is in general a functor, covariant or contravariant, from $\mathbf{C}$ to $\mathbf{Set}$. The condition characterizing a *left* universal pair (R, u) is that for every $X \in \mathrm{Ob}(\mathbf{C})$ and $x \in F(X)$, there should be a unique morphism $f: R \to X$ such that $F(f)(u) = x$. This condition – which we see requires a covariant F – is equivalent to saying that for each object X, the set of morphisms $f \in \mathbf{C}(R, X)$ is sent bijectively to the set of elements of $F(X)$ by the map $f \mapsto F(f)(u)$. The bijectivity of this correspondence for each X leads to an isomorphism between the functor $\mathbf{C}(R, -)$, i.e., $h_R: \mathbf{C} \to \mathbf{Set}$, and the given functor $F: \mathbf{C} \to \mathbf{Set}$. Thus, the universal property of R can be formulated as a statement of this isomorphism:

Theorem 7.2.1. *Let* $\mathbf{C}$ *be a category, and* $F: \mathbf{C} \to \mathbf{Set}$ *a functor. Then the following data are equivalent:*

(i) *An object* $R \in \mathrm{Ob}(\mathbf{C})$ *and an element* $u \in F(R)$ *having the* universal property *that for all* $X \in \mathrm{Ob}(\mathbf{C})$ *and all* $x \in F(X)$, *there exists a unique* $f \in \mathbf{C}(R, X)$ *such that* $F(f)(u) = x$.

(ii) *An* initial object (R, u) *in the category whose objects are all ordered pairs* (X, x) *with* $X \in \mathrm{Ob}(\mathbf{C})$ *and* $x \in F(X)$, *and whose morphisms are morphisms among the first components of these pairs which respect the second components.*

(iii) *An object* R *and an isomorphism of functors* $i: h_R \cong F$ *in* $\mathbf{Set}^{\mathbf{C}}$.

Namely, given (R, u) *as in* (i) *or* (ii), *one obtains a pair* (R, i) *as in* (iii) *by letting* $i(X)$ *take* $f \in h_R(X)$ *to* $F(f)(u) \in F(X)$, *while in the reverse direction, one obtains* u *from* i *as* $i(R)(\mathrm{id}_R)$.

Sketch of Proof. The equivalence of the structures described in (i) and (ii) is immediate.

Concerning our description of how to pass from these structures to that of (iii), it is a straightforward verification that for any $u \in F(R)$, the map i described there gives a *morphism of functors* $h_R \to F$. That this is an isomorphism is then the content of the universal property of (i). In the opposite direction, given an isomorphism i as in (iii), if u is defined as indicated, then the universal property of (i) is just a restatement of the bijectivity of the maps $i(X): h_R(X) \to F(X)$.

Finally, it is easy to check that if one goes as above from universal element to isomorphism of functors and back, one recovers the original element, and if one goes from isomorphism to universal element and back, one recovers the original isomorphism. □

Exercise 7.2:1. Write out the ''straightforward verifications'' referred to in the second sentence of the above proof.

Dualizing, we get

Theorem 7.2.2. *Let* $\mathbf{C}$ *be a category, and* F *a contravariant functor from* $\mathbf{C}$ *to* **Set** (*i.e., a functor* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$). *Then the following data are equivalent:*

(i) *An object* $R \in \mathrm{Ob}(\mathbf{C})$ *and an element* $u \in F(R)$ *with the universal property that for any* $X \in \mathrm{Ob}(\mathbf{C})$ *and* $x \in F(X)$, *there exists a unique* $f \in \mathbf{C}(X, R)$ *such that* $F(f)(u) = x$.

(ii) *A terminal object* (R, u) *in the category whose objects are all ordered pairs* (X, x) *with* $X \in \mathrm{Ob}(\mathbf{C})$ *and* $x \in F(X)$, *and whose morphisms are morphisms among the first components of these pairs which respect the second components.*

(iii) *An object* R *and an isomorphism of contravariant functors* $i\colon h^R \cong F$ *in* $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

Namely, given (R, u) *as in* (i) *or* (ii), *one obtains a pair* (R, i) *as in* (iii) *by letting* $i(X)$ *take* $f \in h^R(X)$ *to* $F(f)(u) \in F(X)$, *while in the reverse direction, one obtains* u *from* i *as* $i(R)(\mathrm{id}_R)$. □

Note that the last phrase of Theorem 7.2.1(ii), ''which respect second components'', means that for a morphism $f: X \to Y$ to be considered a morphism $(X, x) \to (Y, y)$, we require $F(f)(x) = y$, while in Theorem 7.2.2(ii), the corresponding condition is $F(f)(y) = x$.

The reader who has done Exercise 6.8:24 will see that the auxiliary categories used in point (ii) of the above two theorems are comma categories, $(1 \downarrow F)$.

The properties described above have names:

Definition 7.2.3. *Let* $\mathbf{C}$ *be a category.*

A covariant functor $F\colon \mathbf{C} \to \mathbf{Set}$ *is said to be* representable *if it is isomorphic to a covariant hom-functor* h_R *for some* $R \in \mathrm{Ob}(\mathbf{C})$.

A contravariant functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *is said to be* representable *if it is isomorphic to a contravariant hom-functor* h^R *for some* $R \in \mathrm{Ob}(\mathbf{C})$.

In each case, R *is called the* representing object *for* F, *and if* i *is the given isomorphism of functors, then* $i(R)(\mathrm{id}_R)$ *is called the associated* universal element *of* $F(R)$.

So from this point of view, universal problems in a category $\mathbf{C}$ are questions of the *representability* of certain set-valued functors on $\mathbf{C}$. Let us examine a few set-valued functors, and see which of them are representable.

If U is the underlying-set functor on **Group**, a representing object for U should be a group with a universal element of its underlying set. The object with this property is the free group on one generator. More generally, if a category has free objects with respect to a concretization U, then U will be represented by the free object on one generator, while the free object on a general set I can be characterized as representing the functor U^I (Definition 6.9.8).

The functor associating to every group the set of its elements of exponent 2 is represented by the group $\mathbf{Z}_2$. More generally, the group with presentation by generators and relations $<X \mid R>$ represents the functor associating to every group G the set of X-tuples of members of G which satisfy the relations R.

Is the functor associating to every commutative ring K the set $|K[t]|$ of all polynomials over

K in one indeterminate t representable? A representing object would be a ring R with a universal polynomial $u(t) \in R[t]$. The universal property says that given any polynomial $p(t)$ over any ring K, there should exist a unique homomorphism $R \to K$ which, applied coefficient-wise to polynomials, carries $u(t)$ to $p(t)$. But clearly there is a problem here: The polynomial u will have some degree n, and if we choose a polynomial p of degree $> n$, it cannot be obtained from u in this way. So the set-of-polynomials functor is not representable.

However, there is a concept related to that of polynomial but not subject to the restriction that only finitely many of the coefficients be nonzero, that of a *formal power series* $a_0 + a_1 t + a_2 t^2 + \ldots$. If K is a ring, then the ring of formal power series over K is denoted $K[[t]]$; its underlying set $|K[[t]]| = \{a_0 + a_1 t + a_2 t^2 + \ldots\}$ can be identified with the set of all sequences $(a_0, a_1, \ldots)$ of elements of K, i.e., with $|K|^\omega$. We know that the functor $K \mapsto |K|^\omega$ is represented by the free commutative ring on an ω-tuple of generators, that is, the polynomial ring $\mathbf{Z}[A_0, A_1, \ldots]$. And indeed, the formal power series ring over this polynomial ring contains the element $A_0 + A_1 t + A_2 t^2 + \ldots$, which clearly has the property of a universal power series.

Exercise 7.2:2. (i) Show that the functor associating to every monoid S the set of its invertible elements is representable, but that the functor associating to S the set of its right-invertible elements is not.

(ii) What about the functor associating to every monoid S the set of pairs (x, y) such that $xy = e$ and $yx = e$? The set of pairs (x, y) merely satisfying $xy = e$? The set of 3-tuples (x, y, z) such that $xy = xz = e$?

(iii) Determine which, if any, of the functors mentioned in (i) and (ii) are isomorphic to one another.

Exercise 7.2:3. Let P denote the contravariant power-set functor, associating to every set X the set $\mathbf{P}(X)$ of its subsets, and E the contravariant functor associating to every set X the set $\mathbf{E}(X)$ of equivalence relations on X. Determine whether each of these is representable.

Exercise 7.2:4. Let A, B be objects of a category $\mathbf{C}$. Describe a set-valued functor F on $\mathbf{C}$ such that a *product* of A and B, if it exists in $\mathbf{C}$, means a representing object for F, and likewise a functor G such that a *coproduct* of A and B in $\mathbf{C}$ means a representing object for G. (One of these will be covariant and the other contravariant.)

Students who know some Lie group theory might try

Exercise 7.2:5. Let **LieGp** denote the category of Lie groups and continuous group homomorphisms. Let T: **LieGp** → **Set** denote the functor associating to a Lie group L the set of tangent vectors to L at the neutral element. Which of the following covariant functors **LieGp** → **Set** are representable? (a) the functor T, (b) the functor T^2: $L \mapsto T(L) \times T(L)$, (c) the functor $L \mapsto \{(x, y) \in T(L) \times T(L) \mid [x, y] = x\}$.

Exercise 7.2:6. Given a set X, let $\mathrm{GpStruct}(X)$ denote the set of all group-structures on X (consisting of a composition operation μ, an inverse operation ι, and a neutral element e). A group can be considered as a set X given with an element $s \in \mathrm{GpStruct}(X)$, and the category **Group** has an initial object. This looks as though it should mean the underlying set of this group is a representing object for GpStruct; but something is clearly wrong, since a map from this set into a set X surely does not determine a group structure on X. Resolve this paradox.

The equivalence, in each of Theorems 7.2.1 and 7.2.2, of parts (ii) and (iii) shows that the concept of representable functor can be characterized in terms of initial and terminal objects. The reverse is also true:

Exercise 7.2:7. Let $\mathbf{C}$ be any category. Construct a covariant functor F and a contravariant functor G from $\mathbf{C}$ to $\mathbf{Set}$ such that an initial, respectively a terminal object of $\mathbf{C}$ is equivalent to a representing object for F, respectively G.

The implication (i)⇒(iii) in Theorem 7.2.1 shows that an isomorphism between the hom-functor h_R associated with an object R, and an arbitrary functor F, is equivalent to a specification of an element of $F(R)$ with the universal property given in (i). In fact, *every* morphism (invertible or not) from a hom-functor h_R to a functor F corresponds to a choice of *some* element of $F(R)$. Though utterly simple to prove, this is an important tool. We give both this result and its contravariant dual in

Lemma 7.2.4 (Yoneda's Lemma)**.** *Let* $\mathbf{C}$ *be a category, and* R *an object of* $\mathbf{C}$.

If $F\colon \mathbf{C} \to \mathbf{Set}$ *is a covariant functor, then morphisms* $f\colon h_R \to F$ *are in one-to-one correspondence with elements of* $F(R)$, *under the map* $f \mapsto f(R)(\mathrm{id}_R)$.

Likewise, if $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *is a contravariant functor, morphisms* $f\colon h^R \to F$ *are in one-to-one correspondence with elements of* $F(R)$, *again under the map* $f \mapsto f(R)(\mathrm{id}_R)$.

Proof. In the covariant case, we must describe how to get from an element $x \in F(R)$ an appropriate morphism $f_x\colon h_R \to F$. We define f_x to carry $a \in h_R(X) = \mathbf{C}(R, X)$ to $F(a)(x) \in F(X)$. The verification that this is a morphism of functors, and that this construction is inverse to the indicated map from morphisms of functors to elements of $F(R)$, is immediate.

The contravariant case follows by duality (or by the dualized argument). □

Again –

Exercise 7.2:8. Show the verifications omitted in the proof of the above result.

The following line of thought yields some intuition on Yoneda's Lemma. Recall that if G is a group, then a G-set, i.e., a functor from the category $G_{\mathbf{cat}}$ to $\mathbf{Set}$, can be looked at as a (possibly non-faithful) representation of G by permutations. In the same way, for any category $\mathbf{C}$, a functor $F\colon \mathbf{C} \to \mathbf{Set}$ can be thought of as a (possibly non-faithful) representation of $\mathbf{C}$ by sets and set maps. Like a G-set, such a representation F can be regarded as a mathematical ''object'', whose ''elements'' are the members of the sets $F(X)$ $(X \in \mathrm{Ob}(\mathbf{C}))$. This was the point of view of our development of Cayley's Theorem for small categories. In proving that result, we constructed such an object by introducing one generator in each set $F(X)$, and no relations; in the discussion that followed we observed that if one introduced only a generator in the set $F(X)$ corresponding to a *particular* $X \in \mathrm{Ob}(\mathbf{C})$, and again no relations, the resulting ''freely generated'' object would be essentially the hom-functor which we named h_X. Yoneda's Lemma is the statement of the universal property of this ''free'' construction – that a morphism from this ''representation of $\mathbf{C}$ by sets'' to any other ''representation of $\mathbf{C}$ by sets'' is uniquely determined by specifying where the one generator, the identity element $\mathrm{id}_X \in h_X(X)$, is to be sent. We make this formulation explicit in

Corollary 7.2.5. *Let* $\mathbf{C}$ *be a category and* R *an object of* $\mathbf{C}$.

In the (*large*) *category whose objects are pairs* (F, x) *where* F *is a functor* $\mathbf{C} \to \mathbf{Set}$ *and* x *an element of* $F(R)$, *the pair* (h_R, id_R) *is the initial object. Equivalently, the object* $h_R \in \mathrm{Ob}(\mathbf{Set}^{\mathbf{C}})$ *is a representing object for the ''evaluation at* R'' *functor* $\mathbf{Set}^{\mathbf{C}} \to \mathbf{Set}$, *the universal element being* $\mathrm{id}_R \in h_R(R)$.

Likewise, in the category whose objects are pairs (F, x) *where* F *is a functor* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$

and x *an element of* $F(R)$, *the pair* (h^R, id_R) *is the initial object; equivalently, the object* $h^R \in \mathrm{Ob}(\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}})$ *represents the* (*again covariant!*) *''evaluation at* R*'' functor* $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}$. □

What if we apply Yoneda's Lemma (covariant or contravariant) to the case where the arbitrary functor F is another hom-functor h_S, respectively h^S? We get

Corollary 7.2.6. *Let* $\mathbf{C}$ *be a category.*

Then for any two objects $R, S \in \mathrm{Ob}(\mathbf{C})$, *the morphisms from* h_R *to* h_S *as functors* $\mathbf{C} \to \mathbf{Set}$ *are in one-to-one correspondence with morphisms* $S \to R$. *Thus, the mapping* $R \mapsto h_R$ *gives a* contravariant full embedding *of* $\mathbf{C}$ *in* $\mathbf{Set}^{\mathbf{C}}$, *the ''Yoneda embedding''.*

Likewise, morphisms from h^R *to* h^S *as functors* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *correspond to morphisms* $R \to S$, *giving a* covariant *full ''Yoneda embedding'' of* $\mathbf{C}$ *in* $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

These two embeddings may both be obtained from the bivariant hom-functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ *by distinguishing one or the other argument, i.e., regarding this bifunctor in one case as a functor* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$, *and in the other as a functor* $\mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

Sketch of Proof. By Lemma 6.10.1 the bivariant hom functor does indeed yield functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$ and $\mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ on distinguishing one or the other argument, and we see that the object R is sent to h_R, respectively h^R. Given a morphism $f: S \to R$ in $\mathbf{C}$, one verifies that the induced morphism of functors $h_f: h_R \to h_S$ takes id_R to $f \in h_S(R)$. It follows from Yoneda's Lemma with $F = h_S$ that the map $f \mapsto h_f$ is one-to-one and onto, so our functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$ is full and faithful. The contravariant case follows by duality. □

Exercise 7.2:9. Verify the above characterization of the morphism of functors induced by a morphism $f: S \to R$.

Exercise 7.2:10. Show how to answer most of the parts of Exercise 6.9:4 using Yoneda's Lemma.

Remark 7.2.7. It may seem paradoxical that we get the *contravariant* Yoneda embedding using *covariant* hom-functors, and the *covariant* Yoneda embedding using *contravariant* hom-functors, but there is a simple explanation. When we write the hom bifunctor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ as a functor to a functor category, $\mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ or $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$, by distinguishing one variable, the variance in that variable determines the variance of the resulting Yoneda embedding, while the variance in the other variable determines the variance of the hom-functors this embedding takes on as its values. Whichever way we slice it, we get covariance in one, and contravariance in the other.

In the last chapter, we saw that systems of universal constructions could frequently be linked together, by natural morphisms among the constructed objects, to give functors. From the above corollary, we see that this should correspond to situations where the functors that these universal objects are constructed to *represent* are linked by a corresponding system of morphisms of functors, in other words (by Lemma 6.10.1) where they form the components of a *bifunctor*. There is a slight complication in formulating this precisely, because the given representable functors are not themselves the hom-functors h_R or h^R, but only isomorphic to these, and the choice of representing objects R is itself determined only up to isomorphism. To get around this, let us prove a lemma showing that a system of objects separately isomorphic to the values of a functor in fact form the values of an isomorphic functor.

Lemma 7.2.8. *Let $F: \mathbf{C} \to \mathbf{D}$ be a functor, and for each $X \in \mathrm{Ob}(\mathbf{C})$, let $i(X)$ be an isomorphism of $F(X)$ with another object $G(X) \in \mathrm{Ob}(\mathbf{D})$.*

Then there is a unique way to assign to each morphism of $\mathbf{C}$, $f \in \mathbf{C}(X, Y)$ a morphism $G(f) \in \mathbf{D}(G(X), G(Y))$ so that the objects $G(X)$ and morphisms $G(f)$ constitute a functor G: $\mathbf{C} \to \mathbf{D}$, and i constitutes an isomorphism of functors, $F \cong G$.

Proof. If G is to be a functor and i a morphism of functors, then for each $f \in \mathbf{C}(X, Y)$ we must have $G(f)\, i(X) = i(Y) F(f)$. Since $i(X)$ is an isomorphism, we can rewrite this as $G(f) = i(Y) F(f)\, i(X)^{-1}$. It is straightforward to verify that G, so defined on morphisms, is indeed a functor. This definition of $G(f)$ insures that i is a morphism of functors, and it clearly has an inverse i^{-1}: $G \to F$ defined by $i^{-1}(X) = i(X)^{-1}$. □

Exercise 7.2:11. Write out the verification that G, constructed as above, is a functor.

We can now get our desired result about tying representing objects together into a functor.

Lemma 7.2.9. *Let A: $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ be a bifunctor, and suppose that for each $X \in \mathrm{Ob}(\mathbf{C})$ the induced functor $A(X, –)$: $\mathbf{D} \to \mathbf{Set}$ is representable, with a representing object $F(X) \in \mathrm{Ob}(\mathbf{D})$, and isomorphism $i(X)$: $A(X, –) \cong h_{F(X)}$. Then F can be made a covariant functor $\mathbf{C} \to \mathbf{D}$ in a unique way so that the isomorphisms $i(X)$ constitute an isomorphism of bifunctors*

$$i:\ A(–, –)\ \cong\ \mathbf{D}(F(–), –).$$

Conversely, suppose we are given a family of covariant set-valued functors $A(X, –)$: $\mathbf{D} \to \mathbf{Set}$, one for each object X of $\mathbf{C}$, such that each $A(X, –)$ is representable, say with representing object $F(X)$ and isomorphism $i(X)$: $A(X, –) \cong h_{F(X)}$. Then if the objects $F(X)$ can be made the values on objects of a functor F: $\mathbf{C} \to \mathbf{D}$, we can make the family of functors $A(X, –)$ into a bifunctor A: $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ in a unique way so that the isomorphisms $i(X)$ together give an isomorphism of bifunctors, as above.

Proof. In the situation of the first paragraph, note that since A: $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ is a bifunctor, the induced system of functors $A(X, –)$: $\mathbf{D} \to \mathbf{Set}$ will together constitute a single functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{D}}$ (Lemma 6.10.1); let us call this B. For each $X \in \mathrm{Ob}(\mathbf{C})$ we have an isomorphism $i(X)$ of $B(X)$ with a hom-functor $h_{F(X)}$, so by the preceding lemma we get an isomorphic functor C: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{D}}$, such that $C(X) = h_{F(X)}$, and the isomorphism i: $B \cong C$ is made up of the $i(X)$'s. Now by Corollary 7.2.6, the covariant hom-functors h_Y $(Y \in \mathrm{Ob}(\mathbf{D}))$ form a full subcategory of $\mathbf{Set}^{\mathbf{D}}$ contravariantly isomorphic to $\mathbf{D}$ via the Yoneda embedding $Y \mapsto h_Y$. Hence the contravariant functor C: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{D}}$ is induced by composing this embedding $\mathbf{D}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{D}}$ with a unique *covariant* functor F: $\mathbf{C} \to \mathbf{D}$, and this F is the functor of the statement of the lemma.

In the situation of the second paragraph, let us similarly consider each functor $A(X, –)$ as an object $B(X)$ of $\mathbf{Set}^{\mathbf{D}}$. Then for each X we have an isomorphism $i(X)$: $B(X) \cong h_{F(X)}$, and applying the preceding lemma to the isomorphisms $i(X)^{-1}$, we conclude that the objects $B(X)$ are the values of a functor B: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{D}}$, which we may regard as a bifunctor A: $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$, and again the values of i become an isomorphism of bifunctors. □

The above lemma concerns systems of objects representing covariant hom-functors; let us state the corresponding result for contravariant hom-functors. A priori, this means replacing $\mathbf{D}$ by $\mathbf{D}^{\mathrm{op}}$. But it is then natural to replace the ''parametrizing'' category $\mathbf{C}^{\mathrm{op}}$ by $\mathbf{C}$ so as to keep the parametrization of the constructed objects of $\mathbf{D}$ covariant. And having done that much, why not interchange the names of $\mathbf{C}$ and $\mathbf{D}$ so as to get an initial set-up formally identical to that of the preceding case? Doing so, we get

Lemma 7.2.10. *Let* $A: \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ *be a bifunctor, and suppose that for each* $Y \in \mathrm{Ob}(\mathbf{D})$ *the induced contravariant functor* $A(-, Y): \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *is representable, with a representing object* $U(Y) \in \mathrm{Ob}(\mathbf{C})$, *and isomorphism* $j(Y): A(-, Y) \cong h^{U(Y)}$. *Then* U *can be made a covariant functor* $\mathbf{D} \to \mathbf{C}$ *in a unique way so that the isomorphisms* $j(Y)$ *constitute an isomorphism of bifunctors*

$$j: A(-,-) \cong \mathbf{C}(-, U(-)).$$

Conversely, suppose we are given a family of contravariant set-valued functors $A(-, Y): \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, *one for each object* Y *of* $\mathbf{D}$, *each of which is representable, say with representing object* $U(Y)$ *and isomorphism* $j(Y): A(-, Y) \cong h^{U(Y)}$. *Then if the objects* $U(Y)$ *can be made the values on objects of a functor* $U: \mathbf{C} \to \mathbf{D}$, *we can make the family of functors* $A(-, Y)$ *into a bifunctor* $A: \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ *in a unique way so that the isomorphisms* $j(Y)$ *together give an isomorphism of bifunctors, as above.* □

7.3. Adjoint functors. Let us look at some examples of the situation of the two preceding lemmas – families of objects that we characterized individually as the representing objects for certain naturally occurring functors, but that turned out, themselves, to fit together into a functor. By those lemmas, this means that the system of functors that these objects represented fit together into a bifunctor. We shall see that in each of these cases, this structure of bifunctor was actually present in the original situation, providing an explanation of why our constructions yielded functors.

The free group on each set X is the object of $\mathbf{Group}$ representing the functor $G \mapsto |G|^X = \mathbf{Set}(X, U(G))$. So the free group *functor* arises by representing the family of functors $\mathbf{Group} \to \mathbf{Set}$ obtained by inserting all sets as the first argument of the *bifunctor*

$$\mathbf{Set}(-, U(-)): \mathbf{Set}^{\mathrm{op}} \times \mathbf{Group} \to \mathbf{Set}.$$

The analogous description obviously applies in any category $\mathbf{C}$ having free objects with respect to a concretization $U: \mathbf{C} \to \mathbf{Set}$.

If G is a group, the *abelianization* of G is the object of $\mathbf{Ab}$ representing the functor $\mathbf{Ab} \to \mathbf{Set}$ given by $A \mapsto \mathbf{Group}(G, A)$. The symbol $\mathbf{Group}(G, A)$ makes sense because $\mathbf{Ab}$ is a subcategory of $\mathbf{Group}$, but to put this example in the context of the general pattern, let us write V for the inclusion functor of $\mathbf{Ab}$ in $\mathbf{Group}$. We then see that the abelianization functor arises by representing the family of set-valued functors obtained by inserting values in the first argument of the bifunctor

$$\mathbf{Group}(-, V(-)): \mathbf{Group}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Set}.$$

In the same way, if W denotes the forgetful functor $\mathbf{Group} \to \mathbf{Monoid}$, then the functor taking a monoid to its universal enveloping group arises by representing the family of set-valued functors obtained by inserting values in the first argument of the bifunctor

$$\mathbf{Monoid}(-, W(-)): \mathbf{Monoid}^{\mathrm{op}} \times \mathbf{Group} \to \mathbf{Set}.$$

The above were ‘‘left universal’’ examples, that is, constructions $F: \mathbf{C} \to \mathbf{D}$ such that each object $F(X)$ represented a covariant functor $\mathbf{D} \to \mathbf{Set}$. We see that in each such case, the bifunctor from which these covariant functors were extracted had the form

$$\mathbf{C}(-,\ U(-))\colon\ \mathbf{C}^{\mathrm{op}} \times \mathbf{D}\ \to\ \mathbf{Set}, \tag{7.3.1}$$

for some functor $U\colon \mathbf{D} \to \mathbf{C}$. Taking (7.3.1) to be the A in the display of Lemma 7.2.9, we see that the universal property of F in terms of U can be formulated in each of these cases as

$$\mathbf{C}(-,\ U(-))\ \cong\ \mathbf{D}(F(-),\ -)$$

– a strikingly symmetrical condition!

Let us consider one ‘‘right universal’’ example. Given a monoid S, we have already considered the construction of the universal group G with a homomorphism of S into G_{md}; but there is also a universal group G with a homomorphism of G_{md} into S, namely the group $G = S_{\mathrm{inv}}$ of invertible elements (‘‘units’’) of S. If we write $F\colon$ **Group** → **Monoid** for the forgetful functor $G \mapsto G_{\mathrm{md}}$, and call the above group-of-units functor $U\colon$ **Monoid** → **Group**, we see that $U(S)$ represents the contravariant functor associating to each group G the set $\mathbf{Monoid}(F(G),\ S)$. If we write $\mathbf{C}$ and $\mathbf{D}$ for **Group** and **Monoid**, then on taking $\mathbf{D}(F(-),\ -)$ for the bifunctor A in the last formulation of Lemma 7.2.10, we get an isomorphism characterizing this right universal construction U:

$$\mathbf{D}(F(-),\ -)\ \cong\ \mathbf{C}(-,\ U(-)).$$

This is exactly the same as the isomorphism characterizing our examples of left universal constructions – except that the sides have been written in reverse order, and the relation is looked at as characterizing U in terms of F, rather than F in terms of U! The fact that for these two situations we got the same isomorphism, but with the roles of U and F reversed, means that a functor F gives objects representing the covariant functors $\mathbf{C}(X,\ U(-))$ *if and only if* U gives objects representing the contravariant functors $\mathbf{D}(F(-),\ Y)$.

Let us test this conclusion, by turning our characterization of the free group construction upside down. Since the free group $F(X)$ on a set X is left universal among groups G with set maps of X into their underlying sets $U(G)$, the *underlying set* $U(G)$ of a group G should be right-universal among all sets X with group homomorphisms from the free group $F(X)$ into G. And indeed, though it may seem bizarre to treat the free-group construction as given, and the underlying-set construction as something to be characterized, the universal property certainly holds: For any group G, $U(G)$ is a set with a homomorphism $u\colon F(U(G)) \to G$, such that given any homomorphism f from a free group $F(X)$ on any set into G, there is a unique set map $h\colon X \to U(G)$ (which you should be able to describe) such that $f = uF(h)$. This property of underlying sets is sometimes even useful. For instance, in showing that every group can be presented by generators and relations, one wishes to write an arbitrary group G as a homomorphic image of a free group on some set X. The above property says that there is a universal choice of such X, namely the underlying set $U(G)$ of G.

Before setting out to tie together all our ways of describing these universal constructions, let us prove a lemma that will allow us to relate isomorphisms of bifunctors as above and systems of maps $X \to U(F(X))$ and $F(U(Y)) \to Y$.

Lemma 7.3.2. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories and* $U\colon \mathbf{D} \to \mathbf{C}$, $F\colon \mathbf{C} \to \mathbf{D}$ *functors, and consider the two bifunctors* $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$,

$$\mathbf{C}(-,\ U(-)), \qquad \mathbf{D}(F(-),\ -).$$

Then a morphism of bifunctors

$$a\colon\ \mathbf{C}(-,\,U(-))\ \to\ \mathbf{D}(F(-),\,-)$$

is determined by its values on identity morphisms $\mathrm{id}_{U(D)}\in\mathbf{C}(U(D),U(D))$ $(D\in\mathrm{Ob}(\mathbf{D}))$. *In fact, given* a, *if we write* $\alpha(D) = a(U(D),D)(\mathrm{id}_{U(D)})\in\mathbf{D}(F(U(D)),D)$, *then* α *is a morphism* $FU\to\mathrm{Id}_{\mathbf{D}}$; *and this construction yields a bijection between morphisms* $a\colon\mathbf{C}(-,\,U(-))\to\mathbf{D}(F(-),\,-)$ *and morphisms* $\alpha\colon FU\to\mathrm{Id}_{\mathbf{D}}$. *Inversely, given* α, *the morphism* a *can be described as acting on* $f\in\mathbf{C}(C,\,U(D))$ *by first applying* F *to get* $F(f)\colon F(C)\to FU(D)$, *then composing this with* $\alpha(D)\colon FU(D)\to D$, *getting* $a(f)=\alpha(D)\,F(f)\colon F(C)\to D$.

Likewise, a morphism of bifunctors

$$b\colon\ \mathbf{D}(F(-),\,-)\ \to\ \mathbf{C}(-,\,U(-))$$

is determined by its values on identity morphisms, in this case morphisms $\mathrm{id}_{F(C)}\in\mathbf{D}(F(C),F(C))$ $(C\in\mathrm{Ob}(\mathbf{C}))$, *and writing* $\beta(C) = b(C,F(C))(\mathrm{id}_{F(C)})\in\mathbf{C}(C,U(F(C)))$, *we get a bijection between morphisms* $b\colon\mathbf{D}(F(-),\,-)\to\mathbf{C}(-,\,U(-))$ *and morphisms* $\beta\colon\mathrm{Id}_{\mathbf{C}}\to UF$. *Given* β, *the morphism* b *can be described as taking* $f\in\mathbf{D}(F(C),\,D)$ *to* $U(f)\beta(C)\in\mathbf{C}(C,\,U(D))$.

Sketch of Proof. Consider a morphism $a\colon\mathbf{C}(-,\,U(-))\to\mathbf{D}(F(-),\,-)$. For each $D\in\mathrm{Ob}(\mathbf{D})$ this gives a morphism of functors $\mathbf{C}(-,\,U(D))\to\mathbf{D}(F(-),\,D)$. Since the first of these functors is $h^{U(D)}$, the Yoneda Lemma says this morphism is determined by its value on the identity morphism of $U(D)$. It is straightforward to verify that the condition that these morphisms of functors $\mathbf{C}(-,\,U(D))\to\mathbf{D}(F(-),D)$ should comprise a single morphism of bifunctors a: $\mathbf{C}(-,\,U(-))\to\mathbf{D}(F(-),\,-)$ is equivalent to the condition that the values of these morphisms on identities should comprise a morphism of functors $\alpha\colon FU\to\mathrm{Id}_{\mathbf{D}}$. The reader can easily check that the description of how to recover a from α also leads to a morphism of functors, and that this construction is inverse to the first.

The second paragraph follows by duality. □

Exercise 7.3:1. Write out the ''straightforward'' verification and the ''easy check'' referred to in the proof of the lemma.

To get a feel for the above construction, you might start with the morphism of bifunctors a that associates to every set map from a set X to the underlying set $U(G)$ of a group G the induced group homomorphism from the free group $F(X)$ into G. Determine the morphism of functors α that the above construction yields, and check explicitly that the ''inverse'' construction described does indeed recover a. In this example, one finds that a is invertible; calling its inverse b, similarly work out for this b the constructions of the second assertion of the lemma.

We now give several descriptions of the type of universal construction discussed at the beginning of this section.

Theorem 7.3.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. Then the following data are equivalent:*

(i) *A pair of functors* $U\colon\mathbf{D}\to\mathbf{C}$, $F\colon\mathbf{C}\to\mathbf{D}$, *and an isomorphism*

$$i\colon\ \mathbf{C}(-,\,U(-))\ \cong\ \mathbf{D}(F(-),\,-)$$

of functors $\mathbf{C}^{\mathrm{op}}\times\mathbf{D}\to\mathbf{Set}$.

(ii) *A functor* $U\colon\mathbf{D}\to\mathbf{C}$, *and for every* $C\in\mathrm{Ob}(\mathbf{C})$, *an object* $R_C\in\mathrm{Ob}(\mathbf{D})$ *and an element* $u_C\in\mathbf{C}(C,\,U(R_C))$ *which are universal among such object-element pairs, i.e., which represent the* *covariant functor* $\mathbf{C}(C, U(–))\colon \mathbf{D} \to \mathbf{Set}$.

(ii*) *A functor* $F\colon \mathbf{C} \to \mathbf{D}$, *and for every* $D \in \mathrm{Ob}(\mathbf{D})$, *an object* $R_D \in \mathrm{Ob}(\mathbf{C})$ *and an element* $v_D \in \mathbf{D}(F(R_D), D)$ *which are universal among such object-element pairs, i.e., which represent the contravariant functor* $\mathbf{D}(F(–), D)\colon\ \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

(iii) *A pair of functors* $U\colon \mathbf{D} \to \mathbf{C}$, $F\colon \mathbf{C} \to \mathbf{D}$, *and a pair of morphisms of functors*

$$\eta\colon \mathrm{Id}_{\mathbf{C}} \to UF, \qquad \varepsilon\colon FU \to \mathrm{Id}_{\mathbf{D}},$$

such that the two composites

$$U \xrightarrow{\eta \circ U} UFU \xrightarrow{U \circ \varepsilon} U, \qquad F \xrightarrow{F \circ \eta} FUF \xrightarrow{\varepsilon \circ F} F,$$

are the identity morphisms of U *and* F *respectively.* (*For the ``*$\circ$*'' notation see Lemma 6.10.2.*)

Sketch of Proof. The equivalence of (i) with (ii) and (ii*) is given by Lemmas 7.2.9 and 7.2.10 respectively. By Lemma 7.3.2, an isomorphism of bifunctors as in (i) must correspond to a pair of morphisms of functors $\eta\colon \mathrm{Id}_{\mathbf{C}} \to UF$, $\varepsilon\colon FU \to \mathrm{Id}_{\mathbf{D}}$ which induce mutually inverse morphisms of bifunctors. It is straightforward to verify that the conditions needed for these induced morphisms to be mutually inverse are those shown diagrammatically in (iii). □

Exercise 7.3:2. (i) Write down the verification referred to in the last sentence of the above proof.

(ii) Show that η will be composed of the ``universal morphisms'' u_C associated with the left universal properties of the objects $F(C)$ stated in point (ii) of the theorem, and ε will be composed of the universal morphisms v_D associated with the right universal properties of the objects $U(D)$ stated in point (ii*).

(iii) Take one universal construction, e.g., that of free groups, write down the equalities expressed diagrammatically in part (iii) of the above theorem for this construction in terms of maps of set- and group-elements, and explain why they hold in *this case*.

Definition 7.3.4. *Given categories* $\mathbf{C}$ *and* $\mathbf{D}$ *and functors* $U\colon \mathbf{D} \to \mathbf{C}$, $F\colon \mathbf{C} \to \mathbf{D}$, *an isomorphism*

$$i\colon\ \mathbf{C}(–, U(–)) \cong \mathbf{D}(F(–), –)$$

of bifunctors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$, *or equivalently, a pair of morphisms of functors* ε, η *satisfying the condition of point* (iii) *of the above theorem, is called an* adjunction *between* U *and* F.

In this situation, U *is called the ``right adjoint'' of* $F$, *and* $F$ *the ``left adjoint'' of* U (*referring to their occurrence in the right and left slots of the hom-bifunctors in the above isomorphism*). *The morphisms of functors* η *and* ε *are called, respectively, the* unit *and* counit *of the adjunction.*

Historical note: The term ``adjoint'' was borrowed from analysis, where the adjoint of a bounded operator between Hilbert spaces, $A\colon X \to Y$, is the operator $B\colon Y \to X$ characterized by the condition on inner products $(x, By) = (Ax, y)$.

The student who finds condition (iii) of Theorem 7.3.3 hard to grasp will be happy to know that we will not make much use of it in the next few chapters. (I have trouble with it myself.) But we *will* use the morphisms η and ε named in that condition, so you should get clear on how these act. (What we will seldom use is the fact that the indicated compositional condition on a pair of morphisms η, ε is equivalent to their being the unit and counit of an adjunction. Nevertheless, I recommend working Exercise 7.3:2 this once.)

The terms ‘‘unit’’ and ‘‘counit’’ will be easier to explain when we consider the concepts of *monad* and *comonad* in Chapter 10 (not yet written).

We can now characterize many of the universal constructions we are familiar with as right or left adjoints. The three diagrams below show the cases we used to motivate the concept. In each of these, a pair of successive vertical arrows between two categories represents a pair of mutually adjoint functors, the right adjoint being shown on the right and the left adjoint on the left.

$$\begin{tikzcd}
\mathbf{Group} \arrow[d, shift left=2, "\text{under-lying set}"] \\
\mathbf{Set} \arrow[u, shift left=2, "\text{free group}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Group} \arrow[d, "\text{for-get-ful}"] \\
\mathbf{Monoid} \arrow[u, shift left=4, "S \mapsto S^{\mathrm{gp}}"] \arrow[u, shift right=4, "S \mapsto U(S)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Ab} \arrow[d, shift left=2, "\text{for-get-ful}"] \\
\mathbf{Group} \arrow[u, shift left=2, "\text{abel-ianiz-ation}"]
\end{tikzcd}$$

The middle diagram is interesting in that the forgetful functor there (in the notation of §3.11, $G \mapsto G_{\mathrm{md}}$) has both a left and a right adjoint. In the first diagram, we can replace **Group** with any category $\mathbf{C}$ having free objects with respect to a concretization U. A still wider generalization is noted in the next exercise.

Exercise 7.3:3. If you did not do Exercise 6.8:10, prove that if $\mathbf{C}$ is a category with small coproducts and $U: \mathbf{C} \to \mathbf{Set}$ a functor, then U has a left adjoint if and only if it is *representable*.

(Exercise 6.8:10 was essentially the case of this result where U was faithful, so that it could be called a ‘‘concretization’’ and its left adjoint could be called a ‘‘free object’’ construction; but faithfulness played no part in the proof. In Chapter 9 we shall extend the concept of ‘‘representable functor’’ from set-valued functors to algebra-valued functors, and generalize the above result to this much wider context.)

Exercise 7.3:4. Show that the left (or right) adjoint of a functor, if one exists, is unique up to canonical isomorphism, and conversely, that if A and B are isomorphic functors, then any functor which can be made a left (or right) adjoint of A can also be made a left (or right) adjoint of B.

Exercise 7.3:5. Show that if $A: \mathbf{C} \to \mathbf{D}$, $B: \mathbf{D} \to \mathbf{C}$ give an equivalence of categories, then B is both a right and a left adjoint to A.

The next exercise is a familiar example in disguise.

Exercise 7.3:6. Let $\mathbf{C}$ be the category with $\mathrm{Ob}(\mathbf{C}) = \mathrm{Ob}(\mathbf{Group})$, but such that for groups G and H, $\mathbf{C}(G,H) = \mathbf{Set}(|G|, |H|)$. Thus **Group** is a subcategory of $\mathbf{C}$, with the same object set but smaller morphism sets. Does the inclusion functor $\mathbf{Group} \to \mathbf{C}$ have a left and/or a right adjoint?

There are many other constructions whose universal properties translate into adjointness statements: The forgetful functor $\mathbf{Ring}^1 \to \mathbf{Monoid}$ that remembers only the multiplicative structure has as left adjoint the *monoid ring* construction. The forgetful functor $\mathbf{Ring}^1 \to \mathbf{Ab}$ that remembers only the additive structure has for left adjoint the *tensor ring* construction. (These two constructions were discussed briefly toward the end of §3.12.) The forgetful functor from compact Hausdorff spaces to arbitrary topological spaces has for left adjoint the Stone-Čech compactification functor (§3.17). The functor associating to every commutative ring its Boolean ring of idempotent elements has as left adjoint the construction asked for in Exercise 3.14:3(iv). The forgetful functors going from **Lattice** to ∨**-Semilattice** and ∧**-Semilattice**, and from these in turn to **POSet**, have left adjoints which you were asked to construct in Exercise 5.1:8.

The student familiar with Lie algebras will note that the functor associating to an associative algebra A the Lie algebra A_{Lie} with the same underlying vector space as A, and with the commutator operation of A for Lie bracket, has for left adjoint the *universal enveloping algebra* construction. (The Poincaré-Birkhoff-Witt Theorem gives a normal form for this universal object; I hope to treat such results in a much later chapter.)

Suppose $\mathbf{C}$ is a category having *products* and *coproducts* of all pairs of objects. We know that each of these constructions will give a functor $\mathbf{C}\times\mathbf{C} \to \mathbf{C}$. Can these functors be characterized as adjoints of some functors $\mathbf{C} \to \mathbf{C}\times\mathbf{C}$? Similarly, can the *tensor product* functor $\mathbf{Ab}\times\mathbf{Ab} \to \mathbf{Ab}$ be characterized as an adjoint of some functor $\mathbf{Ab} \to \mathbf{Ab}\times\mathbf{Ab}$?

The universal property of the product functor $\mathbf{C}\times\mathbf{C} \to \mathbf{C}$ is a right universal one, so if it arises as an adjoint, it should be a right adjoint to some functor $A\colon \mathbf{C} \to \mathbf{C}\times\mathbf{C}$. No such functor was evident in our definition of products. However, the product functor will be right adjoint to a functor F if and only if F is left adjoint to the product functor, so let us pose the universal problem whose solution should be such a *left* adjoint: Given $X\in\mathrm{Ob}(\mathbf{C})$, will there exist $(Y,Z)\in\mathrm{Ob}(\mathbf{C}\times\mathbf{C})$ with a universal example of a morphism $X \to Y\times Z$? Since a morphism $X \to Y\times Z$ corresponds to a morphism $X \to Y$ and a morphism $X \to Z$, this asks whether there exists a pair (Y, Z) of objects of $\mathbf{C}$ universal for having a morphism from X to each member of this pair. In fact, the pair (X, X) is easily seen to have the desired universal property. This leads us to define the ''diagonal functor'' $\Delta\colon \mathbf{C} \to \mathbf{C}\times\mathbf{C}$ taking each object X to (X,X), and each morphism f to (f, f). It is then easy to check that the universal property of the direct product construction is that of a right adjoint to Δ. Moreover, similar reasoning shows that the universal property of the coproduct is that of a left adjoint of Δ. So in a category $\mathbf{C}$ having both products and coproducts, we have the diagram of adjoint functors

$$\begin{tikzcd}
\mathbf{C} \arrow[d, "\Delta"] \\
\mathbf{C}\times\mathbf{C} \arrow[u, shift left=6, "\amalg"] \arrow[u, shift right=6, "\prod"']
\end{tikzcd} .$$

We recall that if $\mathbf{C}$ is $\mathbf{Ab}$, the constructions of pairwise products and coproducts (''direct products and direct sums'') coincide. So in that case we get a ''cyclic'' diagram of adjoints.

Exercise 7.3:7. Does the direct product construction on **Set** have a *right* adjoint? Does the coproduct construction have a *left* adjoint?

The next exercise is one of my favorites:

Exercise 7.3:8. Recall that $\mathbf{2}$ denotes the category with two objects, 0 and 1, and exactly one nonidentity morphism, $0 \to 1$, so that for any category $\mathbf{C}$, an object of $\mathbf{C}^{\mathbf{2}}$ corresponds to a choice of two objects $A_0, A_1 \in\mathrm{Ob}(\mathbf{C})$ and a morphism $f\colon A_0 \to A_1$.

Let $p_0\colon \mathbf{Group}^{\mathbf{2}} \to \mathbf{Group}$ denote the functor taking each object (A_0, A_1, f) to its first component A_0, and likewise every morphism $(a_0, a_1)\colon (A_0, A_1, f) \to (B_0, B_1, g)$ of $\mathbf{Group}^{\mathbf{2}}$ to its first component a_0.

Investigate whether p_0 has a left adjoint, and whether it has a right adjoint. If a left adjoint is found, investigate whether this in turn has a left adjoint (clearly it has a right adjoint – namely

p_0); likewise if p_0 has a right adjoint, investigate whether this in turn has a right adjoint; and so on, as long as further adjoints can be found.

Exercise 7.3:9. Let G be a group, and G-**Set** the category of all G-sets.

You can probably think of several very easily described functors from **Set** to G-**Set**, or vice versa. Choose one of them, and apply the idea of the preceding exercise; i.e., look for a left adjoint and/or a right adjoint, and for further adjoints of these, as long as you can find any.

When you are finished, does the chain of functors you have gotten contain all the ‘‘easily described functors’’ between these two categories that you were able to think of? If not, take one that was missed, and do the same with it.

Exercise 7.3:10. Translate the idea indicated in observation (a) following Exercise 3.8:1 into questions of the existence of adjoints to certain functors between categories G_1-**Set** and G_2-**Set**, determine whether these adjoints do in fact exist, and describe them as well as you can, if they do.

Let us now consider the case of the tensor product construction, $\otimes: \mathbf{Ab}\times\mathbf{Ab}\to\mathbf{Ab}$. It is the solution to a left universal problem, and we can characterize this problem as arising, in the sense of Lemma 7.2.9, from the bifunctor Bil: $(\mathbf{Ab}\times\mathbf{Ab})^{\mathrm{op}}\times\mathbf{Ab} \to \mathbf{Ab}$, where for abelian groups A, B, C we let $\mathrm{Bil}((A,B),C)$ denote the set of bilinear maps $(A,B)\to C$. From the preceding examples, we might expect $\mathrm{Bil}((A,B),\,C)$ to be expressible in the form $(\mathbf{Ab}\times\mathbf{Ab})((A,B),\,U(C))$ for some functor $U: \mathbf{Ab}\to\mathbf{Ab}\times\mathbf{Ab}$.

But, in fact, it cannot be so expressed; in other words, the tensor product construction $\mathbf{Ab}\times\mathbf{Ab}\to\mathbf{Ab}$, though it is a left universal construction, is *not* a left adjoint. The details (and a different sense in which the tensor product *is* a left adjoint functor construction) are something you can work out:

Exercise 7.3:11. (i) Show that the functor $\otimes: \mathbf{Ab}\times\mathbf{Ab}\to\mathbf{Ab}$ has no left or right adjoint.

(ii) On the other hand, show that for any fixed abelian group A, the functor $A\otimes -$: $\mathbf{Ab}\to\mathbf{Ab}$ is left adjoint to the functor $\mathrm{Hom}(A,\,-)$: $\mathbf{Ab}\to\mathbf{Ab}$. (I am writing $\mathrm{Hom}(A,B)$ for the *abelian group* of homomorphisms from A to B, in contrast to $\mathbf{Ab}(A,B)$ the *set* of such homomorphisms – an admittedly arbitrary and ad hoc notational choice.)

(iii) Investigate whether the functor $A\otimes -$: $\mathbf{Ab}\to\mathbf{Ab}$ has a left adjoint, and whether $\mathrm{Hom}(A,\,-)$: $\mathbf{Ab}\to\mathbf{Ab}$ has a right adjoint. If such adjoints do not *always* exist, do they exist for *some* choices of A?

If you are familiar enough with ring theory, generalize the above problems to modules over a fixed commutative ring k, or to bimodules over pairs of noncommutative rings.

Exercise 7.3:12. For a fixed set A, does the functor $\mathbf{Set}\to\mathbf{Set}$ given by $S\mapsto S\times A$ have a left or right adjoint?

A situation which is similar, in that the question of whether a construction is an adjoint depends on what we take as the variable, is considered in

Exercise 7.3:13. In this exercise ‘‘ring’’ will mean commutative ring with 1; recall that we denote the category of such rings $\mathbf{CommRing}^1$.

If R is a ring and X any set, $R[X]$ will denote the polynomial ring over R in an X-tuple of indeterminates.

(i) Show that for X a nonempty set, the functor P_X: $\mathbf{CommRing}^1\to\mathbf{CommRing}^1$ taking each ring R to $R[X]$ has neither a right nor a left adjoint, and similarly that for R a ring, the functor Q_R: $\mathbf{Set}\to\mathbf{CommRing}^1$ taking each set X to $R[X]$ has neither a right nor a left adjoint.

(ii) On the other hand, show that the functor $\mathbf{CommRing}^1\times\mathbf{Set}\to\mathbf{CommRing}^1$ taking a pair (R,X) to $R[X]$ is an adjoint (on the appropriate side) of an easily described functor.

(iii) For any ring R, let $\mathbf{CommRing}^1_R$ denote the category of commutative R-algebras (rings S given with homomorphisms $R \to S$), and R-algebra homomorphisms (ring homomorphisms making commuting triangles with R. In the notation of the paragraph following Exercise 6.8:24, this is the comma category $(R \downarrow \mathbf{CommRing}^1)$.)

Similarly, for any set X, let $\mathbf{CommRing}^1_X$ denote the category of rings S given with set maps $X \to |S|$, and again having for morphisms the ring homomorphisms making commuting triangles. (This is the comma category $(X \downarrow U)$, where U is the underlying set functor of $\mathbf{CommRing}^1$. Incidentally, note that to keep our names for these categories unambiguous, we must remember to use distinct symbols for rings and sets.)

Show that for any R, the functor $\mathbf{Set} \to \mathbf{CommRing}^1_R$ taking X to $R[X]$ can be characterized as an adjoint, and that for any X, the functor $\mathbf{CommRing}^1 \to \mathbf{CommRing}^1_X$ taking R to $R[X]$ can also be characterized as an adjoint.

(iv) Investigate similar questions for the formal power series construction, $R[[X]]$; in particular, whether the analog of (i) is true.

Here is still another way to make the tensor product construction into an adjoint functor:

Exercise 7.3:14. (i) Let **Bil** be the category whose objects are all 4-tuples (A, B, β, C) where A, B, C are abelian groups, and $\beta\colon (A, B) \to C$ is a bilinear map, and with morphisms defined in the natural way. (Say what this natural way is!) Show that the forgetful functor $\mathbf{Bil} \to \mathbf{Ab} \times \mathbf{Ab}$, taking each such 4-tuple to its first two components, has a left adjoint, which is ‘‘essentially’’ the tensor-product construction.

(ii) Show that an analogous trick can be used to convert any isomorphism of bifunctors as in the Lemma 7.2.9 into an adjunction. (Between what categories?) Do the same for the situation of Lemma 7.2.10.

Exercise 7.3:15. Describe all pairs of adjoint functors at least one member of which is a *constant* functor, i.e., takes all objects of its domain category to a single object X of its codomain category, and all morphisms of its domain category to id_X.

What happens when we compose two functors arising from adjunctions?

Note that the *abelianization* of the *free group* on a set X is a *free abelian group* on X. That is, when we compose these two functors, each of which is a left adjoint, we get another functor with that property. The general statement is delightfully easy to prove.

Theorem 7.3.5. *Suppose* $\begin{tikzcd}\mathbf{E} \arrow[r, shift left, "U"] & \mathbf{D} \arrow[l, shift left, "F"] \arrow[r, shift left, "V"] & \mathbf{C} \arrow[l, shift left, "G"]\end{tikzcd}$ *are pairs of adjoint functors, with* U *and* V *the right adjoints,* F *and* G *the left adjoints. Then* $\begin{tikzcd}\mathbf{E} \arrow[r, shift left, "VU"] & \mathbf{C} \arrow[l, shift left, "FG"]\end{tikzcd}$ *are also adjoint, with* VU *the right adjoint and* FG *the left adjoint.*

Proof. $\mathbf{C}(-, VU(-)) \cong \mathbf{D}(G(-), U(-)) \cong \mathbf{E}(FG(-), -)$. □

Exercise 7.3:16. Suppose U, V, F and G are as above, η and ε are the unit and counit of the adjoint pair U, F, and η' and ε' are the unit and counit of the adjoint pair V, G. Describe the unit and counit of the adjoint pair VU, FG.

For further examples of the above theorem, consider two ways we can factor the forgetful functor from $\mathbf{Ring}^1$ to $\mathbf{Set}$. We can first pass from a ring to its multiplicative monoid, then go to the underlying set thereof, or we can first pass from the ring to its additive group, and then to the underlying set:

$$\begin{tikzcd}
\mathbf{Ring}^1 \arrow[r, "\times"] \arrow[d, "+"'] \arrow[dr] & \mathbf{Monoid} \arrow[d] \\
\mathbf{Ab} \arrow[r] & \mathbf{Set}
\end{tikzcd}$$

Taking left adjoints, we get the two decompositions of the *free ring* construction noted in §3.12: as the free-monoid functor followed by the monoid-ring functor, and as the free abelian group functor followed by the tensor algebra functor:

$$\begin{tikzcd}
\mathbf{Ring}^1 & \mathbf{Monoid} \arrow[l] \\
\mathbf{Ab} \arrow[u] & \mathbf{Set} \arrow[l] \arrow[u] \arrow[ul]
\end{tikzcd}$$

7.4. Number-theoretic interlude: the *p*-adic numbers, and related constructions. While you digest the concept of adjunction (fundamentally simple, yet daunting in its multiple facets), let us look at some constructions of a different sort, which we have not studied so far. In this section we will develop a particular case important in number theory; the general category-theoretic concept will be defined in the next section. A much broader generalization, which also embraces several constructions we *have* studied, will be developed in the section after that.

Suppose we are interested in solving the equation

$$x^2 = -1 \qquad (7.4.1)$$

in the integers, $\mathbf{Z}$. Of course, we know it has no solution in the real numbers, let alone the integers, but we will ignore that dreary fact for the moment.

We may observe that the above equation has a solution in the finite ring $\mathbf{Z}_5$; in fact, two solutions, 2 and 3. Up to sign, these are the same, so let us look for a solution of (7.4.1) in $\mathbf{Z}$ satisfying

$$x \equiv 2 \pmod 5.$$

An integer x which is $\equiv 2 \pmod 5$ has the form $5y+2$, so we rewrite (7.4.1) as

$$(5y+2)^2 = -1$$

and expand. We get $25y^2+20y = -5$. Hence $20y \equiv -5 \pmod{25}$, and dividing by 5 we get $4y \equiv -1 \pmod 5$. This has the unique solution

$$y \equiv 1 \pmod 5,$$

which, substituted back, determines x modulo 25:

$$x = 5y+2 \equiv 5\cdot 1+2 = 7 \pmod{25}.$$

We continue in the same fashion: At the next stage, putting $x = 25z+7$ we have

$(25z+7)^2 = -1$. You should verify that this implies

$$z \equiv 2 \pmod 5,$$

which leads to

$$x \equiv 57 \pmod{125}.$$

Can we go on indefinitely? This is answered in

Exercise 7.4:1. (i) Show that given $i>0$, and $c\in\mathbf{Z}$ such that $c^2 \equiv -1 \pmod{5^i}$, there exists $c'\in\mathbf{Z}$ such that $c'^2 \equiv -1 \pmod{5^{i+1}}$, and $c' \equiv c \pmod{5^i}$.
(ii) Show that any integer is uniquely determined by its residues modulo 5, 5^2, 5^3, ... , 5^i, ...

Part (ii) of the above exercise shows that if there *were* an integer satisfying (7.4.1), the sequence of residues arising by repeated application of part (i) would determine it. But now let us return to our senses, and remember that (7.4.1) has no real solution, and ask what, if anything, we *have* found.

Clearly, we have shown that there exists a sequence of residues, $x_1\in|\mathbf{Z}_5|$, $x_2\in|\mathbf{Z}_{5^2}|$, ... , $x_i\in|\mathbf{Z}_{5^i}|$, ... , each of which satisfies (7.4.1) in the appropriate ring, and which are ''consistent'', in the sense that each x_{i+1} is a ''lifting'' of x_i, under the series of natural ring homomorphisms

$$\ldots \to \mathbf{Z}_{5^{i+1}} \to \mathbf{Z}_{5^i} \to \ldots \to \mathbf{Z}_{5^2} \to \mathbf{Z}_5\,.$$

Let us name the ith homomorphism in the above sequence $f_i: \mathbf{Z}_{5^{i+1}} \to \mathbf{Z}_{5^i}$; thus, f_i takes the residue of any integer n modulo 5^{i+1} to the residue of n modulo 5^i. Now note that the set of all strings

(7.4.2) $(\ldots, x_i, \ldots, x_2, x_1)$ such that $x_i\in|\mathbf{Z}_{5^i}|$ and $f_i(x_{i+1}) = x_i$ $(i = 1, 2, \ldots)$

forms a ring under componentwise operations. What we have shown is that *this ring* contains a square root of -1. (''If the fool would persist in his folly, he would become wise,'' William Blake [**44**].) Since, as we have noted, an integer n is determined by its residues modulo the powers of 5, the ring $\mathbf{Z}$ is *embedded* in this ring, though the square root, in this ring, of $-1\in|\mathbf{Z}|$ is of course *not* in the subring $\mathbf{Z}$.

The ring of sequences (7.4.2) is called the *ring of* 5*-adic integers*. The corresponding object constructed for any prime p, using the system of maps

(7.4.3) $$\ldots \to \mathbf{Z}_{p^{i+1}} \to \mathbf{Z}_{p^i} \to \ldots \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p,$$

is called the ring of p*-adic* integers, and these rings are of fundamental importance in modern number theory, and come up in many other areas as well. The notation for them is not uniform; the symbol we will use here is $\hat{\mathbf{Z}}_{(p)}$. (The (p) in parenthesis denotes the *ideal* of the ring $\mathbf{Z}$ generated by the element p. What is meant by putting it as a subscript of $\mathbf{Z}$ and adding a hat will be seen a little later. Many number-theorists simply write $\mathbf{Z}_p$ for the p-adic integers, denoting the field of p elements by $\mathbf{Z}/p\mathbf{Z}$ or $\mathbf{F}_p$; cf. [**21**, p.272], [**28**, p. 162, Example].)

The construction of this ring is in some ways analogous to the construction of the real numbers from the rationals. Real numbers are entities that can be approximated by rational numbers under the *distance* metric; p-adic integers are entities that can be approximated by integers via *congruences* modulo arbitrarily high powers of p. This analogy is made stronger in

Exercise 7.4:2. Let p be a fixed prime number. If n is any integer, let $v_p(n)$ denote the greatest integer e such that p^e divides n, or the symbol $+\infty$ if $n = 0$. The *p-adic metric* on $\mathbf{Z}$ is defined by $d_p(m, n) = p^{-v_p(m-n)}$. Thus, it makes m and n ''close'' if they are congruent modulo a high power of p.

(i) Verify that d_p is a metric on $\mathbf{Z}$, and that the ring operations are continuous in this metric, and deduce that the *completion* of $\mathbf{Z}$ with respect to this metric (the set of Cauchy sequences modulo the usual equivalence relation) can be made a ring containing $\mathbf{Z}$.

(ii) Show that this completion is isomorphic to $\hat{\mathbf{Z}}_{(p)}$.

(iii) Show that every element x of this completion has a unique ''left-facing base-p expression'' $x = \Sigma_{0 \leq i < \infty} c_i p^i$, where each $c_i \in \{0, 1, \ldots, p-1\}$, indicating why this infinite sum is convergent in the p-adic metric. What is the expression for -1 in this form?

We showed above that one could find a solution to the equation $x^2 = -1$ in $\hat{\mathbf{Z}}_{(5)}$. Let us note some simpler equations one can solve:

Exercise 7.4:3. (i) Show that every integer n not divisible by p is invertible in $\hat{\mathbf{Z}}_{(p)}$.

(ii) Will the ''base-p expressions'' (in the sense of the preceding exercise) for the elements n^{-1} be eventually periodic?

It follows from point (i) of the above exercise that we can embed into the p-adic integers not only $\mathbf{Z}$, but the subring of $\mathbf{Q}$ consisting of all fractions with denominators not divisible by p. Now when one adjoins to a commutative ring R inverses of all elements not lying in some prime ideal P, the resulting ring (which, if R is an integral domain, is a subring of the field of fractions of R) is denoted R_P, so what we have embedded in the p-adic integers is the ring $\mathbf{Z}_{(p)}$. In $\mathbf{Z}_{(p)}$, every nonzero element is clearly an invertible element times a power of p, from which it follows that the nonzero ideals are precisely the ideals (p^i). It is easy to verify that the factor-ring $\mathbf{Z}_{(p)}/(p^i)$ is isomorphic to $\mathbf{Z}_{p^i}$; hence the system of finite rings and homomorphisms (7.4.3) can be described as consisting of all the *proper* factor-rings of $\mathbf{Z}_{(p)}$, together with the canonical maps among them. Hence the p-adic integers can be thought of as elements which can be approximated by members of $\mathbf{Z}_{(p)}$ modulo all *nonzero ideals* of that ring. Ring-theorists call the ring of such elements the *completion* of $\mathbf{Z}_{(p)}$ with respect to the system of its nonzero ideals. This is the explanation for the symbol $\hat{\mathbf{Z}}_{(p)}$.

We will not go into a detailed study of what algebraic equations have solutions in the ring of p-adic integers. A general result applicable to a large class of rings including the p-adics is *Hensel's Lemma*; see [**21**, Theorem 8.5.6] or [**19**, §III.4.3] for the statement.

Let us characterize abstractly the relation between the diagram (7.4.3) and the ring of p-adic integers which we have constructed from it. Since a p-adic integer is by definition a sequence $(\ldots, x_i, \ldots, x_2, x_1)$ with each $x_i \in \mathbf{Z}_{p^i}$, the ring of p-adic integers has *projection* homomorphisms p_i onto each ring $\mathbf{Z}_{p^i}$. (Apologies for the double use of the letter ''p''!) Since the components x_i of each element satisfy the compatibility conditions $f_i(x_{i+1}) = x_i$, these projection maps satisfy

$$f_i\, p_{i+1} = p_i,$$

i.e., they make a commuting diagram

(7.4.4)
$$\begin{tikzcd}
 & \widehat{\mathbf{Z}}_{(p)} \arrow[d] \arrow[dr] \arrow[drrr] \arrow[drrrr] & & & & \\
\dots \arrow[r] & \mathbf{Z}_{p^{i+1}} \arrow[r] & \mathbf{Z}_{p^i} \arrow[r] & \dots \arrow[r] & \mathbf{Z}_{p^2} \arrow[r] & \mathbf{Z}_p .
\end{tikzcd}$$

Moreover, $\widehat{\mathbf{Z}}_{(p)}$ will be universal for these properties: Given any ring R with homomorphisms $r_i : R \to \mathbf{Z}_{p^i}$ which are ''compatible'', i.e., satisfy $f_i r_{i+1} = r_i$, we see that for any $a \in R$, the system of images $(\dots, r_i(a), \dots, r_2(a), r_1(a))$ defines an element $r(a) \in \widehat{\mathbf{Z}}_{(p)}$. The resulting map $r: R \to \widehat{\mathbf{Z}}_{(p)}$ will be a homomorphism such that $r_i = p_i r$ for all i, and uniquely determined by these equations. This universal property is expressed by saying that $\widehat{\mathbf{Z}}_{(p)}$ is the *inverse limit* of the system (7.4.3); one writes

$$\widehat{\mathbf{Z}}_{(p)} = \varprojlim_i \mathbf{Z}_{p^i} .$$

We will give the formal definition of this concept in the next section.

A very similar example of an inverse limit is that of the system

(7.4.5) $$\dots \to k[x]/(x^{i+1}) \to k[x]/(x^i) \to \dots \to k[x]/(x^2) \to k[x]/(x) ,$$

where $k[x]$ is the ring of polynomials in x over a field k, and (x^i) the ideal of all multiples of x^i. A member of $k[x]/(x^i)$ can be thought of as a polynomial in x specified modulo terms of degree $\geq i$. If we take a sequence of such partially specified polynomials, each extending the next, these determine a *formal power series* in x. So the inverse limit of the above system is the formal power series ring $k[[x]]$. This ring is well known as a place where one can solve various sorts of equations. Some of these results are instances of Hensel's Lemma, referred to above; others, such as the existence of formal-power-series solutions to appropriate sorts of differential equations, fall outside the scope of that lemma.

We constructed the p-adic integers using the canonical *surjections* $\mathbf{Z}_{p^{i+1}} \to \mathbf{Z}_{p^i}$. Now there are also canonical *embeddings* $\mathbf{Z}_{p^i} \to \mathbf{Z}_{p^{i+1}}$, sending the residue of n modulo p^i to the residue of pn modulo p^{i+1}. These respect addition but not multiplication, i.e., they are homomorphisms of abelian groups but not of rings. If we write out this system of groups and embeddings,

(7.4.6) $$\mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \dots \to \mathbf{Z}_{p^i} \to \mathbf{Z}_{p^{i+1}} \to \dots$$

it is natural to think of each group as a subgroup of the next, and to try to take their ''union'' G. But they are not literally subgroups of one another, so we need to think further about what we want this G to be.

Clearly, for every element x of each group in the above system, we want there to be an element of G representing the image of x. Furthermore, if an element x of one of the above groups is mapped to an element y of another by some composite of our maps, then these two elements should have the same image in G. Hence to get our G, let us form a disjoint union of the underlying sets of the given groups, and divide out by the equivalence relation that equates two elements if the image of one under a composite of the given maps is the other. It is straightforward to verify that this *is* an equivalence relation on the disjoint union, and that because the maps in the above diagram are group homomorphisms, the quotient by this relation inherits a group structure. If we call the maps in (7.4.6) $e_i : \mathbf{Z}_{p^i} \to \mathbf{Z}_{p^{i+1}}$, and the maps to the group we have constructed $q_i : \mathbf{Z}_{p^i} \to G$, then the identifications we have made have the effect that for each i,

$$q_{i+1} e_i \ = \ q_i ,$$

i.e., that the diagram

(7.4.7)
$$\begin{tikzcd}
\mathbf{Z}_p \arrow[r] \arrow[ddrrrrr] & \mathbf{Z}_{p^2} \arrow[r] \arrow[ddrrrr] & \dots \arrow[r] & \mathbf{Z}_{p^i} \arrow[r] \arrow[ddrr] & \mathbf{Z}_{p^{i+1}} \arrow[r] \arrow[ddr] & \dots \\
 & & & \dots & & \dots \\
 & & & & & G
\end{tikzcd}$$

commutes. Since we have made *only* these identifications, G will have the universal property that given any group H and family of homomorphisms $r_i : \mathbf{Z}_{p^i} \to H$ satisfying $r_{i+1} e_i = r_i$ for each i, there will exist a unique homomorphism $r: G \to H$ such that $r_i = r q_i$ for all i. This universal property is expressed by saying that the group G is the *direct* limit, $\varinjlim_i \mathbf{Z}_{p^i}$, of the given system of groups.

Group theorists denote the direct limit G of the system (7.4.6) by the suggestive symbol $\mathbf{Z}_{p^\infty}$.

Exercise 7.4:4. (i) Show that $\mathbf{Z}_{p^\infty}$ is isomorphic to the subgroup of $\mathbf{Q}/\mathbf{Z}$ generated by the elements p^{-1}, p^{-2}, ...

(ii) Show that the ring of endomorphisms of the abelian group $\mathbf{Z}_{p^\infty}$ is isomorphic to $\hat{\mathbf{Z}}_{(p)}$.

Exercise 7.4:5. Let us call an element x of a group G *completely divisible* if for every positive integer n there is a $y \in |G|$ such that $y^n = x$ (or if G is written additively, $ny = x$).

(i) Show that no nonzero element of the additive group of $\hat{\mathbf{Z}}_{(p)}$ is completely divisible.

On the other hand

(ii) Show that if A is any nonzero subgroup of $\hat{\mathbf{Z}}_{(p)}$ such that $\hat{\mathbf{Z}}_{(p)}/A$ is torsion-free, then *every* element of $\hat{\mathbf{Z}}_{(p)}/A$ is completely divisible; in fact, that $\hat{\mathbf{Z}}_{(p)}/A$ is the underlying additive group of a $\mathbf{Q}$-vector-space.

7.5. Direct and inverse limits. Before we give formal definitions of our two types of limits, let us give an example showing that one may want to consider limits of systems indexed by more general partially ordered sets than the natural numbers. Consider the concept of a *germ of a function* at a point z of the complex plane or any other topological space X. This arises by considering, for every neighborhood S of z, the set $F(S)$ of functions of the desired sort on the set S (for instance, analytic functions if X is the complex plane), and observing that when one goes from a neighborhood S to a smaller neighborhood T, one gets a restriction map $F(S) \to F(T)$ (not in general one-to-one, since distinct functions on the set S may have the same restriction to the subset T, and not necessarily onto, since not every admissible function on T extends to S). To get germs of functions at z, one intuitively wants to ‘‘follow’’ this system of sets of functions over smaller and smaller neighborhoods of z, and ‘‘take the limit’’. To do this formally, one takes a disjoint union of all the sets $F(S)$, and divides out by the equivalence relation that makes two functions $a \in F(S_1)$, $b \in F(S_2)$ equivalent if and only if they have the same image in $F(T)$ for some neighborhood of z, $T \subseteq S_1 \cap S_2$.

If the sets of functions $F(S)$ are given with some algebraic structure (structures of groups, rings, etc.) for which the above restriction maps are homomorphisms, we find that an algebraic structure of the same sort is induced on the direct limit set. The key point is that given functions a, b defined on different neighborhoods S and T of z, both will have images in the neighborhood $S \cap T$ of z, and these images can be added, multiplied, etc. there, allowing us to define the sum, product, etc., of the images of a and b in the limit set.

If we look for the conditions on a general partially ordered index set that allow us to reason in this way, we get

Definition 7.5.1. *Let P be a partially ordered set.*

P is said to be directed (*or* upward *directed*) *if for any two elements x, y of P, there exists an element z majorizing both x and y.*

P is said to be inversely directed (*or* downward *directed*) *if for any two elements x, y of P, there exists an element $z \leq$ both x and y; equivalently, if P^{op} is directed.*

(*The word ''filtered'' is sometimes used instead of ''directed'' in these definitions.*)

(If you did Exercise 5.2:9, you will find that these conditions are two of the nine ''interpolation'' properties of that exercise.)

We can now give the general definitions of direct and inverse limits. The formulations we give below assume that the morphisms of our given systems go in the ''upward'' direction with respect to the ordering on the indexing set. It happens that in our initial example of $\hat{\mathbf{Z}}_{(p)}$, the standard ordering on the positive integers is such that the morphisms went the *opposite* way; in our construction of $\mathbf{Z}_{p^\infty}$ they went the ''right'' way; while in the case of germs of analytic functions, if one orders neighborhoods of z by inclusion, the morphisms again go the ''wrong'' way (namely, from the set of functions on a larger neighborhood to the set of functions on a smaller neighborhood). This can be corrected formally by using, when necessary, the opposite partial ordering on the index set. Informally, in discussing direct and inverse limits one often just specifies the system of *objects and maps*, and understands that for application of the formal definition, the set indexing the objects should be partially ordered so as to make maps among them go ''upward''.

Definition 7.5.2. *Let $\mathbf{C}$ be a category, and suppose we are given a family of objects $X_i \in \mathrm{Ob}(\mathbf{C})$ $(i \in I)$, a partial ordering on the index set I, and a system of morphisms, $f_{ij} \in \mathbf{C}(X_i, X_j)$ $(i < j,\ i,j \in I)$ such that for $i < j < k$, one has $f_{jk} f_{ij} = f_{ik}$. (In brief, suppose we are given a partially ordered set I, and a functor $F\colon I_{\mathbf{cat}} \to \mathbf{C}$.)*

If I is inversely directed, then $(X_i, f_{ij})_I$ is called an inversely directed system *of objects and maps in $\mathbf{C}$. An* inverse limit *of this system means an object L given with morphisms $p_i\colon L \to X_i$ which are compatible, in the sense that for all $i < j \in I$, $p_j = f_{ij}\, p_i$, and which is* universal *for this property, in the sense that given any object W and morphisms $w_i\colon W \to X_i$ such that for all $i < j \in I$, $w_j = f_{ij}\, w_i$, there exists a unique morphism $w\colon W \to L$ such that $w_i = p_i w$ for all $i \in I$.*

Likewise, if I is directed, then $(X_i, f_{ij})_I$ is called a directed system *in $\mathbf{C}$; and a* direct limit *of this system means an object L given with morphisms $q_i\colon X_i \to L$ such that for all $i < j \in I$, $q_i = q_j f_{ij}$, and which is universal in the sense that given any object Y and morphisms $y_i\colon X_i \to Y$ such that for all $i < j \in I$, $y_i = y_j f_{ij}$, there exists a unique morphism $y\colon L \to Y$ such that $y_i = y q_i$ for all $i \in I$.*

(*Synonyms sometimes used for inverse and direct limit are* projective *and* inductive *limit respectively.*)

Loosely, one often writes the inverse limit object $\varprojlim_i X_i$, and the direct limit object $\varinjlim_i X_i$. More precisely, letting F denote the functor $I_{\mathbf{cat}} \to \mathbf{C}$ corresponding to the inversely directed or directed system (X_i, f_{ij}), one writes these objects as $\varprojlim F$ and $\varinjlim F$ respectively.

The morphisms $p_j\colon \varprojlim_i X_i \to X_j$ *are called the* projection *maps associated with this inverse limit, and the* $q_j\colon X_j \to \varinjlim_i X_i$ *the* coprojection *maps associated with the direct limit.*

In the above definition, by the ''functor corresponding to the system (X_i, f_{ij})'' we understand the functor which takes on the value X_i at the object i, the value f_{ij} at the morphism (i,j) ($i<j$ in I), and the value id_{X_i} at the morphism (i,i). Note that in the case where the indexing partially ordered set consists of the positive or negative integers, the full system of morphisms is determined by the morphisms $f_{i,\,i+1}$, hence in such cases one generally specifies only these morphisms in the description of the system.

One may ask what the point is, in the above definitions, of the restriction that the partially ordered set I be directed or inversely directed. One can set up the definitions without that restriction, and in most natural cases one can, in fact, construct objects which satisfy the resulting condition. But the behavior of these constructions tends to be quite different from those we have discussed, unless these directedness assumptions are made. In any case, such a generalized definition would be subsumed by a still more general definition to be made in the next section! So the value of the definition in the form given above is that it singles out a situation in which the limit objects can be studied by certain techniques.

Exercise 7.5:1. Let (X_i, f_{ij}) be a directed system in a category $\mathbf{C}$, and J a subset of I.

(i) Show that if J is *cofinal* in I, then $\varinjlim_J X_j \cong \varinjlim_I X_i$; precisely, that any object with the universal property of the direct limit of the first system can be made into a direct limit of the second in a natural way, and vice versa.

(ii) Show that the isomorphism of (i) is an instance of a morphism (in one direction or the other) between $\varinjlim_J X_j$ and $\varinjlim_I X_i$ which can be defined whenever both limits exist, whether J is cofinal or not.

(iii) State the result corresponding to (i) for inverse limits. (For this we need a term for a subset of a partially ordered set which has the property of being cofinal under the opposite ordering; let us use ''downward cofinal''. When speaking of inverse systems, one also sometimes just says ''cofinal'', with the understanding that this is meant in the only sense that is relevant to such systems.)

(iv) What can you deduce from (i) and (iii) about direct limits over directed partially ordered sets having a greatest element, and inverse limits over inversely directed partially ordered sets having a least element?

(v) Given any directed partially ordered set I and any *non*cofinal subset J of I, show that there exists a directed system of sets, (X_i, f_{ij}), indexed by I, such that $\varinjlim_I X_i \ncong \varinjlim_J X_j$.

The next few exercises concern direct and inverse limits of *sets*. The direct limit of a directed system (X_i, f_{ij}) of sets and set maps may be constructed in the manner indicated in the preceding section, by forming the disjoint union of the X_i and dividing out by the relation that makes $x \in X_i$ and $x' \in X_{i'}$ equivalent if they have the same image in some X_j $(j > i, i')$. The inverse limit of an inversely directed system of sets and set maps (X_i, f_{ij}) can likewise be constructed as we constructed the p-adic integers:

$$(7.5.3)\qquad \begin{array}{l} \varprojlim X_i \;=\; \{(x_i)\in \prod_I X_i \mid x_j = f_{ij}(x_i) \ \text{ for } \ i<j\in I\}, \ \text{ with} \\ \text{the } \ p_j \ \text{ given by projection maps, } \ \varprojlim X_i \subseteq \prod X_i \to X_j . \end{array}$$

We shall show in the next chapter that direct and inverse limits of algebraic objects have as their underlying sets the direct or inverse limits of the objects' underlying sets. Hence the results obtained in the exercises below on limits of sets will be applicable to algebras. (The above claim about underlying sets of *direct* limits of algebras will require that the algebras have only finitary operations.)

The construction of the p-adic integers was based on a system of *surjective* homomorphisms. The first point of the next exercise looks at inverse systems with the opposite property, and the second considers the dual situation for direct limits.

Exercise 7.5:2. (i) Let (S_i, f_{ij}) be an inversely directed system in **Set** such that all the morphisms f_{ij} are one-to-one, and let us choose any element $i_0 \in I$. Show that $\underleftarrow{\mathrm{Lim}}_i\, S_i$ can be identified with the intersection, in S_{i_0}, of the sets $f_{i i_0}(S_i)$ $(i < i_0)$.

(ii) Let (S_i, f_{ij}) be a directed system in **Set** such that all the morphisms f_{ij} are onto, and let us choose any element $i_0 \in I$. Show that $\underrightarrow{\mathrm{Lim}}_i\, S_i$ can be identified with the quotient set of S_{i_0} by the union of the equivalence relations induced by the maps $f_{i_0 i}: S_{i_0} \to S_i$ $(i > i_0)$.

Exercise 7.5:3. (i) Show that the inverse limit of any inverse system of *finite nonempty* sets is nonempty.

(Suggestions: Either build up the description of an element of the inverse limit ‘‘from below’’, by looking at partial assignments satisfying appropriate extendibility conditions, and applying Zorn’s Lemma to get a maximal such assignment, or else ‘‘narrow down on an element from above’’, by looking at ‘‘subsystems’’ of the given inverse system, i.e., systems of nonempty subsets of the given sets carried into one another by the given mappings, and use Zorn’s Lemma to get a minimal such subsystem. You might find it instructive to work out both of these proofs.)

(ii) Show that (i) can fail if the condition ‘‘finite’’ is removed, even for inverse limits over the totally ordered set of negative integers.

(iii) If you have some familiarity with general topology, see whether you can generalize statement (i) to a result on topological spaces, with ‘‘compact Hausdorff’’ replacing ‘‘finite’’.

As an application of part (i) of the above exercise, suppose we are given a subdivision of the plane into regions, possibly infinitely many, and are studying the problem of coloring these regions with n colors so that no two adjacent regions are the same color. Let the set of all our regions be denoted R, the adjacency relation $A \subseteq R \times R$ (i.e., $(r_1, r_2) \in A$ if and only if r_1 and r_2 are adjacent regions), and the set of colors C. For any subset $S \subseteq R$, let X_S denote the set of all colorings of S (maps $S \to C$) under which no two adjacent regions have the same color; let us call these ‘‘permissible colorings of S’’. If $S \subseteq T$, then the restriction to S of a permissible coloring of T is a permissible coloring of S; thus we have a restriction map $X_T \to X_S$. Now –

Exercise 7.5:4. (i) Show that in the above situation, the sets X_S, as S ranges over the *finite* subsets of R, form an inversely directed system, and that X_R may be identified with the inverse limit of this system in **Set**.

(ii) Deduce using Exercise 7.5:3(i) that if each finite family $S \subseteq R$ can be colored, then the whole picture R can be colored. (Note: the assumption that every finite family S can be colored does *not* say that *every* permissible coloring of a finite family S can be extended to a permissible coloring of every larger finite family T!)

Exercise 7.5:5. (i) Show that if (X_i, f_{ij}) is a directed system of sets, and each f_{ij} is one-to-one, then the canonical maps $q_j: X_j \to \underrightarrow{\mathrm{Lim}}\, X_i$ are all one-to-one.

(ii) Let (X_i, f_{ij}) be an inversely directed system of sets such that each f_{ij} is surjective. Show that if I is *countable*, then the canonical maps $p_j: \underleftarrow{\mathrm{Lim}}\, X_i \to X_j$ are surjective. (Suggestion: First prove this in the case where I is the set of negative integers. Then show that any countable inversely directed partially ordered set either has a least element, or has a downward-cofinal subset order-isomorphic to the negative integers, and apply Exercise 7.5:1.)

(iii) Does this result remain true for uncountable I? In particular, what if I is the opposite of the first uncountable ordinal?

Exercise 7.5:6. Show that every group is a direct limit of finitely presented groups.

The proof of Exercise 7.5:6 is not specific to groups. We shall be able to extend it to more general algebraic structures when we develop the necessary language in the next chapter.

The remaining exercises in this section develop some particular examples and applications of direct and inverse limits (including some further results concerning the p-adic integers). In these exercises you may assume the result which, as noted earlier, will be proved in the next chapter, that a direct or inverse limit of algebras can be constructed by forming the corresponding limit of underlying sets, and giving this an induced algebra structure. None of these exercises, or the remarks connecting them, is needed for the subsequent sections of these notes.

One can sometimes achieve interesting constructions by taking direct limits of systems in which all objects are the same; this is illustrated in the next three exercises. The first shows a sophisticated way to get a familiar construction; in the next two, direct limits are used to get curious counterexamples.

Exercise 7.5:7. Consider the directed system (X_i, f_{ij}) in $\mathbf{Ab}$, where I is the set of positive integers, partially ordered by divisibility (i considered less than or equal to j if and only if i divides j), each object X_i is the additive group $\mathbf{Z}$, and for $j = ni$, $f_{ij}\colon \mathbf{Z} \to \mathbf{Z}$ is given by multiplication by n.

(i) Show that $\underrightarrow{\mathrm{Lim}}\, X_i$ may be identified with the additive group of rational numbers.

(ii) Can you construct the ring multiplication of $\mathbf{Q}$ in terms of this description?

(iii) Show that if you perform the construction of (i) starting with an arbitrary abelian group A in place of $\mathbf{Z}$, the result is a $\mathbf{Q}$-vector-space which can be characterized by a universal property relative to A.

Exercise 7.5:8. For this exercise, assume known the facts that every subgroup of a free group is free, and in particular, that in the free group on two generators x, y, the subgroup generated by the two commutators $x^{-1}y^{-1}xy$ and $x^{-2}y^{-1}x^2y$ is free on those two elements.

Let F denote the free group on x and y, and f the endomorphism of F taking x to $x^{-1}y^{-1}xy$ and y to $x^{-2}y^{-1}x^2y$. Let G denote the direct limit of the system $F \to F \to F \to \ldots$, where all the arrows shown are the above morphism f.

Show that G is a nontrivial group such that every finitely generated subgroup of G is free, but that G is equal to its own commutator, $G = [G, G]$; i.e., that the abelianization of G is the trivial group. Deduce that though G is ‘‘locally free’’, it is not free.

Exercise 7.5:9. Let k be a field. Let R denote the direct limit of the system of k-algebras $k[x] \to k[x] \to k[x] \to \ldots$, where each arrow is the homomorphism sending x to x^2. Show that R is an integral domain in which every finitely generated ideal is principal, but not every ideal is finitely generated. (Thus, for each ideal, the minimum cardinality of a generating set is either 0, 1 or infinite.)

For the student familiar with the Galois theory of finite-dimensional field extensions, the next exercise shows how the Galois groups of *infinite-dimensional* Galois extensions can be characterized in terms of the finite-dimensional case.

Exercise 7.5:10. Suppose E/K is a normal algebraic field extension, possibly of infinite degree. Let I be the set of subfields of E normal and of *finite* degree over K. If $F_2 \subseteq F_1$ in I, let $f_{F_1,F_2}\colon \mathrm{Aut}_K F_1 \to \mathrm{Aut}_K F_2$ denote the map which acts by *restricting* automorphisms of F_1 to the subfield F_2.

(i) Show that the definition of f_{F_1,F_2} makes sense, and gives a group homomorphism.

(ii) Show that if we order I by reverse inclusion of fields, then the groups $\mathrm{Aut}_K F$ $(F\in I)$ and homomorphisms f_{F_1, F_2} $(F_1 \leq F_2)$ form an inversely directed system of groups.

(iii) Show that $\mathrm{Aut}_K E$ is the inverse limit of this system in **Group**.

(iv) Can you find a normal algebraic field extension whose automorphism group is isomorphic to the additive group of the p-adic integers?

The following is equivalent to an outstanding open question.

Exercise 7.5:11. (i) Suppose a group G is the inverse limit of a system of finite groups. If G is a torsion group (i.e., if all elements of G are of finite order), must G have finite exponent (i.e., must there exist an integer n such that $x^n = e$ is an identity of G)?

Though the above question is very difficult, the next two parts are reasonable exercises, and may help render that question more tractable:

(ii) Show that (i) is equivalent to the corresponding question in which we assume that G is the inverse limit of a system of finite groups indexed by the negative integers (under the natural ordering), with all connecting morphisms surjective.

(iii) Translate (i) (possibly with the help of (ii)) into a question on finite groups which you could pose to a person not familiar with the concept of inverse limit. (The more natural-sounding, the better.)

Back to the p-adic integers, now.

Exercise 7.5:12. (i) Show that the function v_p of Exercise 7.4:2 satisfies $v_p(xy) = v_p(x)+v_p(y)$ and $v_p(x+y) \geq \min(v_p(x), v_p(y))$ $(x, y\in\mathbf{Z})$.

(ii) Deduce that $\hat{\mathbf{Z}}_{(p)}$ is an integral domain.

(iii) Show that v_p can be extended in a unique manner to a $\mathbf{Z}\cup\{+\infty\}$-valued function on $\mathbf{Q}$ satisfying the properties noted in (i).

(iv) Show that the completion of $\mathbf{Q}$ with respect to the metric d_p induced by the above extended function v_p is the field of fractions of $\hat{\mathbf{Z}}_{(p)}$.

(v) Show that elements of this field have expansions $x = \Sigma_i\, c_i p^i$, where again $c_i\in\{0, 1, \ldots, p-1\}$, and where i now ranges over all integer values (not necessarily positive), but subject to the condition that the set of i such that c_i is nonzero is bounded below.

This field is called the field of *p-adic rationals*, and denoted $\hat{\mathbf{Q}}_{(p)}$ (or $\mathbf{Q}_p$).

Is the ‘‘adic’’ construction limited to primes p, or can one construct, say, a ring of ‘‘10-adic integers’’, $\hat{\mathbf{Z}}_{(10)}$? One encounters a trivial difficulty in that there are two ways of interpreting this symbol. But we shall see that they lead to the same ring; so there is a well-defined object to which we can give this name. However, its properties will not be as nice as those of the p-adic integers for prime p.

Exercise 7.5:13. Let $\mathbf{Z}_{(10)}$ denote the ring of all rational numbers which can be written with denominators relatively prime to 10.

(i) Determine all nonzero ideals $I \subseteq \mathbf{Z}_{(10)}$. Sketch the diagram of the inverse system of all factor-rings $\mathbf{Z}_{(10)}/I$ and canonical maps among them.

(ii) Show that the inverse system $\ldots \to \mathbf{Z}_{10^i} \to \ldots \to \mathbf{Z}_{100} \to \mathbf{Z}_{10}$ constitutes a downward *cofinal* subsystem of the above inverse system.

Hence by Exercise 7.5:1 the inverse limits of these two systems are isomorphic, and we shall denote their common value $\hat{\mathbf{Z}}_{(10)}$. It is clear from the form of the second inverse system that elements of $\hat{\mathbf{Z}}_{(10)}$ can be described by ‘‘infinite decimal expressions to the left of the decimal point’’.

(iii) Show that the relation $2\cdot 5 = 0$ in $\mathbf{Z}_{10}$ can be lifted to get a pair of nonzero elements which have product 0 in $\mathbf{Z}_{100}$, that these can be lifted to such elements in $\mathbf{Z}_{1000}$, and so on, and deduce that $\hat{\mathbf{Z}}_{(10)}$ is not an integral domain.

(iv) Prove, in fact, that $\hat{\mathbf{Z}}_{(10)} \cong \hat{\mathbf{Z}}_{(2)} \times \hat{\mathbf{Z}}_{(5)}$.

A variant construction often used in number theory is

Exercise 7.5:14. Show that the inverse limit of the system of all factor-rings of $\mathbf{Z}$ by nonzero ideals is isomorphic to $\prod_p \hat{\mathbf{Z}}_{(p)}$, where the direct product is taken over all primes p. (This ring is denoted $\hat{\mathbf{Z}}$.)

A feature we have not yet mentioned, but which is important in the study of inverse limits, is topological structure. We have constructed the inverse limit of a system of sets and set maps (X_i, f_{ij}) as a subset of $\prod X_i$. Regarding each X_i as a discrete topological space, we may give $\prod X_i$ the product topology. In general, a product of discrete spaces is not discrete; however, a product of compact spaces *is* compact, so if our discrete spaces X_i are *finite*, their product will be compact. It is not hard to show that the subset $\varprojlim X_i \subseteq \prod X_i$ will be closed in the product topology, and hence, if the X_i are finite, will be compact in the induced topology.

Exercise 7.5:15. (i) Verify the assertion that $\varprojlim X_i \subseteq \prod X_i$ is always closed in the product topology, and therefore compact if all X_i are finite.

(ii) Show that Exercise 7.5:3(i) (and hence Exercise 7.5:4(ii)) can be deduced using the compactness of $\varprojlim X_i$.

(iii) Show that the compact topology described above agrees in the case of $\hat{\mathbf{Z}}_{(p)}$ with the topology arising from the metric d_p of Exercise 7.4:2.

In fact, results like Exercise 7.5:4(ii), saying that a family of conditions can be satisfied simultaneously if all finite subfamilies of these conditions can be so satisfied, are called by logicians ‘‘compactness’’ results, because the proofs can generally be formulated in terms of the compactness of some topological space.

I can now say that the usual formulation of the open question raised in Exercise 7.5:11 is, ‘‘If a compact group is torsion, must it have finite exponent?’’ The equivalence of this with the question of that exercise follows from a deep result, that any compact group is an inverse limit of surjective maps of compact Lie groups (see [**85**, Theorem IV.4.6, p.175]), combined with the observation that if any of these Lie groups had positive dimension, we would get elements of infinite order. Thus, compact torsion groups are inverse limits of 0-dimensional compact Lie groups, i.e., finite discrete groups, under the product topology.

An inverse limit of finite structures is called *pro*finite (based on the synonym ‘‘projective limit’’ for ‘‘inverse limit’’). I hope to eventually add to these notes a chapter treating profinite algebras, and objects with related conditions, such as pro-finite-dimensionality. Let us look briefly at the latter condition in

Exercise 7.5:16. Let V be a vector space over a field k.

(i) Show that the dual space V^* is the inverse limit, over all finite-dimensional subspaces $V_0 \subseteq V$, of the spaces $V_0{}^*$.

(ii) Can you get the result of (i) as an instance of a general result describing *duals* of *direct limits* of vector spaces?

(iii) If you did Exercise 5.5:5(ii)-(iii), show that the topology described there is that of the inverse limit of the finite-dimensional discrete spaces $V_0{}^*$ referred to above. Show moreover that the only linear functionals $V^* \to k$ continuous in this topology are those induced by the elements of V.

The remainder of this section constitutes a digression for curiosity's sake.

Ordinary real numbers expressed in base p have expansions going endlessly to the right, and finitely many steps to the left of the decimal point; p-adic rationals (Exercise 7.5:12) have expansions going endlessly to the left, and finitely many steps to the right. Is it possible to define an arithmetic of elements with formal base-p expansions going endlessly in both directions?

Exercise 7.5:17. Let p be a prime. For every integer n, we have a subgroup $p^n\mathbf{Z} \subseteq \mathbf{R}$, hence we can form the quotient group $\mathbf{R}/p^n\mathbf{Z}$. Observe that these groups are each isomorphic to the circle group $\mathbf{R}/\mathbf{Z}$, and are connected by homomorphisms $\ldots \to \mathbf{R}/p^2\mathbf{Z} \to \mathbf{R}/p\mathbf{Z} \to \mathbf{R}/\mathbf{Z} \to \ldots$, taking the residue of a real number modulo $\mathbf{Z}_p{}^{i+1}$ to its residue modulo $\mathbf{Z}_p{}^{i}$. Let G be the *inverse limit* of this system of groups.

(i) Show how to express elements of G as formal doubly infinite series $\Sigma_{i\in\mathbf{Z}}\, c_i p^i$, where $c_i \in \{0, 1, \ldots\, p-1\}$, $(i = \ldots, -1, 0, 1, \ldots)$. Show that such a representation is unique except for the cases where for all sufficiently small i, c_i either becomes constant with value 0 or constant with value $p-1$.

(ii) Show that $\hat{\mathbf{Q}}_{(p)}$ and $\mathbf{R}$ both embed as dense subgroups of G.

Groups of the above sort appear in the theory of locally compact abelian groups, where they are called ''solenoids'', from a term in electronics meaning ''a hollow tightly wound coil of wire''. For students familiar with Pontryagin duality, the solenoid G constructed above will be seen to be the dual of the *discrete* additive group of $\mathbf{Z}[p^{-1}]$ (the ring of rational numbers of the form np^{-i}).

The above group G may also be obtained as a completion: For p a prime, let us define a function v_p on the real numbers, by letting $v_p(x)$ be the supremum of all integers n such that $x\in p^n\mathbf{Z}$. This will be $+\infty$ if $x = 0$, a nonnegative integer if x is a nonzero integer, a negative integer if x is a noninteger rational number of the form m/p^i, and $-\infty$ if none of these cases hold. (This does not agree with the definition of $v_p(x)$ we gave in Exercise 7.5:12 for rational x, though it does for x in the subring $\mathbf{Z}[p^{-1}]$.) Now for any two real numbers x, y, define $d_{p,|\,|}(x,y) = \inf_{z\in\mathbf{R}} (p^{-v_p(x-z)} + |z-y|)$. Observe that although $p^{-v_p(x-z)}$ takes on the value $+\infty$ for most z, there exist values of z for which it is finite, so the infimum shown will be finite for all x and y.

Exercise 7.5:18. (i) Show that $d_{p,|\,|}$ is a metric on the real line $\mathbf{R}$, and is bounded above.

(ii) Show how to obtain from a doubly infinite series $\Sigma_{i\in\mathbf{Z}}\, c_i p^i$ a Cauchy sequence in $\mathbf{R}$ under this metric, and show that all elements of the completion of $\mathbf{R}$ in the metric $d_{p,|\,|}$ can be represented by such series.

(iii) Deduce that this completion is isomorphic to the solenoid G of the preceding exercise.

Exercise 7.5:19. (i) Show that the topology on G arising from the above metric agrees with that obtained by regarding G as an inverse limit of compact groups $\mathbf{R}/p^n\mathbf{Z}$. Deduce that the additive group operations of $\mathbf{R}$ extend continuously to this completion.

(ii) Let r be a real number, and $\bar{r}: \mathbf{R} \to \mathbf{R}$ the operation of multiplication by r. Show that $\bar{r}$ is continuous in the metric $d_{p,|\,|}$ if and only if $r\in\mathbf{Z}[p^{-1}]$. Deduce that multiplication as a map $\mathbf{R}\times\mathbf{R} \to \mathbf{R}$ is not bicontinuous in this metric. Hence the ring structure on $\mathbf{R}$ does not extend to the solenoid.

(iii) Can addition of elements of the solenoid be performed by the same operations on digits that one uses to add ordinary real numbers in base p? What happens if we try to apply the ordinary procedure for *multiplying* numbers in base p?

(iv) If n is a positive integer not a power of p, show that the elements ''n^{-1}'' of $\mathbf{R}$ and of $\hat{\mathbf{Q}}_{(p)}$ have distinct images under the embeddings of Exercise 7.5:17(ii). Deduce that the additive group of the solenoid has nonzero elements of finite order.

(v) Show that the solenoid described above is isomorphic to the group $\mathbf{Ab}(\mathbf{Z}[p^{-1}], \mathbf{R}/\mathbf{Z})$ (as asserted in the paragraph following Exercise 7.5:17(ii)).

7.6. Limits and colimits. Direct and inverse limits are similar in their universal properties to several other constructions we have seen. Let us recall these.

Given two objects X_1, X_2 of a category $\mathbf{C}$, a *product* of X_1 and X_2 in $\mathbf{C}$ is an object P given with morphisms p_1 and p_2 into X_1 and X_2, and universal for this property.

Given a pair of parallel morphisms $X_1 \rightrightarrows X_2$ in $\mathbf{C}$, a *difference kernel* of this system is an object K given with a morphism k into X_1 having the same composite with the two given morphisms, and again universal. To improve the parallelism with similar constructions, let us rename the morphism k as k_1, and let $k_2: K \to X_2$ denote the common value of the composites of k_1 with the two morphisms $X_1 \rightrightarrows X_2$. Then we can describe K as having a morphism into *each* of X_1, X_2, such that the composite of $k_1: K \to X_1$ with each of the two given morphisms $X_1 \to X_2$ is the morphism $k_2: K \to X_2$, and as being universal for these properties. We see that this is exactly like the universal property of an inverse limit, except that the indexing category $\cdot \rightrightarrows \cdot$ is not of the form $P_{\mathbf{cat}}$.

In the same way, a *pullback* of a pair of morphisms $f_1: X_1 \to X_3$, $f_2: X_2 \to X_3$ can be redefined as an object P given with morphisms p_1, p_2, p_3 into X_1, X_2, X_3 respectively, satisfying $f_1 p_1 = p_3$ and $f_2 p_2 = p_3$, and universal for this property.

Let us look at a case we haven't discussed so far. If G is a group and S a G-set, then the *fixed-point set* of the action of G on S means $\{x \in |S| \mid (\forall\, g \in |G|)\ gx = x\}$. If we denote the action of each $g \in |G|$ on S by $g_S: |S| \to |S|$, then the fixed-point set is universal among sets A with maps $i: A \to |S|$ such that for all $g \in |G|$, $i = g_S i$. Given any object X of a category $\mathbf{C}$ and an action of a group G on X, we can look for an object with the same universal property, and, if it exists, call it the ''fixed object'' of the action.

We have also seen constructions dual to those of product, difference kernel and pullback. A construction dual to that of ''fixed object'' should take an object X of $\mathbf{C}$ with an action of G on it to an object B of $\mathbf{C}$ with a map $j: X \to B$ unchanged under composition on the right with the actions of elements of G, and universal for this property. Examples of this concept are examined in

Exercise 7.6:1. Let G be a group.

(i) If X is a set on which G acts by permutations, and x an element of X, one defines the *orbit* of x under G to be the set $Gx = \{gx \mid g \in |G|\}$. Let B be the set of such orbits Gx, called the *orbit space* of X. Show that this set B, together with the map $X \to B$ taking x to Gx, has the universal property discussed above.

(ii) Show that if G acts by automorphisms on (say) a ring R, then there is an object S in the category of rings with this same universal property, but that its underlying set will not in general be the orbit space of the action of G on the underlying set of R.

(iii) If G acts by automorphisms on an object X of $\mathbf{POSet}$, again show the existence of an object B with the above universal property. Show moreover that if G is finite, the underlying set of B will be the orbit space of the underlying set of X, and the universal map $X \to B$ will be *strictly* isotone; but that if G is infinite, neither statement need be true.

(iv) Do the assertions of (iii) about the case where G is finite remain true if we replace $\mathbf{POSet}$ by $\mathbf{Lattice}$?

The universal properties we have been discussing are all cases of two patterns, whose statements are formally identical with the definitions of direct and inverse limits, except that the partially ordered set I of that definition is replaced by a general category $\mathbf{D}$. (For instance, in the examples noted above, the categories occurring as $\mathbf{D}$ included the two-object category $\cdot\rightrightarrows\cdot$ and the one-object category $G_{\mathbf{cat}}$.) As names for the general concepts, one uses modified versions of the terms ''direct and inverse limits''.

Definition 7.6.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories, and* $F\colon \mathbf{D}\to\mathbf{C}$ *a functor.*

Then a limit *of* F, *written* $\underleftarrow{\mathrm{Lim}}\ F$ *or* $\underleftarrow{\mathrm{Lim}}_{\mathbf{D}}\ F(X)$, *means an object* $L\in\mathrm{Ob}(\mathbf{C})$ *given with morphisms* $p(X)\colon L\to F(X)$ *for all* $X\in\mathrm{Ob}(\mathbf{D})$, *such that for* $f\in\mathbf{D}(X,Y)$ *one has* $p(Y)=F(f)\,p(X)$, *and universal for this property, in the sense that given any object* $M\in\mathrm{Ob}(\mathbf{C})$ *and family of morphisms* $m(X)\colon M\to F(X)$ $(X\in\mathrm{Ob}(\mathbf{C}))$ *which similarly make commuting triangles with the morphisms* $F(f)$, *there exists a unique morphism* $h\colon M\to L$ *such that for all* X, $m(X)=p(X)\,h$.

Likewise, a colimit *of* F, *written* $\underrightarrow{\mathrm{Lim}}\ F$ *or* $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\ F(X)$, *means an object* $L\in\mathrm{Ob}(\mathbf{C})$ *given with morphisms* $q(X)\colon F(X)\to L$ *for all* $X\in\mathrm{Ob}(\mathbf{D})$ *such that for* $f\in\mathbf{D}(X,Y)$ *one has* $q(X)=q(Y)\,F(f)$, *and universal for this property, in the sense that given* $M\in\mathrm{Ob}(\mathbf{C})$ *and morphisms* $m(X)\colon F(X)\to M$ $(X\in\mathrm{Ob}(\mathbf{C}))$ *making commuting triangles with the morphisms* $F(f)$, *there exists a unique morphism* $h\colon L\to M$ *such that for all* X, $m(X)=h\,q(X)$.

The morphisms $p(X)$ *in the definition of a limit may be called the* projection *morphisms, and the* $q(X)$ *in the definition of colimit may be called the* coprojection *morphisms.*

One says that a category $\mathbf{C}$ *''has small limits'' if all functors from small categories* $\mathbf{D}$ *into* $\mathbf{C}$ *have limits; likewise* $\mathbf{C}$ *''has small colimits'' if all functors from small categories into* $\mathbf{C}$ *have colimits.*

Remarks on terminology. Since the above concepts generalize not only direct and inverse limits, but also a large number of other pairs of constructions, they might just as well have been given names suggestive of one of the other pairs. I think that the reason ''limit'' and ''colimit'' were chosen is that each of the other universal constructions of this sort involves a more or less fixed diagram, while the diagrams involved in direct and inverse limits are varied. Hence in developing the latter concepts, people were forced to formulate a more general definition, and just a little more generality gave the concepts noted above.

But though the choice is historically explainable, I think it is unfortunate. As we can see from the examples of products and coproducts, or of kernels and cokernels, the objects given by limit and colimit constructions over diagram categories other than directed partially ordered sets are not ''approximated arbitrarily closely'' by the objects from which they are constructed. The cases that best exemplify the general concepts are, I think, those of pullback and pushout, so it would be preferable if the limit and colimit of $F\colon\mathbf{D}\to\mathbf{C}$ could be renamed the *pullback* and the *pushout* of F (regarded as a system of objects and maps in $\mathbf{C}$). But it seems too late to turn the tide of usage. (Note, incidentally, the initially confusing fact that *limits* generalize *inverse limits*, while *colimits* generalize *direct limits*. The explanation is that the words ''direct'' and ''inverse'' refer to forward and backward orientation with respect to a partial ordering, while the relation between the terms ''limit'' and ''colimit'' is based on looking at which is left and which is right universal, by analogy with ''products and coproducts'' and ''kernels and cokernels''. There is no reason why two such principles of naming should agree as to which concept gets the ''plain'' and which the ''modified'' name, and in this case, they do not.)

There is another set of words for the same constructions: Freyd has named them ''roots'' and ''coroots'', probably because if one pictures a system of objects and morphisms as a graph, the addition of the universal object makes it a *rooted* graph, with the universal object at the root.

However there is no evident connection with roots of equations etc., and this terminology has not caught on.

Following the associations of the word ‘‘limit’’, Mac Lane calls a category $\mathbf{C}$ *complete* if it has small limits, *cocomplete* if it has small colimits.

Exercise 7.6:2. If S is a monoid, then as for groups, an S-set is equivalent to a functor F: $S_{\mathbf{cat}} \to \mathbf{Set}$. Show how to construct the limit and colimit of such a functor.

A useful observation is

Lemma 7.6.2. *Let* $\mathbf{D}$ *be a category and* X_0 *an object of* $\mathbf{D}$ *such that there are morphisms from* X_0 *to every object of* $\mathbf{D}$. *Let* F: $\mathbf{D} \to \mathbf{C}$ *be a functor having a limit* L. *Then the projection morphism* $p(X_0)$: $L \to F(X_0)$ *is a monomorphism. In particular, all difference kernel maps are monomorphisms.*

Likewise, if $\mathbf{D}$ *is a category having an object* X_0 *such that there are morphisms from every object of* $\mathbf{D}$ *to* X_0, *and* F: $\mathbf{D} \to \mathbf{C}$ *is a functor having a colimit* L, *then the coprojection morphism* $q(X_0)$: $F(X_0) \to L$ *is an epimorphism. In particular, difference cokernel maps are epimorphisms.*

Proof. Assume the first situation. The universal property of L implies that a morphism h: $M \to L$ in $\mathbf{C}$ is uniquely determined by the system of morphisms $p(X)h$: $M \to F(X)$ $(X \in \mathrm{Ob}(\mathbf{D}))$. But for any $X \in \mathrm{Ob}(\mathbf{D})$, we can find a morphism $f: X_0 \to X$ in $\mathbf{D}$, and we then have $p(X) = F(f)p(X_0)$. Thus any h: $M \to L$ in $\mathbf{C}$ is uniquely determined by the single morphism $p(X_0)h$. This is equivalent to saying $p(X_0)$ is a monomorphism. The result for colimits follows by duality. □

We have seen that the constructions of pairwise product and coproduct, when they exist for all pairs of objects of a category $\mathbf{C}$, give right and left adjoints to the ‘‘diagonal’’ functor Δ: $\mathbf{C} \to \mathbf{C} \times \mathbf{C}$. These statements generalize to limits and colimits.

Proposition 7.6.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. Let* Δ: $\mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ *denote the ‘‘diagonal’’ functor, taking every object* $X \in \mathrm{Ob}(\mathbf{C})$ *to the functor* $\Delta(X) \in \mathrm{Ob}(\mathbf{C}^{\mathbf{D}})$ *with value* X *at all objects of* $\mathbf{D}$ *and value* id_X *at all morphisms of* $\mathbf{D}$, *and likewise taking each morphism* $f \in \mathbf{C}(X, Y)$ *to the morphism of functors* $\Delta(f)$: $\Delta(X) \to \Delta(Y)$ *with value* f *at all objects of* $\mathbf{D}$.

Then a limit of a functor F: $\mathbf{D} \to \mathbf{C}$ *is the same as an object* L *representing the contravariant functor* $\mathbf{C}^{\mathbf{D}}(\Delta(-), F)$: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. *In particular, if for a given* $\mathbf{D}$ *all functors* $\mathbf{D} \to \mathbf{C}$ *have limits, then the construction* $\underleftarrow{\mathrm{Lim}}$: $\mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ *is a* right adjoint *to the diagonal functor* Δ: $\mathbf{C} \to \mathbf{C}^{\mathbf{D}}$.

Likewise, a colimit of F: $\mathbf{D} \to \mathbf{C}$ *is an object* L *representing the covariant functor* $\mathbf{C}^{\mathbf{D}}(F, \Delta(-))$: $\mathbf{C} \to \mathbf{Set}$. *Thus, when all functors* $\mathbf{D} \to \mathbf{C}$ *have colimits, the construction* $\underrightarrow{\mathrm{Lim}}$: $\mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ *is a* left *adjoint to the diagonal functor* Δ: $\mathbf{C} \to \mathbf{C}^{\mathbf{D}}$. □

These adjointness relationships are shown below.

$$\begin{tikzcd} \mathbf{C} \arrow[d, "\Delta"] \\ \mathbf{C}^{\mathbf{D}} \arrow[u, bend left=60, "\underrightarrow{\mathrm{Lim}}"] \arrow[u, bend right=60, "\underleftarrow{\mathrm{Lim}}"'] \end{tikzcd}$$

Note that if, as above, $\mathbf{C}$ has colimits of all functors $F \in \mathbf{C}^{\mathbf{D}}$, then our observation that $\underrightarrow{\mathrm{Lim}} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ is left adjoint to Δ, tells us, in particular, that it is a *functor*. Thus, given a morphism

$$f : F \to G$$

in $\mathbf{C}^{\mathbf{D}}$, we get an induced morphism

$$\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, f : \underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, F \to \underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, G$$

in $\mathbf{C}$. This will be characterized by the equations

$$(7.6.4) \qquad (\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, f)\, q_F(X) \;=\; q_G(X)\, f(X) \qquad (X \in \mathrm{Ob}(\mathbf{D})).$$

Similarly, if functors in $\mathbf{C}^{\mathbf{D}}$ have limits, then $\underleftarrow{\mathrm{Lim}}_{\mathbf{D}} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ becomes a functor, with

$$\underleftarrow{\mathrm{Lim}}_{\mathbf{D}}\, f : \underleftarrow{\mathrm{Lim}}_{\mathbf{D}}\, F \to \underleftarrow{\mathrm{Lim}}_{\mathbf{D}}\, G$$

characterized by

$$(7.6.5) \qquad p_G(X)(\underleftarrow{\mathrm{Lim}}_{\mathbf{D}}\, f) \;=\; f(X)\, p_F(X) \qquad (X \in \mathrm{Ob}(\mathbf{D})).$$

In drawing a picture of a morphism $\Delta(M) \to F$ or $F \to \Delta(M)$ $(M \in \mathrm{Ob}(\mathbf{C}))$, we can for convenience collapse the copies of the object M and the identity arrows among these into a single ‘‘M’’. (I.e., we can collapse the picture representing Proposition 7.6.3 into the picture representing Definition 7.6.1.) What we have then looks like a ‘‘cone’’ of maps, with M at the apex. Hence a morphism of functors $\Delta(M) \to F$ or $F \to \Delta(M)$ is often called a ‘‘cone’’ from the object M to the functor F, or from the functor F to the object M. So limits and colimits may be described as objects with ‘‘universal cones’’ to or from given functors.

Exercise 7.6:3. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. By Lemma 6.10.1 (‘‘Law of Exponents for Functors’’), the functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ corresponds to some functor $\mathbf{D} \times \mathbf{C} \to \mathbf{C}$. Describe this functor.

Our construction in §7.5 of the inverse limit of an inverse system of sets (X_i, f_{ij}) as the subset of $\prod X_i$ determined by ‘‘compatibility’’ conditions can be generalized to give a construction of general limits in any category having appropriate products and difference kernels, and it dualizes to a construction of colimits in categories with appropriate coproducts and difference cokernels. (The latter construction may be thought of as generalizing our construction of the direct limit of a directed system of sets as the quotient of a disjoint union by an equivalence relation, though the simple way that equivalence relation could be described when $\mathbf{D}$ was a directed partially ordered set, and $\mathbf{C}$ was $\mathbf{Set}$, does not go over to the general situation.) In the case of inverse limits, the compatibility conditions said that for all $i < j$ in I, the pair of maps $(p_j, f_{ij} p_i)$ had to agree on elements of our subset of $\prod X_i$. Such a family of conditions can in fact be translated to a condition saying that a *single* pair of maps into an appropriate product object should agree. Using this construction, we get

Proposition 7.6.6. *Let* $\mathbf{C}$ *be a category and* $\mathbf{D}$ *a small category, and let* α *be an infinite cardinal such that* $\mathbf{D}$ *has* $<\alpha$ *objects and* $<\alpha$ *morphisms.*

Then if $\mathbf{C}$ *has products of all families of* $<\alpha$ *objects, and has difference kernels, then every functor* $\mathbf{D}\to\mathbf{C}$ *has a limit.*

Likewise, if $\mathbf{C}$ *has coproducts of families of* $<\alpha$ *objects, and has difference cokernels, then every functor* $\mathbf{D}\to\mathbf{C}$ *has a colimit.*

Proof. Under the hypotheses of the first assertion, let

$$P \ = \ \prod\nolimits_{X\in\mathrm{Ob}(\mathbf{D})} F(X),$$

$$P' \ = \ \prod\nolimits_{X,Y\in\mathrm{Ob}(\mathbf{D}),\ f\in\mathbf{D}(X,Y)} F(Y).$$

(If we required categories to have disjoint hom-sets, we could write the latter definition more simply as $P' = \prod_{f\in\mathrm{Ar}(\mathbf{D})} F(\mathrm{cod}(f))$.) Denote the projection morphisms associated with these two product objects by $p_X: P\to F(X)$ $(X\in\mathrm{Ob}(\mathbf{D}))$ and $p'_{X,Y,f}: P'\to F(Y)$ $(X,\ Y\in\mathrm{Ob}(\mathbf{D}),$ $f\in\mathbf{C}(X,\ Y))$. We shall construct L as the difference kernel of two maps $a,b: P\to P'$. Since a and b are to be morphisms into a direct product object P', they may be defined by specifying their composites with the projection morphisms $p'_{X,Y,f}: P'\to F(Y)$. Define them so that $p'_{X,Y,f}\,a = p_Y$, $p'_{X,Y,f}\,b = F(f)\,p_X$. If L is the difference kernel of a and b, and $k: L\to P$ the canonical morphism, we see that the universal property of L as a difference kernel is equivalent to the statement that the morphisms $p_X\,k: L\to F(X)$ form commuting triangles with the morphisms $F(f)$ and are universal for this property. It is immediate to verify that the object L together with the morphisms $p_X\,k$ has the property characterizing $\underleftarrow{\mathrm{Lim}}\ F$.

The result for colimits follows by duality. □

Exercise 7.6:4. Give the ‘‘immediate verification’’ referred to near the end of the above proof.

Of course, certain limits or colimits may exist even if the category does not have enough (co)products and difference (co)kernels to obtain them by the above lemma. Such a case is noted in point (iv) of the next exercise. (But the most useful part of this exercise is (i), and the most challenging is (ii).)

Exercise 7.6:5. Let $\mathbf{C}$ be a category.

(i) Show that an initial object of $\mathbf{C}$ is equivalent to a *colimit* of the unique functor from the empty category into $\mathbf{C}$.

(ii) Show that such an initial object is also equivalent to a *limit* of the identity functor of $\mathbf{C}$.

(iii) State the corresponding results for a terminal object.

(iv) Give an example where the limit of (ii) exists, but $\mathbf{C}$ does not satisfy the hypotheses needed to get this from Proposition 7.6.6.

Here is another degenerate case of the concept of limit:

Exercise 7.6:6. Find conditions on a category $\mathbf{D}$ which imply that any constant functor from $\mathbf{D}$ to a category $\mathbf{C}$, i.e., any functor of the form $\Delta(C): \mathbf{D}\to\mathbf{C}$ $(C\in\mathrm{Ob}(\mathbf{C}))$ has a limit given by the object C itself, with universal cone consisting of identity morphisms of C. State the corresponding result for colimits.

We have seen that a product or coproduct of objects in one category may or may not coincide with their product or coproduct in a subcategory to which they belong. E.g., the coproduct of two abelian groups in the category of all groups and their coproduct in the category of all abelian groups are different, since the former is generally nonabelian. We note below that for full subcategories, such phenomena occur *only* when the constructed object in the larger category fails to lie in the subcategory.

Lemma 7.6.7. *Let* $\mathbf{C}$ *be a category,* $\mathbf{B}$ *a full subcategory of* $\mathbf{C}$*,* $I\colon \mathbf{B} \to \mathbf{C}$ *the inclusion functor, and* $F\colon \mathbf{D} \to \mathbf{B}$ *a functor from an arbitrary category into* $\mathbf{B}$*.*

If $\varprojlim IF$ *exists* (*loosely, if there exists ''a limit of the system of objects* $F(X)$ *in the larger category* $\mathbf{C}$*''*), *and if as an object it belongs to* $\mathbf{B}$*, then this same object, with the same cone to the objects* $F(X)$*, constitutes a limit* $\varprojlim F$ (*loosely, ''a limit of the given system within the subcategory* $\mathbf{B}$*''*).

The same is true for colimits $\varinjlim IF$ *and* $\varinjlim F$*.* □

Exercise 7.6:7. Prove the above lemma.

We indicated in the last two paragraphs of §6.10 that if a category $\mathbf{C}$ has finite products, then any functor category $\mathbf{C}^{\mathbf{E}}$ will also have products, which can be computed ''objectwise''. This is true generally for limits and colimits; you should find it easy to verify

Lemma 7.6.8. *Let* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{E}$ *be categories. Then if all functors* $\mathbf{D} \to \mathbf{C}$ *have limits, so do all functors* $\mathbf{D} \to \mathbf{C}^{\mathbf{E}}$*. Namely, given* $F\colon \mathbf{D} \to \mathbf{C}^{\mathbf{E}}$*, the object* $L = \varprojlim_{\mathbf{D}} F$ *of* $\mathbf{C}^{\mathbf{E}}$ *can be described as the functor taking each* $E \in \mathrm{Ob}(\mathbf{E})$ *to* $\varprojlim_{\mathbf{D}} p_E \circ F$*, and each* $f \in \mathbf{E}(E_1, E_2)$ *to* $\varprojlim_{\mathbf{D}} p_f \circ F$*, where* $p_E\colon \mathbf{C}^{\mathbf{E}} \to \mathbf{C}$ *is the ''*E*th projection functor'', taking functors and morphisms of functors to their values at the object* E*.*

Likewise, if all functors $\mathbf{D} \to \mathbf{C}$ *have colimits, then all functors* $\mathbf{D} \to \mathbf{C}^{\mathbf{E}}$ *have colimits, which are similarly constructed ''object- and morphism-wise''.* □

Exercise 7.6:8. Prove Lemma 7.6.8 for the case of limits.

7.7. What respects what.

7.7. What respects what. It is natural to ask what one can say about *limits* and *colimits* of systems of objects constructed by *adjoint functors*, about the values of *adjoint functors* on objects constructed by *limits* and *colimits*, and similar questions for other sorts of universal constructions.

Some quick examples: It is not hard to see that the free group on a disjoint union of sets, $X \sqcup Y$, will be the coproduct of the free groups on X and Y. If we look similarly at the free group on the difference cokernel of a pair of set maps, $f, g\colon X \rightrightarrows Y$ we find that it is the difference cokernel of the induced maps of free groups, $F(f), F(g)\colon F(X) \rightrightarrows F(Y)$. On the other hand, a direct product of free groups is in general not a free group, in particular not the free group on the direct product set. So the *free group* construction seems to respect *colimits*, but not limits.

If we look at its right adjoint, the underlying set functor, we find the opposite: The underlying set of a product or difference kernel of groups is the product or difference kernel of the underlying sets of the groups (that is how one constructs products and difference kernels of groups), but the underlying set of a coproduct of groups is not the coproduct (disjoint union) of their underlying sets, both because the group operation within this coproduct generally produces new elements from the elements of the two given groups, and because the images of the two identity elements fall together. Similarly, when we take a difference cokernel of two group homomorphisms

$f,\ g\colon G \rightrightarrows H,$ more identifications of elements are forced than in the set-theoretic difference cokernel; not only must pairs of elements $f(a)$ and $g(a)$ $(a\in|G|)$ fall together, but also pairs such as $f(a)b$ and $g(a)b$ $(a\in|G|,\ b\in|H|)$.

These examples suggest the general principle that ‘‘left universal constructions respect left universal constructions, and right universal constructions respect right universal constructions’’. We shall prove a series of theorems of that form in this and the next section.

We have seen left universal constructions in four guises: initial objects, representing objects for covariant set-valued functors, left adjoint functors, and colimits. Since an initial object of a category may be described as the object representing a certain trivial set-valued functor (Exercise 7.2:7) or as the colimit of a functor from a certain trivial category (Exercise 7.6:5), let us focus on relations among the remaining three types of constructions. Logically, these give us six combinations to consider. But I see no way that one can speak of the construction of an object representing a covariant set-valued functor U ‘‘respecting’’ the construction of an object representing another such set-valued functor V, so let us move on to the next case, the relation between left adjoint functors and representing objects for covariant set-valued functors. We give this, along with its dual, as

Theorem 7.7.1. *Suppose* $\mathbf{D} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{C}$ *are adjoint functors, with* U *the right adjoint and* F *the left adjoint, and with unit* η *and counit* ε.

If $A\colon \mathbf{C} \to \mathbf{Set}$ *is a representable functor, with representing object* $R\in\mathrm{Ob}(\mathbf{C})$ *and universal element* $u\in A(R)$, *then* $AU\colon \mathbf{D} \to \mathbf{Set}$ *is also representable, with representing object* $F(R)$ *and universal element* $A(\eta(R))(u)\in A(U(F(R)))$.

Likewise, if $B\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$ *is representable, with representing object* $R\in\mathrm{Ob}(\mathbf{D})$ *and universal element* $u\in B(R)$, *then* $BF\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *is representable, with representing object* $U(R)$ *and universal element* $B(\varepsilon(R))(u)\in B(F(U(R)))$.

Proof. In the first situation, $AU(-) \cong \mathbf{C}(R,\ U(-)) \cong \mathbf{D}(F(R),\ -)$, showing that AU is represented by $F(R)$. The identification of the universal element, corresponding to the identity morphism in $\mathbf{D}(F(R), F(R))$ is straightforward. The second situation is the dual of the first. □

As an example, suppose we wish to construct the ring with a universal pair of elements $x,\ y$ satisfying the relation $xy = yx^2$. We notice that this ring-theoretic relation ‘‘is actually a monoid relation’’; the formal statement is that the functor we want to represent can be written AU, where U is the forgetful functor from $\mathbf{Ring}^1$ to $\mathbf{Monoid}$, and A the functor associating to any *monoid* S the set of pairs of elements $x,\ y\in|S|$ satisfying $xy = yx^2$. The observation that we can construct our ring by first forming the monoid R presented by these generators and relation, and then passing to the monoid ring $\mathbf{Z}R$, i.e., applying the left adjoint to U, is an instance of the above theorem. We see that in this situation, the universal pair of ring elements satisfying the given equation is the image of the corresponding universal pair of monoid elements, under the canonical map $\eta(R)\colon R \to U(F(R)) = U(\mathbf{Z}R)$ (informally, the inclusion map $R \to \mathbf{Z}R$).

The above example makes it clear that Theorem 7.7.1 is a powerful tool, and that it indeed deserves to be described as saying that ‘‘left adjoint functors respect the construction of objects representing covariant set-valued functors’’.

Note, however, that, the sense in which this is true is rather idiosyncratic; the statement involves both the left adjoint functor and its right adjoint, and it does not appear to be a special case of any natural concept of a left adjoint functor respecting a general construction, or of a general functor respecting the construction of representing objects. There is a similarly idiosyncratic sense in which ‘‘left adjoint functors respect left adjoint functors’’; this is Theorem 7.3.5, already proved, which says that the composite of the left adjoints of two functors is the left adjoint of their composite (in the opposite order).

In contrast, when one looks at questions of how one or another sort of construction interacts with colimits (which are all we have left, of our six possible sorts of interaction among left universal constructions), one finds that there *is* a natural definition of an arbitrary functor’s respecting limits or colimits. We will examine that concept in the next section, and verify the corresponding cases of our observation that left universal constructions respect left universal constructions, and likewise that right universal constructions respect right universal constructions.

Exercise 7.7:1. Prove the following converse to the first assertion of Theorem 7.7.1: If $U\colon \mathbf{D} \to \mathbf{C}$ is a functor such that for every representable functor $A\colon \mathbf{C} \to \mathbf{Set}$, the composite functor $AU\colon \mathbf{D} \to \mathbf{Set}$ is representable, then U has a left adjoint. Also state the dual result.

7.8. Functors respecting limits and colimits.

Here is the definition of a functor ‘‘respecting’’ a limit or colimit.

Definition 7.8.1. *Let* $\mathbf{C}$, $\mathbf{C}'$ *be categories, and* $F\colon \mathbf{C} \to \mathbf{C}'$ *a functor.*

Then if $S\colon \mathbf{E} \to \mathbf{C}$ *is a functor into* $\mathbf{C}$, *having a limit* $\varprojlim S$, *with projection maps* $p_E\colon \varprojlim S \to S(E)$ $(E\in\mathrm{Ob}(\mathbf{E}))$, *one says that* F respects *the limit of* S *if the object* $F(\varprojlim S)$, *together with the cone given by the morphisms* $F(p_E)\colon F(\varprojlim S) \to F(S(E))$ *from this object to the functor* $FS\colon \mathbf{E} \to \mathbf{C}'$, *is a limit of* FS.

We shall say that F respects small limits *if for every functor* S *from a small category* $\mathbf{E}$ *to* $\mathbf{C}$ *which has a limit,* F *respects the limit of* S. *We shall say that* F *respects* possibly large *limits if this is true without the restriction that* $\mathbf{E}$ *be small. Likewise, we shall say that* F *respects pullbacks, terminal objects, small products, possibly large products, small inverse limits, possibly large inverse limits, etc., if it respects all instances of the sorts of limits that these names describe.*

Dually, if $S\colon \mathbf{E} \to \mathbf{C}$ *is a functor having a colimit* $\varinjlim S$, *with coprojection maps* $q_E\colon S(E) \to \varinjlim S$, *then we shall say that* F *respects the colimit of* S *if the object* $F(\varinjlim S)$, *with the cone from* FS *given by the* $F(q_E)\colon F(S(E)) \to F(\varinjlim S)$ *is a colimit of* FS; *and we will say that* F *respects small colimits, possibly large colimits, pushouts, initial objects, small or possibly large direct limits, etc., if it respects all colimits having these respective descriptions.*

In all of these situations, one may use ‘‘commutes with’’ as a synonym for ‘‘respects’’.

(Many authors, e.g., Mac Lane [**14**], again following the topological associations of the word ‘‘limit’’, call a functor respecting limits ‘‘continuous’’, and one respecting colimits ‘‘cocontinuous’’. But we will not use these terms here.)

The distinctions between the ‘‘small’’ and ‘‘possibly large’’ cases of the above definition are technically necessary, but there are situations where they can be ignored:

Observation 7.8.2. *Suppose all functors* F *having a certain property* P *respect small limits* (*respectively small colimits, or small limits or colimits of a particular sort, such as products or coproducts*). *Suppose, moreover, that the condition* P *does not depend on the choice of universe; or more generally, that if we enlarge our universe, any functor that satisfied* P *relative to the old universe continues to satisfy* P *with respect to the new one. Then any functor satisfying* P *in fact respects possibly large limits* (*respectively, possibly large colimits, products, coproducts, etc.*).

Hence, in discussing properties P which are preserved under enlarging the universe, we may say that functors satisfying P ''respect limits'' etc., without specifying ''small'' or ''possibly large''. □

Since properties such as ''F: $\mathbf{C} \to \mathbf{D}$ is a left adjoint functor'' do not depend on one's choice of universe, we will be able to ignore the small / possibly-large distinction in formulating the results of this section.

Theorem 7.8.3. *Left adjoint functors respect colimits, and right adjoint functors respect limits.*

Proof. Let $\mathbf{D} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{C}$ be adjoint functors, with U the right and F the left adjoint, and suppose S: $\mathbf{E} \to \mathbf{C}$ has a colimit L, with coprojection maps q_E ($E \in \mathrm{Ob}(\mathbf{E})$). Recall that L represents the functor $\mathbf{C}^{\mathbf{E}}(S, \Delta(-))$: $\mathbf{C} \to \mathbf{Set}$, i.e., the construction taking each object $C \in \mathrm{Ob}(\mathbf{C})$ to the set of cones $\mathbf{C}^{\mathbf{E}}(S, \Delta(C))$ and acting correspondingly on morphisms, and that the cone $(q_E)_{E \in \mathrm{Ob}(\mathbf{E})}$ is the universal element for this representing object.

Applying Theorem 7.7.1, we see that $F(L)$ will represent the functor $\mathbf{D} \to \mathbf{Set}$ given by

$$\text{(7.8.4)} \qquad \mathbf{C}^{\mathbf{E}}(S, \Delta(U(-))) \;=\; \mathbf{C}^{\mathbf{E}}(S, U\Delta(-)) \;\cong\; \mathbf{D}^{\mathbf{E}}(FS, \Delta(-));$$

in other words, it will be a colimit of FS.

The universal cone could hardly be anything but $(F(q_E))$; but we need to check this formally. By Theorem 7.7.1, to get this universal element we apply to L the unit η of our adjunction, getting a morphism $\eta(L)$: $L \to UF(L)$, apply the functor $\mathbf{C}^{\mathbf{E}}(S, \Delta(-))$ to it, getting a set map

$$\mathbf{C}^{\mathbf{E}}(S, \Delta(\eta(L))): \; \mathbf{C}^{\mathbf{E}}(S, \Delta(L)) \;\to\; \mathbf{C}^{\mathbf{E}}(S, \Delta(UF(L))),$$

and apply this set map to our original universal cone. Now the above set map is given by left composition with $\eta(L)$, so it transforms our original cone (q_E) from S to L into the cone $(\eta(L)q_E)$ from S to $UF(L)$. But in (7.8.4) we identify cones from the functor S to objects $U(D)$ ($D \in \mathrm{Ob}(\mathbf{D})$) with cones from FS to the objects D by use of the given adjunction. This identification works by applying F to the given morphisms, then applying the counit of the adjunction to the codomains of the resulting morphisms. So the morphisms $\eta(L)q_E$ of our cone are first transformed to $F(\eta(L)q_E) = F(\eta(L))\,F(q_E)$, then composed on the left with $\varepsilon(F(L))$. By Theorem 7.3.3(iii), the latter morphism is left inverse to $F(\eta(L))$, so the composite is $F(q_E)$, as claimed.

The assertion about right adjoint functors and limits follows by duality. □

For example, suppose $(\mathbf{C}, U)$ is a concrete category having free objects on all sets, i.e., such that U has a left adjoint F. Then we see by applying the above theorem to appropriate colimits in $\mathbf{Set}$ that a free object in $\mathbf{C}$ on a disjoint union of sets is a coproduct of the free objects on the given sets, and that a free object on the empty set is an initial object. (These facts were noted for particular cases in Chapter 3.) The fact that right adjoints respect limits tells us, likewise, that for $\mathbf{C}$ and U as above, if we call $U(X)$ the ''underlying set'' of $X \in \mathrm{Ob}(\mathbf{C})$, then underlying sets of product objects, terminal objects, difference kernels, and inverse limits are, respectively, direct products of underlying sets, the one-element set, difference kernels of underlying sets, and inverse limits of underlying sets. This explains why, in so many familiar cases, the construction of the latter objects begins by applying the corresponding construction to underlying sets. (The perceptive reader may note that what this actually does is reduce these many facts to the one unexplained fact that the underlying set functors of the categories arising in algebra tend to have left adjoints –

though they rarely have right adjoints.)

We should next say something about how limits and colimits interact with objects that represent functors. In this form, there is not an obvious question to ask; but we can ask whether *representable functors* respect limits and colimits. However, our definition of a functor F respecting a limit or colimit assumed F covariant, so to include the case of contravariant representable functors, we need to formally extend that definition.

Definition 7.8.5. *Let* $\mathbf{C}$, $\mathbf{C}'$ *be categories, and* $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}'$ *a contravariant functor.*

Then if $S\colon \mathbf{E} \to \mathbf{C}$ *is a functor into* $\mathbf{C}$ *having a limit* $\varprojlim S$, *with projection maps* $p_E\colon \varprojlim S \to S(E)$, *one says that* F turns the limit of S into a colimit *if the object* $F(\varprojlim S)$, *together with the cone from the functor* $FS\colon \mathbf{E}^{\mathrm{op}} \to \mathbf{C}'$ *to this object given by the morphisms* $F(p_E)\colon F(S(E)) \to F(\varprojlim S)$, *is a colimit of that functor* (*equivalently, if, viewing* $\varprojlim S$ *and* (p_E) *as an object and a cone of morphisms in* $\mathbf{C}^{\mathrm{op}}$ *which comprise a colimit of the functor* $S^{\mathrm{op}}\colon \mathbf{E}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$, *the functor* F *respects this colimit*).

This yields the obvious definitions of statements such as that F *‘‘turns small limits into colimits’’, ‘‘turns possibly large products into coproducts’’, ‘‘turns pullbacks into pushouts’’, ‘‘turns terminal objects into initial objects’’, etc..*

We define analogously the concept of F *turning the* colimit *of a functor* S *into a* limit, *and thus of turning coproducts to products, pushouts to pullbacks, etc..*

We can now state

Theorem 7.8.6. *Let* $\mathbf{C}$ *be a category. Then covariant representable functors* $V\colon \mathbf{C} \to \mathbf{Set}$ *respect limits, and contravariant representable functors* $W\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *turn colimits to limits.*

Sketch of Proof. The second statement is equivalent to the first applied to the category $\mathbf{C}^{\mathrm{op}}$, so it suffices to prove the first assertion.

Without loss of generality we may take $V = h_R$ where $R \in \mathrm{Ob}(\mathbf{C})$. Let L be the limit of a functor $S\colon \mathbf{E} \to \mathbf{C}$. Then $h_R(L) = \mathbf{C}(R, L)$, which by the universal property of L can be identified with the set of cones from R to S, i.e., ‘‘compatible’’ systems $(r_E)_{E\in\mathrm{Ob}(\mathbf{E})}$ of morphisms $r_E \in \mathbf{C}(R, S(E))$. On the other hand, limits over $\mathbf{E}$ in $\mathbf{Set}$ are given by $\mathrm{Ob}(\mathbf{E})$-tuples of elements also satisfying compatibility conditions (cf. proof of Proposition 7.6.6), and we see that the compatibility conditions defining elements of $\varprojlim_{\mathbf{E}} h_R(S(E))$ agree with those defining cones $R \to S$; so $h_R(\varprojlim_{\mathbf{E}} S)$ can be naturally identified with $\varprojlim_{\mathbf{E}} h_R(S(E))$.

That h_R also carries the universal cone of morphisms from the object L to the objects $S(E)$ to the corresponding cone of morphisms from $h_R(L)$ to the $h_R(S(E))$ is straightforward to verify. □

Exercise 7.8:1. (i) Show by example that covariant representable functors $\mathbf{Ab} \to \mathbf{Set}$ need not respect colimits. In fact, give examples of failure to respect coproducts, failure to respect difference cokernels, and failure to respect direct limits over directed systems.

(ii) Similarly, show by examples that contravariant representable functors on $\mathbf{Ab}$ in general fail to turn products, difference kernels, and inverse limits into coproducts, difference cokernels and direct limits respectively.

Finally, we come to the interaction of colimits with colimits, and of limits with limits. Suppose $B\colon \mathbf{D}\times\mathbf{E} \to \mathbf{C}$ is a bifunctor. Then each object D of $\mathbf{D}$ induces a functor $B(D,-)\colon \mathbf{E} \to \mathbf{C}$, and each morphism $f\colon D \to D'$ in $\mathbf{D}$ yields a morphism of functors, $B(f,-)\colon B(D,-) \to$

$B(D', -)$. (Cf. Lemma 6.10.1 and preceding discussion.) If for each D the functor $B(D, -)$ has a colimit, let us write these objects $\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E) \in \mathrm{Ob}(\mathbf{C})$. The morphisms between functors $B(D, -)$ induce morphisms among these colimit objects (cf. (7.6.4) and preceding display), so that the construction of $\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E)$ from D becomes a functor $\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(-, E)\colon \mathbf{D} \to \mathbf{C}$. Suppose this functor in turn has a colimit, which we write $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, (\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E))$. Then the composites of coprojections

$$(7.8.7) \qquad B(D, E) \to \underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E) \to \underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, (\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E))$$

constitute a cone of morphisms from the $B(D, E)$ to our iterated colimit, and it is straightforward to verify that the latter object, together with this cone, has the universal property of $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}\times\mathbf{E}}\, B(D, E)$.

Exercise 7.8:2. Prove the above claim, that if $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, (\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E))$ exists, it satisfies the universal property of $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}\times\mathbf{E}}\, B(D, E)$.

This gives the first isomorphism in the first display of the next theorem. By symmetry, we likewise have the second isomorphism of that display if the rightmost colimit exists. The isomorphisms of the second display similarly hold under the dual hypotheses.

Theorem 7.8.8. *Colimits commute with colimits, and limits commute with limits.*

Precisely, let $B\colon \mathbf{D}\times\mathbf{E} \to \mathbf{C}$ *be a bifunctor. Then*

$$(7.8.9) \qquad \underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, (\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E)) \cong \underrightarrow{\mathrm{Lim}}_{\mathbf{D}\times\mathbf{E}}\, B(D, E) \cong \underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, (\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, B(D, E)),$$

in the sense that if the left side of the above display is defined, then this object also has the universal property of the middle object, via the cone of morphisms (7.8.7)*, and similarly, if the right side is defined, it has the property of the middle object via the analogous cone. Hence, if both sides are defined, they are isomorphic.*

Likewise

$$(7.8.10) \qquad \underleftarrow{\mathrm{Lim}}_{\mathbf{D}}\, (\underleftarrow{\mathrm{Lim}}_{\mathbf{E}}\, B(D, E)) \cong \underleftarrow{\mathrm{Lim}}_{\mathbf{D}\times\mathbf{E}}\, B(D, E) \cong \underleftarrow{\mathrm{Lim}}_{\mathbf{E}}\, (\underleftarrow{\mathrm{Lim}}_{\mathbf{D}}\, B(D, E))$$

in the same sense. □

As formulated, (7.8.9) is not an instance of a functor ''respecting'' colimits in the precise sense of Definition 7.8.1, because the minimalist hypotheses that we assumed do not make $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}$ a functor on all of $\mathbf{C}^{\mathbf{D}}$. If we in fact assume that all functors from $\mathbf{D}$ to $\mathbf{C}$ have colimits (e.g., if $\mathbf{C}$ has small colimits and $\mathbf{D}$ is small), then the isomorphism $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, (\underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, (B(D, E))) \cong \underrightarrow{\mathrm{Lim}}_{\mathbf{E}}\, (\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}\, (B(D, E)))$ becomes a case of Theorem 7.8.3, since $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}}$ becomes a left adjoint functor $\mathbf{C}^{\mathbf{D}} \to \mathbf{C}$. However, the identification of the common value of the two iterated colimits as $\underrightarrow{\mathrm{Lim}}_{\mathbf{D}\times\mathbf{E}}\, B(D, E)$ must still be stated and proved separately. (In the same spirit, if $\mathbf{C}$ has small coproducts, the covariant case of Theorem 7.8.6 follows from Theorem 7.8.3 and Exercise 7.3:3.)

The case of (7.8.9) where $\mathbf{E}$ is the empty category says that colimits respect initial objects; i.e., that if I is an initial object of $\mathbf{C}$, then the diagram $\Delta(I) \in \mathbf{C}^{\mathbf{D}}$ has colimit I. For instance, the coproduct in $\mathbf{Ring}^1$ of two copies of $\mathbf{Z}$ is again $\mathbf{Z}$. The next exercise examines variants of this result.

Exercise 7.8:3. (i) Show, conversely, that if an object I of a category $\mathbf{C}$ has the property that for all small categories $\mathbf{D}$, the functor $\Delta(I) \in \mathbf{C}^{\mathbf{D}}$ has a colimit isomorphic to I, then I is an initial object of $\mathbf{C}$.

(ii) Can you characterize those objects I of a category **C** for which the hypothesis of (i) holds for all *nonempty* small categories **D**?

(iii) Show that in $\mathbf{Ring}^1$ (or if you prefer, $\mathbf{CommRing}^1$), every ring of the form $\mathbf{Z}_n$ has the property of (ii).

Despite the similar nomenclature, category-theoretic double limits behave quite differently from double limits in topology. The contrast is explored in

Exercise 7.8:4. (i) For nonnegative integers i, j, define b_{ij} to be 1 if $i > j$, 2 if $i \leq j$. Show that as limits of real-valued functions, $\lim_{i\to\infty}(\lim_{j\to\infty} b_{ij})$ and $\lim_{j\to\infty}(\lim_{i\to\infty} b_{ij})$ exist and are unequal.

(ii) Let the set $\omega\times\omega$ be partially ordered by setting $(i,j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. Show that there exist functors (directed systems) $B\colon (\omega\times\omega)_{\mathbf{cat}} \to \mathbf{Set}$ satisfying $\mathrm{card}(B(i,j)) = b_{ij}$, for b_{ij} as in (i).

(iii) Deduce from Theorem 7.8.8 that a functor as in (ii) can never have the property that for each i, the morphisms $B(i,j) \to B(i,j{+}1)$ and $B(i,j) \to B(i{+}1,j)$ are isomorphisms for all sufficiently large j.

(iv) Establish the result of (iii) directly, without using the concept of category-theoretic limit.

In earlier sections, there were several exercises asking you to determine whether functors were representable or had right or left adjoints. If you go back over the cases where the functors turned out *not* to be representable or, not to have an adjoint, you will find that, whatever ad hoc arguments you may have used at the time, each of these negative results can be deduced from Theorem 7.8.6 or 7.8.3 by noting that the functor in question fails to respect some limit or colimit.

Since limits and colimits come in many shapes and sizes, it is useful to note that to test whether a functor respects these constructions, it suffices to check two basic cases.

Corollary 7.8.11 (to proof of Proposition 7.6.6)**.** *Let* **C**, **D** *be categories and* $F\colon \mathbf{C} \to \mathbf{D}$ *a functor.*

If **C** *has small colimits, then* F *respects such colimits if and only if it respects difference cokernels and respects coproducts of small families of objects.*

Likewise, if **C** *has small limits, then* F *respects these if and only if it respects difference kernels and respects products of small families.* □

One can break things down further, if one wishes:

Exercise 7.8:5. (i) Let **C** be a category having coproducts of pairs of objects, and hence of finite nonempty families of objects. Show that the universal property of a coproduct of an arbitrary family $\coprod_I X_i$ is equivalent to that of a direct limit, over the directed partially ordered set of finite nonempty subsets $I_0 \subseteq I$, of the finite coproducts $\coprod_{I_0} X_i$.

(ii) Deduce that a category has small colimits if and only if it has difference cokernels, pairwise coproducts, and colimits over directed partially ordered sets; and that a functor on such a category will respect small colimits if and only if it respects those three constructions.

State the corresponding result for *limits*.

(iii) For every two of the three conditions ‘‘respects difference kernels’’, ‘‘respects pairwise products’’, ‘‘respects inverse limits over inversely directed partially ordered sets’’ (the conditions occurring in the dual to the result of (ii)), try to find an example of a functor among categories having small limits which satisfies those two conditions but not the third. As far as possible, use naturally occurring examples.

You might look at further similar questions; e.g., whether you can find an example respecting both finite and infinite products, but not inverse limits over inversely directed partially ordered sets; or whether you can still get a full set of examples if you break the two cases of ‘‘finite products and inverse limits’’ into the three cases of *nonempty* finite products, inverse limits over *nonempty* inversely directed partially ordered sets, and the *empty* limit.

One can go into this more deeply. I do not know the answers to most of the questions raised in

Exercise 7.8:6. Let A denote the (large) set of all small categories, and B the (large) set of all legitimate categories. Define a relation $R \subseteq A \times B$ by putting $(\mathbf{E},\ \mathbf{C}) \in R$ if all functors $\mathbf{E} \to \mathbf{C}$ have colimits.

(i) The above relation induces a Galois connection between A and B. Translate results proved about existence of colimits in Proposition 7.6.6 and part (ii) of the preceding exercise into statements about the closure operator $**$ on A.

(ii) Investigate further the properties of the lattice of closed subsets of A. Is it finite, or infinite? Can you characterize the induced closure operator on the subclass of A or of B consisting of categories $P_{\mathbf{cat}}$ for partially ordered sets P?

The above questions concerned *existence* of colimits. To study preservation of colimits, let C denote the class of functors F whose domain and codomain are legitimate categories having small colimits, and let us define a relation $S \subseteq A \times C$ by putting $(\mathbf{E}, F) \in S$ if $F\colon \mathbf{C} \to \mathbf{D}$ respects the colimits of all functors $\mathbf{E} \to \mathbf{C}$.

(iii) The above relation induces a Galois connection between A and C. Can you obtain results relating the lattice of closed subsets of A under this Galois connection and the lattice of subsets of A closed under the Galois connection of part (i)? If they are not identical, investigate the structure of this new lattice. (You will have to use a notation that distinguishes between these two Galois connections.)

In studying situations where we do not know whether one functor respects the limit of another, but where the two limits in question both exist, there is a useful way to compare them:

Definition 7.8.12. *If* $\mathbf{E} \xrightarrow{S} \mathbf{C} \xrightarrow{F} \mathbf{D}$ *are functors such that* $\underrightarrow{\mathrm{Lim}}\ S$ *and* $\underrightarrow{\mathrm{Lim}}\ FS$ *both exist, then by the* comparison morphism

$$\underrightarrow{\mathrm{Lim}}\ FS \ \to\ F(\underrightarrow{\mathrm{Lim}}\ S)$$

we shall mean the unique morphism from the former to the latter object which makes a commuting diagram with the natural cones of maps from the functor FS *to these two objects (namely, the universal cone consisting of the coprojection maps from the objects* $FS(E)$ *to the colimit object* $\underrightarrow{\mathrm{Lim}}\ FS$*, and the cone from these same objects to* $F(\underrightarrow{\mathrm{Lim}}\ S)$ *obtained by applying* F *to the coprojection maps from the* $S(E)$ *to* $\underrightarrow{\mathrm{Lim}}\ S$*).*

Likewise, if $\underleftarrow{\mathrm{Lim}}\ S$ *and* $\underleftarrow{\mathrm{Lim}}\ FS$ *both exist, then by the comparison morphism*

$$F(\underleftarrow{\mathrm{Lim}}\ S) \ \to\ \underleftarrow{\mathrm{Lim}}\ FS$$

we shall mean the unique morphism which makes a commuting diagram with the obvious cones from these two objects to the functor FS*.*

In particular, the term ‘‘comparison morphism’’ will be used with respect to coproducts, products, difference cokernels, difference kernels, etc., regarding these as colimits and limits.

It is clear that these comparison morphisms measure whether the functor F respects these colimits and limits, i.e.,

Lemma 7.8.13. *In the context of the first paragraph of the preceding definition, the functor* F *respects the colimit of* S *if and only if the comparison morphism* $\varinjlim FS \to F(\varinjlim S)$ *is an isomorphism. In the context of the second paragraph,* F *respects the limit of* S *if and only if the comparison morphism* $F(\varprojlim S) \to \varprojlim FS$ *is an isomorphism.* □

Exercise 7.8:7. Suppose $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ are categories such that $\mathbf{C}$ has colimits of all functors $\mathbf{D} \to \mathbf{C}$, and also of all functors $\mathbf{E} \to \mathbf{C}$, so that $\varinjlim_{\mathbf{D}}$ becomes a functor $\mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ and $\varinjlim_{\mathbf{E}}$ a functor $\mathbf{C}^{\mathbf{E}} \to \mathbf{C}$. Show that for any bifunctor $B\colon \mathbf{D} \times \mathbf{E} \to \mathbf{C}$, the above definition yields comparison morphisms $\varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E)) \to \varinjlim_{\mathbf{E}} (\varinjlim_{\mathbf{D}} B(D, E))$ and also $\varinjlim_{\mathbf{E}} (\varinjlim_{\mathbf{D}} B(D, E)) \to \varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E))$, and that these are *inverse* to one another. This gives yet another proof of the isomorphism between the two sides of (7.8.9) under these hypotheses.

Earlier in this section, I said that there was no obvious way to talk about limits or colimits ''respecting'' the construction of objects representing functors, and we looked instead at the subject of representable functors respecting limits and colimits. But there are actually some not-so-obvious results one can get on limits and colimits of objects that represent functors. Conveniently, these reduce to statements that certain functors respect limits and colimits. You can develop these in

Exercise 7.8:8. (i) Show that the covariant Yoneda embedding $\mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ respects small limits, and that the contravariant Yoneda embedding $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$ turns small colimits into limits. (Idea: combine Lemma 7.6.8 and Theorem 7.8.6.)

(ii) Turn the above results into statements on the representability of set-valued functors which are limits or colimits of other representable functors, and characterizations of the objects that represent these.

(iii) Deduce the characterization, noted near the beginning of §3.6, of pairwise coproducts of groups defined by presentations, and the assertion of Exercise 7.5:6, that every group is a direct limit of finitely presented groups.

(iv) Show by example that the covariant Yoneda embedding of a category need not respect small colimits, and that the contravariant Yoneda embedding need not turn small colimits into limits.

(v) Suppose $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ are categories, with $\mathbf{E}$ small, and $U\colon \mathbf{E} \to \mathbf{C}^{\mathbf{D}}$ a functor such that each of the functors $U(E)\colon \mathbf{D} \to \mathbf{C}$ has a left adjoint $F(E)$. Under appropriate assumptions on existence of small limits and/or colimits in one or more of these categories, deduce from preceding parts of this exercise that $\varprojlim_{\mathbf{E}} U(E)$ exists (as an object of $\mathbf{C}^{\mathbf{D}}$), and (as a functor $\mathbf{D} \to \mathbf{C}$) has a left adjoint, constructible from the $F(E)$.

(vi) Show by example that the analogous statement about *colimits* of functors which have left adjoints is false.

7.9. Interaction between limits and colimits.

Since limits are right universal constructions and colimits are left universal, these two sorts of constructions cannot be expected to respect one another in general. However, there are important cases where they do. For instance, we observed in §7.5 (and will prove formally in the next chapter) that one can form the direct limit of any directed system of algebras with finitary operations by taking the direct limit of their underlying sets, and putting operations on this set in a natural manner. The essential reason for this is that algebra structures are given by operations $|A| \times ... \times |A| \to |A|$ on sets, and that in $\mathbf{Set}$, direct limits commute with finite products – although direct limits are colimits, and products are limits.

When we ask whether a given limit and a given colimit commute, there are potentially two comparison morphisms to consider, one a case of the comparison morphism for a limit and a general functor, the other of the comparison morphism for a colimit and a general functor. A priori, one of these might be an isomorphism and the other not, or they might give different isomorphisms between the same objects. Fortunately, these anomalies cannot occur; as we shall now prove, the two comparison morphisms coincide. (Note that these morphisms go in the same direction, because the comparison morphism for limits goes into the global limit object, while the comparison morphism for colimits comes out of the global colimit object. For the case of the interaction between limits and limits or between colimits and colimits, on the other hand, the two comparison morphisms go in opposite directions; cf. Exercise 7.8:7.)

Lemma 7.9.1. *Suppose* $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{E}$ *are categories such that* $\mathbf{C}$ *has* colimits *of all functors with domain* $\mathbf{D}$, *and* limits *of all functors with domain* $\mathbf{E}$, *and let* $B\colon \mathbf{D}\times\mathbf{E} \to \mathbf{C}$ *be a bifunctor. Then the two comparison morphisms*

$$\varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E) \to \varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$$

coincide, their common value being characterizable as the unique morphism c_B *such that for every* $D_0 \in \mathrm{Ob}(\mathbf{D})$ *and* $E_0 \in \mathrm{Ob}(\mathbf{E})$, *the following diagram commutes:*

$$(7.9.2)\qquad \begin{tikzcd} \varprojlim_{\mathbf{E}} B(D_0, E) \arrow[r, "p(D_0{,}E_0)"] \arrow[d, "q(D_0)"] & B(D_0, E_0) \arrow[r, "q(D_0{,}E_0)"] & \varinjlim_{\mathbf{D}} B(D, E_0) \\ \varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E) \arrow[rr, "c_B"] & & \varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E). \arrow[u, "p(E_0)"'] \end{tikzcd}$$

Here $p(D_0, E_0)$ *and* $p(E_0)$ *denote the* E_0*th projection maps of the respective limits* $\varprojlim_{\mathbf{E}} B(D_0, E)$ *and* $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$, *and* $q(D_0, E_0)$, $q(D_0)$ *the* D_0*th coprojections of the colimits* $\varinjlim_{\mathbf{D}} B(D, E_0)$ *and* $\varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$.

Proof. Let c_B denote the comparison map between the objects at the bottom of (7.9.2) which tests whether $\varprojlim_{\mathbf{E}} : \mathbf{C}^{\mathbf{E}} \to \mathbf{C}$, regarded as a functor (and not specifically as a limit), respects the indicated colimit over $\mathbf{D}$. We shall verify that this is the unique morphism making that family of diagrams commute. The dual argument shows the same for the other comparison map, proving the lemma.

The defining property of the colimit-comparison morphism c_B is that it respect the cones from the family of objects $\varprojlim_{\mathbf{E}} B(D_0, E)$ $(D_0 \in \mathrm{Ob}(\mathbf{D}))$ to the two objects in the bottom line of (7.9.2), where the cone to the left-hand object is the universal one, consisting of the left-hand vertical arrows of the diagram, while the cone to the right-hand object consists of morphisms going diagonally across the diagram, the map for each D_0 being obtained by applying $\varprojlim_{\mathbf{E}}(-, E)$ to the family of coprojection maps $(q(D_0, E))_{E \in \mathrm{Ob}(\mathbf{E})}$. Now when we apply $\varprojlim_{\mathbf{E}}(-, E)$ to such a family, the resulting morphism is characterized by the condition that for each E_0, it form a commuting square with the projection maps to the objects indexed by E_0 (cf. 7.6.5). In our case this means that for all E_0, our diagonal map should commute with the top and right-hand arrows of (7.9.2). Hence c_B makes (7.9.2) commute for all D_0 and E_0, and we see from the universal properties involved that it will be the unique morphism with this property. □

Before proving that in certain cases the above comparison morphism is an isomorphism, let us note some easy examples where it is not.

Exercise 7.9:1. Let $\mathbf{D}$ and $\mathbf{E}$ each be the category with two objects, 0 and 1, and no morphisms other than identity morphisms.

(i) Suppose L is a lattice, $U(L)$ its underlying partially ordered set, and $\mathbf{C} = U(L)_{\mathbf{cat}}$. For these choices of $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$, say what it means to give a functor B as in Lemma 7.9.1, verify that the indicated limits and colimits exist, and identify the morphism c_B of the lemma. Show that even if $\mathbf{C}$ is the 2-element lattice, this morphism can fail to be an isomorphism.

(ii) Analyze similarly the case where $\mathbf{C} = \mathbf{Set}$, and $\mathbf{D}$ and $\mathbf{E}$ are as above.

Here, however, is a positive result, generalizing our earlier claim about direct limits of finite products in $\mathbf{Set}$. Note that the proof involves chasing elements; we shall see that the corresponding result with $\mathbf{Set}$ replaced by a general category $\mathbf{C}$ is not true. In thinking about what the result says, you might begin with the cases where $\mathbf{D} = \omega_{\mathbf{cat}}$ (where ω is the partially ordered set of natural numbers), and $\mathbf{E}$ is one of the two- or three-object categories such that limits over $\mathbf{E}$ are difference kernels or pullbacks, or the one-object category $G_{\mathbf{cat}}$ for G a finitely generated group.

Let us note a convenient piece of notation that will be used in the proof. If $B\colon \mathbf{D}\times\mathbf{E} \to \mathbf{C}$ is a bifunctor, D an object of $\mathbf{D}$, and $f\colon E_1 \to E_2$ a morphism of $\mathbf{E}$, then one often writes $B(D,\ f)$ for the induced morphism $B(D,\ E_1) \to B(D,\ E_2)$, which is, strictly, $B(\mathrm{id}_D,\ f)$. Similarly, given a morphism g of $\mathbf{D}$ and an object E of $\mathbf{E}$, one may write $B(g, E)$ for $B(g, \mathrm{id}_E)$.

Proposition 7.9.3. *If* $\mathbf{D}$ *is a category of the form* $P_{\mathbf{cat}}$, *for* P *a directed partially ordered set, and* $\mathbf{E}$ *is a nonempty category which has only finitely many objects, and whose morphism-set is finitely generated under composition, then for any bifunctor* $B\colon \mathbf{D}\times\mathbf{E} \to \mathbf{Set}$, *the morphism* c_B *of Lemma 7.9.1 is an isomorphism.* (*Briefly: ‘‘In* $\mathbf{Set}$, *direct limits commute with finite limits.’’*)

Proof. Let $E_0,\ \ldots\ ,\ E_{m-1}$ be the objects of $\mathbf{E}$, and $f_0,\ \ldots\ ,\ f_{n-1}$ a generating set for the morphisms of $\mathbf{E}$, with $f_j \in \mathbf{E}(E_{u(j)}, E_{v(j)})$. Given elements $D \leq D'$ in the partially ordered set P, let us write $g(D,\ D')$ for the unique morphism $D \to D'$ in $P_{\mathbf{cat}} = \mathbf{D}$. Projection and coprojection morphisms will be named as in (7.9.2).

To show surjectivity of c_B, let x be any element of $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D,\ E)$. For each of the finitely many objects E_i of $\mathbf{E}$, consider $p(E_i)(x) \in \varinjlim_{\mathbf{D}} B(D,\ E_i)$. By the construction of direct limits in $\mathbf{Set}$, there must exist $D(i) \in P = \mathrm{Ob}(\mathbf{D})$ such that the above element arises from some $x_i \in B(D(i), E_i)$, i.e.,

$$q(D(i), E_i)(x_i) \ = \ p(E_i)(x) \qquad (i = 0, \ldots, m{-}1).$$

Since the partially ordered set P is directed, we can find $D_0 \in P$ majorizing all the $D(i)$. Thus we have images of all the x_i at the ‘‘D_0 level’’; let us denote these

$$x'_i \ = \ B(g(D(i),\ D_0),\ E_i)(x_i) \in B(D_0, E_i) \qquad (i = 0, \ldots, m{-}1).$$

Thus,

$$q(D_0, E_i)(x'_i) \ = \ p(E_i)(x) \qquad (i = 0, \ldots, m{-}1). \tag{7.9.4}$$

Now the definition of $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D,\ E)$ as a limit tells us that the system of elements on the right-hand side of (7.9.4) is ‘‘respected’’ by all morphisms of $\mathbf{E}$, equivalently, by the generating family of morphisms f_j. That is,

$$\varinjlim_{\mathbf{D}} B(D, f_j)\colon\ \varinjlim_{\mathbf{D}} B(D, E_{u(j)}) \ \to\ \varinjlim_{\mathbf{D}} B(D, E_{v(j)}) \tag{7.9.5}$$
$$\text{carries}\ \ p(E_{u(j)})(x)\ \ \text{to}\ \ p(E_{v(j)})(x) \qquad (j = 0, \ldots, n\text{–}1).$$

It is not necessarily true that the system of preimages $x'_i \in B(D_0, E_i)$ that we have found for these elements satisfy the corresponding relations, i.e., that $B(D_0, f_j)$ carries $x'_{u(j)}$ to $x'_{v(j)}$; but by the construction of direct limits in **Set**, we can deduce from (7.9.5) that for each j, there is some $D'(j) \geq D_0$ such that the corresponding result holds, namely

$$B(D'(j), f_j)\,(B(g(D_0, D'(j)), E_{u(j)})(x'_{u(j)})) \ = \ B(g(D_0, D'(j)), E_{v(j)})(x'_{v(j)}) \quad (j = 0, \ldots, n\text{–}1).$$

Hence taking D_1 majorizing all the $D'(j)$'s, and letting x''_i denote $B(g(D_0, D_1), E_i)(x'_i) \in B(D_1, E_i)$ for $i = 0, \ldots, m$–1, we have the desired ''lifting'' of the system of equations (7.9.5):

$$B(D_1, f_j)(x''_{u(j)}) \ = \ x''_{v(j)} \qquad (j = 0, \ldots, n\text{–}1).$$

That is, the f's respect the x''_i. Hence, since every morphism of $\mathbf{E}$ is a composite of the f_j, every morphism of $\mathbf{E}$ respects the x''_i; so the x''_i define an element $x'' \in \varprojlim_{\mathbf{E}} B(D_1, E)$. The element $q(D_1)(x'') \in \varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$ is the required inverse image of x under c_B. (Cf. (7.9.2).)

The proof that c_B is one-to-one is similar, but easier; indeed, it does not need the finite generation hypothesis on the morphisms of $\mathbf{E}$, but only the finiteness of the object-set. Suppose $x,\ y \in \varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$ with $c_B(x) = c_B(y)$. Since $\mathbf{D}$ is directed, there will exist $D_0 \in \mathrm{Ob}(\mathbf{D})$ such we can write x and y as the images of some $x_0,\ y_0 \in \varprojlim_{\mathbf{E}} B(D_0, E)$. By assumption, these elements fall together when mapped into $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$, which means that for each i, the projections $p(D_0, E_i)(x_0)$ and $p(D_0, E_i)(y_0)$ fall together in $\varinjlim_{\mathbf{D}} B(D, E_i)$. By the construction of direct limits in **Set**, this means that for each i there is some $D(i) \geq D_0$ such that the images of these elements already agree in $B(D(i), E_i)$. Let $D_1 \in \mathrm{Ob}(\mathbf{D})$ majorize all these $D(i)$. Thus the images of x_0 and y_0 fall together in all the $B(D_1, E_i)$, hence in $\varprojlim_{\mathbf{E}} B(D_1, E)$, hence in $\varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$, i.e., $x = y$. □

Exercise 7.9:2. Show that the above proposition remains true if the condition that $\mathbf{E}$ be nonempty is replaced by the condition that $\mathbf{D}$ be nonempty, but fails when both are empty. The proof we gave for the proposition does not explicitly refer to the nonemptiness of $\mathbf{E}$; where is it used implicitly? (Note that a statement that something is true ''for all E_i'' does not require that the set of E_i be nonempty – it is vacuously true if the set is empty. So you need to find something less obvious than that.)

In the above proposition, neither the assumption that $\mathbf{E}$ has finite object-set nor the assumption that its morphism-set is finitely generated can be dropped:

Exercise 7.9:3. (i) Show that direct limits in **Set** do not commute with infinite products. In fact, give examples both of failure of one-one-ness and of failure of surjectivity.

Now, a product over a set X is a limit over the category $X_{\mathbf{cat}}$ having object-set X and only identity morphisms; thus, the morphism-set of that category may be regarded as generated by the empty set. Hence in the examples you constructed for (i), $\mathbf{E}$ has infinite object-set, but finitely generated morphism-set.

(ii) To show that finite generation of the morphism-set cannot be dropped either, let $\mathbf{E} = G_{\mathbf{cat}}$ for G a non-finitely-generated group, and let P be the partially ordered set of all finitely generated subgroups $H \subseteq G$. Take the direct limit over P of the G-sets G/H, examine the action of $\varprojlim_{\mathbf{E}}$ on this direct limit, and show that this gives the desired counterexample.

A more difficult question is

(iii) Is it also true that for every non-finitely-generated monoid S, there is a directed partially ordered set P and a bifunctor $B\colon P_{\mathbf{cat}} \times S_{\mathbf{cat}} \to \mathbf{Set}$ such that c_B is not invertible?

The above examples show the need for our hypotheses on $\mathbf{E}$. What about the condition that $\mathbf{D}$ have the form $P_{\mathbf{cat}}$ for P a directed partially ordered set? A simple example of a partially ordered set that is not directed is $\begin{tikzcd}[sep=tiny] \bullet \arrow[dr, dash] & & \bullet \arrow[dl, dash] \\ & \bullet & \end{tikzcd}$, while some examples of categories not of the form $P_{\mathbf{cat}}$ for any partially ordered set are the two-object category $\cdot \rightrightarrows \cdot$ and the one-object category $\mathbf{Z}_{\mathbf{cat}}$, where $\mathbf{Z}$ is the infinite cyclic group. So

Exercise 7.9:4. Give examples showing that the fixed-point-set construction on $\mathbf{Z}$-sets (a limit over a one-object category with finitely generated morphism set) respects neither pushouts, nor difference cokernels, nor orbit-sets of actions of $\mathbf{Z}$.

Finally, part (i) of the next exercise shows that we cannot interchange the hypotheses on $\mathbf{D}$ and $\mathbf{E}$. As one can see from part (iii), this is equivalent to saying that Proposition 7.9.3 does not remain true if we replace the category $\mathbf{Set}$ by $\mathbf{Set}^{\mathrm{op}}$; in particular, we cannot replace $\mathbf{Set}$ in that proposition by a general category having small limits and colimits.

Exercise 7.9:5. (i) Show that inverse limits in $\mathbf{Set}$ do *not* commute with difference cokernels.

(ii) Show, on the other hand, that inverse limits in $\mathbf{Set}$ *do* commute with small coproducts.

(iii) Translate the results of (i) and (ii) into statements about constructions in $\mathbf{Set}^{\mathrm{op}}$.

We noted in the last paragraph of the proof of Proposition 7.9.3 that the one-one-ness part of the conclusion did not require finite generation of the morphism set of $\mathbf{E}$. It also did not require the non-emptiness assumption on the object-set; moreover, even the assumption that the object-set was finite can be weakened, using the idea of Lemma 7.6.2, to say that it contains a ‘‘good’’ finite subset. Thus, you can easily verify

Corollary 7.9.6 (to proofs of Proposition 7.9.3 and Lemma 7.6.2)**.** *Let $\mathbf{D}$ be a category of the form $P_{\mathbf{cat}}$, for P a directed partially ordered set, and let $\mathbf{E}$ be a category with only finitely many objects, or more generally, having a finite family of objects $E_0, \dots, E_{m-1}$ such that every object E admits a morphism $E_i \to E$ for some i. Then for any bifunctor $B\colon \mathbf{D} \times \mathbf{E} \to \mathbf{Set}$, the comparison morphism c_B of Lemma 7.9.1 is one-to-one.* □

Let us also note that the role of *finiteness* in the above considerations is easily generalized. The reader will find that under the next definition, the proofs of our proposition and corollary yield the proposition stated below.

Definition 7.9.7. *If α is a cardinal and P a partially ordered set, then P will be called* α-directed *if every subset of P of cardinality $<\alpha$ has an upper bound in P.*

Proposition 7.9.8. *Let α be an infinite cardinal. If $\mathbf{D}$ is a category of the form $P_{\mathbf{cat}}$, for P an α-directed partially ordered set, and $\mathbf{E}$ is a nonempty category which has $<\alpha$ objects, and whose morphism-set is generated under composition by a set of fewer than α morphisms* (*which, except in the case $\alpha = \omega$, is equivalent to saying that $\mathbf{E}$ has $<\alpha$ morphisms*)*, then for any bifunctor $B\colon \mathbf{D} \times \mathbf{E} \to \mathbf{Set}$, the morphism c_B of Lemma 7.9.1 is an isomorphism.* (*Briefly: ‘‘In* **Set**, *α-directed direct limits commute with limits over $<\alpha$-generated categories.’’*)

Further, the one-one-ness of c_B remains true if we weaken the hypothesis on $\mathbf{E}$ to say that there is a set S of $<\alpha$ objects of $\mathbf{E}$ such that every object of $\mathbf{E}$ admits a morphism from a member of S. □

Here are a few more exercises on commuting limits and colimits, some of them open-ended.

Exercise 7.9:6. Generalizing part (ii) of Exercise7.9:5, determine the class of all small categories $\mathbf{E}$ such that limits over $\mathbf{E}$ in $\mathbf{Set}$ commute with coproducts.

Exercise 7.9:7. Let G be a group or monoid, let $(X_i)_{i\in P}$ be an inverse system of G-sets, and let $c_X: \varinjlim_{G_{\mathbf{cat}}} \varprojlim_{i\in P} X_i \to \varprojlim_{i\in P} \varinjlim_{G_{\mathbf{cat}}} X_i$ be the associated comparison morphism. (In the case where G is a group, recall that $\varinjlim_{G_{\mathbf{cat}}}$ is the *orbit-set* construction of Exercise 7.6:1.)

(i) Show that if G is a group and P is countable, then c_X is surjective. (Hint: Use Exercise 7.5:5(ii).)

(ii) Does the result of (i) remain true for G a monoid? For P not necessarily countable? If either of these generalizations fails, can you find any additional conditions under which it again becomes true?

Exercise 7.9:8. (i) In the spirit of Exercise 7.8:6, investigate the Galois connection between small categories $\mathbf{D}$ and small categories $\mathbf{E}$ determined by the relation ''colimits over $\mathbf{D}$ commute with limits over $\mathbf{E}$ in **Set**''.

(ii) Investigate the Galois connections (still on the class of all small categories) obtained by replacing ''**Set**'' in (i) with various other natural categories; e.g., **Ab**.

We have been considering the interaction between limits and colimits. One can also look at the interaction between *limits* and *left adjoint* functors, and between right adjoint functors and colimits. For example

Exercise 7.9:9. Does the abelianization functor $(\ )^{\mathrm{ab}}: \mathbf{Group} \to \mathbf{Ab}$ respect inverse limits? Products? Difference kernels? In each case where the answer is negative, is it one-one-ness, surjectivity, or both properties of the comparison morphism that can fail?

A different sort of comparison morphism is noted in

Exercise 7.9:10. Given functors $\mathbf{D} \xrightarrow{F} \mathbf{E} \xrightarrow{S} \mathbf{C}$ such that S and SF both have colimits in $\mathbf{C}$, describe a natural morphism (in one direction or the other) between these objects, and obtain results about these morphisms, in general or under special hypotheses. (Cf. Exercise 7.5:1.)

7.10. Some existence theorems. Basic results on the existence of *algebras* having various universal properties must wait for the next chapter, where we will set up a general theory of algebras. What we can prove before then are relative results, to the effect that if in a category one can perform certain universal constructions, then one can perform others; for instance, Proposition 7.6.6 was of this sort. With this limitation in mind, can we abstract any of the methods by which we proved the existence of *free groups* in Chapter 2?

The construction by *terms* modulo consequences of the *identities* clearly depends on the fact that one is considering algebras. Generalizing this will be one of the first things we do in Chapter 8.

The *normal form* description is still more specialized. As mentioned toward the end of §2.4, different sorts of algebras vary widely as to whether such results hold. I hope to develop some methods for obtaining normal forms in a later (as yet unwritten) chapter; we are not ready to do anything along that line yet.

But the *subobject of a big direct product* approach of §2.3 seems amenable to a category-theoretic development, and we shall in fact obtain below several results that have evolved from that construction. The approach is due to Peter Freyd.

We know how to translate the concept of direct product into category theoretic terms. There were two other key ideas in the construction of §2.3: a cardinality estimate, which allowed us to find a *small* set of groups to take the direct product of, and the passage to ‘‘the subgroup of the product generated by the given family’’. The first of these will simply be made a hypothesis – that there exists a small set of objects with an appropriate property. What about the concept of ‘‘subalgebra generated’’? We know that there is not a canonical concept of ‘‘subobject’’ in category theory, but is there one that is appropriate to this proof?

We saw at various points in Chapters 2 and 3 that if we had an object satisfying one of our left universal properties, except, possibly, for the *uniqueness* of the factoring maps, then the added condition of uniqueness was equivalent to the object being generated by the appropriate set (e.g., Exercise 2.1:2, and end of proof of Proposition 3.3.3). To put things negatively, in the case of the universal property of a free group on X, we saw in Exercise 2.1:1 that if our candidate for a free group F was not generated by the image of X, then we could get a pair of group homomorphisms from F into some group which agreed on the elements of X, but were *not* equal on all of F. This suggests that the subgroup generated by X may be obtainable as a *difference kernel*, using pairs of morphisms having equal composites with the image of X. That is the idea which we shall abstract below.

Recapitulating the path of the first half of this chapter, let us start with an existence result for initial objects. In reading the next lemma and its proof, you might think of the case where $\mathbf{C}$ is the category of 4-tuples (G, a, b, c) with G a group and $a, b, c \in |G|$, and of the principle that guided us to the subgroup-of-a-product construction, that if one such object (G, a, b, c) is mappable to another such object (H, a', b', c'), then the set of relations satisfied by a, b, c in G is contained in the set of relations satisfied by a', b', c' in H.

Lemma 7.10.1. *Let* $\mathbf{C}$ *be a* (*legitimate*) *category having small limits. Suppose there exists a small set of objects* $S \subseteq \mathrm{Ob}(\mathbf{C})$, *such that for every* $X \in \mathrm{Ob}(\mathbf{C})$ *there is a* $Y \in S$ *with* $\mathbf{C}(Y, X)$ *nonempty. Then* $\mathbf{C}$ *has an initial object.*

Proof. Let $J = \prod_{Y \in S} Y \in \mathrm{Ob}(\mathbf{C})$. For every $X \in \mathrm{Ob}(\mathbf{C})$ there is at least one morphism from J to X, since we can compose the projection of J to some $Y \in S$ with a morphism $Y \to X$. Hence our hypothesis on the set of objects S has been concentrated in this one object J, and we may henceforth forget S and work with J.

We wish to form the ‘‘intersection of the difference kernels of all pairs of maps from J into objects of $\mathbf{C}$’’. If we were working in a category of algebras, this would make sense, for even though all such pairs of maps do not form a small set, the underlying set of J would be small, and hence the set of subobjects that are difference kernels of such pairs of maps would be small, and we could take its intersection. That argument is not available here; but it turns out that, just as we were able to use the family S as a substitute for the class of ‘‘all objects’’ in forming our product J, so it will also serve as a substitute for the class of all objects in this second capacity, though a less obvious argument will be needed. However, since the hypothesis on S has been concentrated in the object J, let us again use J in place of S in this function.

So let us form a product $\prod_{(u,v)} J$ of copies of J indexed by the set of all pairs of morphisms $u, v \in \mathbf{C}(J, J)$. (Since $\mathbf{C}$ is legitimate, such pairs form a small set.) Let $a, b \colon J \rightrightarrows \prod_{(u,v)} J$ be defined by the conditions that for all $u, v \in \mathbf{C}(J, J)$, a followed by the projection of the product onto the (u, v) component gives u, while b followed by that projection gives v. Let us form the difference kernel $i \colon I \to J$ of this pair of morphisms. Note that by the universal property of I, $ui = vi$ for any two endomorphisms u, v of J.

Since J can be mapped to every object of $\mathbf{C}$, we can find a morphism $x\colon J\to I$. Now suppose c is any endomorphism of I. By our preceding observation, the morphisms $I\to J$ given by i, ixi, and $icxi$ are equal. But by Lemma 7.6.2, i is a monomorphism; hence we can cancel it on the left and conclude that id_I, xi, and cxi are equal. Substituting the equation $xi=\mathrm{id}_I$ into $cxi=xi$, we get $c=\mathrm{id}_I$; i.e., I has no nonidentity endomorphisms.

I also inherits from J the property of having morphisms into every object of $\mathbf{C}$, so we can now forget J and work with I only.

We claim that I is an initial object of $\mathbf{C}$. We know it has morphisms into every $X\in\mathrm{Ob}(\mathbf{C})$; consider two such morphisms $u,v\in\mathbf{C}(I,X)$. We may form their difference kernel, $k\colon K\to I$, and take an arbitrary morphism the other way, $d\colon I\to K$. Then kd is an endomorphism of I, hence it is the identity. By choice of k, $uk=vk$, hence $ukd=vkd$, i.e., $u=v$; so I has exactly one morphism into each object of $\mathbf{C}$, as claimed. □

Exercise 7.10:1. The final part of the proof of the above lemma used the facts that (a) the object I of $\mathbf{C}$ had morphisms into all objects, (b) I had no nonidentity endomorphism, and (c) $\mathbf{C}$ had difference kernels. Do (a) and (b) alone imply that I is initial in $\mathbf{C}$?

For some perspective on the above result, recall Exercise 7.6:5, which showed that an initial object of a category $\mathbf{C}$ is equivalent to a *colimit* of the unique functor from the empty category to $\mathbf{C}$, and also to a *limit* of the *identity functor* of $\mathbf{C}$. Now in the study of categories of algebraic objects (for instance, the category of groups with 3-tuples of distinguished elements), one does not have, to begin with, any easy way of constructing colimits, even for as trivial a functor as the one from the empty category! One can, however, construct products and difference kernels using the corresponding constructions on the underlying sets of one’s algebras; hence one can get all *small* limits. Thus, it is not much more unreasonable to try to construct an initial object as a limit of the identity functor of the whole category than it is to try to construct it as a colimit of the unique functor from the empty category! The one difficulty is that the domain of the identity functor of $\mathbf{C}$ is not small. Hence one looks for a small set S of objects of $\mathbf{C}$ which ‘‘get around enough’’ to serve in place of the set of *all* objects.

In fact, if this had been used as our motivation for the above lemma, we would have gotten a proof in which the initial object I was constructed in one step, as the limit of the inclusion functor of the full subcategory with object-set S into $\mathbf{C}$. But I preferred the present approach because the characterization of limits of identity functors is itself not easy to prove. In [**14**] you can find both versions of the proof, as Theorem 1 on p. 116 and Theorem 1 on p. 231 respectively.

In results such as the above, the assumption that there exists a small set S which, for the purposes in question, is ‘‘as good as’’ the set of all objects is known as the ‘‘solution-set condition’’.

On now to the next result in this family. Since a representing object for a functor $U\colon\mathbf{C}\to\mathbf{Set}$ is equivalent to an initial object in an appropriate auxiliary category $\mathbf{C}'$, let us see under what conditions we can apply Lemma 7.10.1 to such an auxiliary category to get a representability result. By Theorem 7.8.6, if U is representable it must respect limits, so the condition of respecting limits must somehow be a precondition for the application of the lemma in this way. The next result shows that for the auxiliary category $\mathbf{C}'$ to have small limits in fact is equivalent to U *respecting* such limits.

Lemma 7.10.2. *Let* $\mathbf{C}$ *be a category,* $U\colon \mathbf{C} \to \mathbf{Set}$ *any functor, and* $\mathbf{C}'$ *the category whose objects are pairs* (X, x) *with* $X \in \mathrm{Ob}(\mathbf{C})$ *and* $x \in U(X)$, *and whose morphisms are morphisms of first components respecting second components* (*in the language of Exercise 6.8:24, the comma category* $(\mathbf{C} \downarrow U)$). *Let* $V\colon \mathbf{C}' \to \mathbf{C}$ *denote the functor forgetting second components. Then*

(a) *If* $\mathbf{D}$ *is a small category and* $G\colon \mathbf{D} \to \mathbf{C}$ *a functor having a limit in* $\mathbf{C}$, *the following conditions are equivalent:*

(i) U *respects the limit of* G; *i.e., the comparison morphism* $c\colon U(\varprojlim G) \to \varprojlim UG$ *is a bijection of sets.*

(ii) *Every functor* $F\colon \mathbf{D} \to \mathbf{C}'$ *satisfying* $VF = G$ *has a limit in* $\mathbf{C}'$.

Hence,

(b) *If* $\mathbf{C}$ *has small limits, then* $\mathbf{C}'$ *will have small limits if and only if* U *respects small limits.*

Sketch of Proof. We shall prove (a), from which (b) will clearly follow.

(i)⇒(ii). A functor F that ''lifts G'' as in (ii) is essentially a compatible way of choosing for each $X \in \mathrm{Ob}(\mathbf{D})$ an element $x \in UG(X)$; hence it corresponds to an element $y \in \varprojlim UG$. By (i), $y = c(z)$ for a unique $z \in U(\varprojlim G)$, and we find that the pair $(\varprojlim G, z)$ will be a limit of F in $\mathbf{C}'$, giving (ii).

(ii)⇒(i). Let y be any element of $\varprojlim UG$. As observed, this corresponds to a functor $F\colon \mathbf{D} \to \mathbf{C}'$, and by (ii) F has a limit (Z, z) in $\mathbf{C}'$. The cone from this limit object to F, applied to first components, gives a cone from Z to the objects $G(X)$, under which the second component, z is carried to the components of y; hence the map $Z \to \varprojlim G$ induced by this cone carries $z \in U(Z)$ to an element $w \in U(\varprojlim G)$, which is taken by c to $y \in \varprojlim UG$. This establishes the surjectivity of c.

Suppose now that c also takes another element $w' \in U(\varprojlim G)$ to y. By the universal property of (Z, z), there is a morphism $\varprojlim G \to Z$ carrying w' to z; composing this with our morphism $Z \to \varprojlim G$ we get an endomorphism of $\varprojlim G$ carrying w' to w. But all these morphisms, and hence this endomorphism in particular, respect cones to G in $\mathbf{D}$, hence by the universal property of $\varprojlim G$, this endomorphism must be the identity morphism of $\varprojlim G$. This shows that $w' = w$, proving one-one-ness of c. □

Exercise 7.10:2. Give the details of the proof of (i)⇒(ii) above.

Exercise 7.10:3. In part (a) of the above lemma, we assumed that the functor G had a limit. We may ask whether this assumption is needed in proving (ii)⇒(i), or whether the existence of the limits assumed in (ii) implies this.

To answer this question, let $\mathbf{C}$ be the category whose objects are pairs (G, S) where G is a group and S a *cyclic* subgroup of G (a subgroup generated by one element), and where a morphism $(G, S) \to (H, T)$ means a homomorphism $G \to H$ which carries the subgroup S *onto* the subgroup T. Let $U\colon \mathbf{C} \to \mathbf{Set}$ be the functor which carries each pair (G, S) to the set of generating elements of S.

Show how to define U on morphisms. Show that $\mathbf{C}$ does not, in general, have products of pairs of objects, but that the category $\mathbf{C}'$, defined as in Lemma 7.10.2, has all small limits, hence, in particular, pairwise products.

The reader should verify that Lemmas 7.10.1 and 7.10.2 now give the desired criterion for representability, namely

Proposition 7.10.3. *Let* **C** *be a category with small limits, and* U: **C** → **Set** *a functor. Then* U *is representable if and only if*

(a) U *respects small limits, and*

(b) *there exists a small set* S *of objects of* **C** *such that for every object* Y *of* **C** *and* $y\in U(Y)$, *there exist* $X\in S$, $x\in U(X)$, *and* $f\in\mathbf{C}(X, Y)$ *such that* $y = U(f)(x)$. □

Finally, let us get from this a condition for the existence of *adjoints*. (Note that for **D** = **Group** and U its underlying-set functor, condition (b) below was precisely what we had to come up with in showing the existence of free groups on arbitrary sets Z.)

Theorem 7.10.4 (Freyd's Adjoint Functor Theorem)**.** *Let* **C** *and* **D** *be categories such that* **D** *has small limits. Then a functor* U: **D** → **C** *has a left adjoint* F: **C** → **D** *if and only if*

(a) U *respects small limits, and*

(b) *for every* $Z\in\mathrm{Ob}(\mathbf{C})$ *there exists a small set* $S(Z)\subseteq\mathrm{Ob}(\mathbf{D})$ *such that for every* $Y\in\mathrm{Ob}(\mathbf{D})$ *and* $y\in\mathbf{C}(Z, U(Y))$, *there exist* $X\in S(Z)$, $x\in\mathbf{C}(Z, U(X))$ *and* $f\in\mathbf{D}(X, Y)$ *such that* $y = U(f)x$.

Proof. The existence of a left adjoint to U is equivalent to the representability, for every $Z\in\mathrm{Ob}(\mathbf{C})$, of the functor $\mathbf{C}(Z, U(–))$: **D** → **Set**. Condition (b) is clearly a translation of condition (b) of the preceding proposition. As for condition (a), we know by Theorem 7.8.3 that it, too, is necessary for the existence of a left adjoint, so it suffices to show that it implies that each set-valued functor $\mathbf{C}(Z, U(–))$ respects limits. If we write this functor as $h_Z\,U$, and recall that covariant representable functors h_Z respect limits, this is immediate. □

I remarked in §7.8 that for every example we had *seen* of a functor that was not representable or did not have a left adjoint, this could be proved by showing that the functor did not respect some limit. We can now understand this better. On a category having small limits, the only way a functor respecting these limits can fail to have a left adjoint or a representing object is if the solution-set condition fails. Since the solution-set condition says ''a *small* set is sufficient'', its failure must involve set-theoretic difficulties, which are rare in algebraic contexts. However, knowing now that this is what we should look for, we can find examples. The next exercise gives a simple, if somewhat artificial example. The example in the exercise after that is more complicated, but more relevant to constructions we are interested in.

Exercise 7.10:4. Let **D** be the subcategory of **Set** whose objects are all sets (or if you prefer, all ordinals; in either case, ''small'' is understood, since by definition **Set** is the category of all small sets), and whose morphisms are the *inclusion maps* of subsets. Show that **D** has small colimits (and has limits over all nonempty categories, though this will not be needed), but has no terminal object.

Hence, letting $\mathbf{C} = \mathbf{D}^{\mathrm{op}}$, the category **C** has small limits (and colimits over nonempty categories) but no initial object. Translate the nonexistence of an initial object for **C** to the nonrepresentability of a certain functor U: **C** → **Set** which respects limits (cf. Exercise 7.2:7).

The results of this section would imply the existence of an initial object of **C**, and of a representing object for U, if a certain solution-set condition held. State this condition, and note why it does not hold.

The next exercise is related to the point mentioned in §5.2, that because the class of *complete lattices* is not defined by a small set of operations, it does not always behave like classes of ''ordinary'' algebras. We shall see below that the solution-set condition required for the existence of the *free* complete lattice on 3 generators fails, and indeed, that there is no such free object.

Exercise 7.10:5. (i) Show that every ordinal has a unique decomposition $\alpha = \beta + n$, where β is a limit ordinal (possibly 0) and $n \in \omega$. Let us call α *even* or *odd* respectively according as the summand n in this decomposition is even or odd.

Now let α be an arbitrary ordinal, let $S = \alpha \cup \{x, y\}$ where x, y are two elements that are not ordinals, and let L be the lattice of all subsets $T \subseteq S$ such that (a) if T contains x and all ordinals less than an odd ordinal $\beta \in \alpha$, then it contains β, and (b) if T contains y and all ordinals less than an even ordinal $\beta \in \alpha$, then it contains β.

(ii) Show that the complete sublattice of L generated by the three elements $\{x\}$, $\{0, y\}$ and α (i.e., the closure of this set of three elements under arbitrary meets and joins within L) has cardinality $\geq \mathrm{card}(\alpha)$. (This is an extension of the trick of Exercise 5.3:9.)

(iii) Deduce that there can be no free complete lattice on 3 generators.

Statement (iii) above was first proved in [**61**], by a different construction. Three proofs of the similar result that there is no free complete Boolean algebra on countably many generators are given in [**57**], [**61**] and [**96**].

But the fact that a class of algebras has a large set of primitive operations does not preclude the existence of free objects, as shown by

Exercise 7.10:6. Complete $\vee$-semilattices with least elements, like complete lattices, have an α-fold join operation for every cardinal α. Nevertheless:

(i) Show that a complete $\vee$-semilattice with least element generated by an X-tuple of elements has at most $\mathrm{card}(\mathbf{P}(X))$ elements.

(ii) Deduce from Freyd's Adjoint Functor Theorem that there exist free complete $\vee$-semilattices with least elements on all sets. (This despite the fact that complete $\vee$-semilattices with least elements are, as partially ordered sets, the same objects as complete lattices!)

(iii) Does the category of $\vee$-complete *lattices* with least element behave, in this respect, like that of complete $\vee$-semilattices with least element, or like that of complete lattices? I.e., does it have free objects on all sets or not?

You may have noticed that in this section, I have not followed my usual practice of stating every result both for left and for right universal constructions. That practice is, of course, logically unnecessary anyway, since one result can always be deduced immediately from the other by putting $\mathbf{C}^{\mathrm{op}}$ for $\mathbf{C}$ and making appropriate notational translations. In earlier sections I nonetheless gave dual pairs of formulations, because both statements were generally of comparable importance. However, when one studies categories of algebras, objects characterized by right universal properties are usually easier to construct directly than those characterized by left universal properties, so we have little need for results obtaining the former from the latter; hence my one-sided presentation. (It is also true that short-term generalizations about what cases are important may fail in the longer run! However, whether we have formally stated them or not, we can always call on the duals of the results of this section if we need them.)

Here is a somewhat vague question, to which I don't know an answer.

Exercise 7.10:7. Suppose a functor U has a left adjoint F, which in turn has a left adjoint G. Can one conclude more about U itself than the results that we have shown follow from the existence of F? In other words, are there any nice necessary conditions for existence of *double* left adjoints, comparable to the property of respecting limits as a condition for existence of a single left adjoint?

7.11. Morphisms involving adjunctions. I am not planning on using the results of this section in subsequent chapters, so the reader may excuse a little sketchiness. (However, the material in the *next* section *will* be referred to in subsequent chapters, and should be read with your usual vigilance.)

Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and $\mathbf{D}\ \underset{F}{\overset{U}{\rightleftarrows}}\ \mathbf{C}$ adjoint functors. We recall the isomorphism which characterizes their adjointness:

$$\mathbf{C}(-,\ U(-))\ \cong\ \mathbf{D}(F(-),\ -). \tag{7.11.1}$$

Suppose now that we have functors from a third category into each of these categories, $P\colon \mathbf{E}\to\mathbf{C}$ and $Q\colon \mathbf{E}\to\mathbf{D}$. It is not hard to verify that if we ''substitute P and Q into the blanks'' in (7.11.1), we get a bijection between sets of morphisms of functors:

$$\mathbf{C}^{\mathbf{E}}(P,\ UQ)\ \longleftrightarrow\ \mathbf{D}^{\mathbf{E}}(FP,\ Q).$$

As one would expect, this bijection is functorial in P and Q, i.e., respects morphisms $P\to P'$, $Q\to Q'$; in other words, writing $F\circ$ and $U\circ$ for the operations of composing on the left with F and U respectively, the above bijection gives an isomorphism of bifunctors $\mathbf{C}^{\mathbf{E}}\times\mathbf{D}^{\mathbf{E}}\to\mathbf{Set}$:

$$\mathbf{C}^{\mathbf{E}}(-,\ U\circ-)\ \cong\ \mathbf{D}^{\mathbf{E}}(F\circ-,\ -).$$

This means that we have an adjoint pair of functors on functor categories, $\mathbf{D}^{\mathbf{E}}\ \underset{F\circ}{\overset{U\circ}{\rightleftarrows}}\ \mathbf{C}^{\mathbf{E}}$. We can also describe this adjunction in terms of its unit and counit; these will be $\eta\circ\colon \mathrm{Id}_{(\mathbf{C}^{\mathbf{E}})}\to(UF)\circ$ and $\varepsilon\circ\colon (FU)\circ\to\mathrm{Id}_{(\mathbf{D}^{\mathbf{E}})}$, where η and ε are the unit and counit of the adjunction between U and F. In fact, the quickest way to prove that $U\circ$ and $F\circ$ are adjoint is to note that the equations in η and ε which establish the adjointness of U and F (Theorem 7.3.3(iii)) give equations in $\eta\circ$ and $\varepsilon\circ$ establishing the adjointness of $U\circ$ and $F\circ$.

The above fits with our comment at the end of §6.9 that a functor category such as $\mathbf{C}^{\mathbf{E}}$ or $\mathbf{D}^{\mathbf{E}}$ behaves very much like its codomain category, $\mathbf{C}$ or $\mathbf{D}$. What that observation does not prepare us for is that analogous results hold for composition on the *right* with adjoint functors. Given adjoint functors U and F, still as in (7.11.1) above, let us take a category $\mathbf{B}$ and functors $R\colon \mathbf{D}\to\mathbf{B}$, $S\colon \mathbf{C}\to\mathbf{B}$. I claim we get a bijection

$$\mathbf{B}^{\mathbf{D}}(SU,\ R)\ \longleftrightarrow\ \mathbf{B}^{\mathbf{C}}(S,\ RF)$$

and thus an isomorphism

$$\mathbf{B}^{\mathbf{D}}(-\circ U,\ -)\ \cong\ \mathbf{B}^{\mathbf{C}}(-,\ -\circ F).$$

i.e., a pair of adjoint functors, $\mathbf{B}^{\mathbf{D}}\ \underset{\circ U}{\overset{\circ F}{\rightleftarrows}}\ \mathbf{B}^{\mathbf{C}}$, where this time $\circ U$ is the left adjoint and $\circ F$ the right adjoint. I don't know a way of seeing this directly from (7.11.1), but it comes out easily if we check the formal properties of the unit and counit $\circ\eta$ and $\circ\varepsilon$.

Let us cook up a random example. We shall take for U and F the familiar case of the underlying set functor on groups and the free group functor. To avoid overlap with the result we proved earlier about composition of adjoints (Theorem 7.3.5), let us take for R and S functors which are not adjoints on either side: Let $S\colon \mathbf{Set}\to\vee\text{-}\mathbf{Semilattice}^0$ take a set X to the upper semilattice of *equivalence relations* on X, and let $R\colon \mathbf{Group}\to\vee\text{-}\mathbf{Semilattice}^0$ take a group G to the upper semilattice of *subgroups* of G. (By $\vee\text{-}\mathbf{Semilattice}^0$ we mean the category of upper semilattices with least elements 0, i.e., with arbitrary finite joins, including the empty join.) A morphism from SU to R thus means a way of associating to every equivalence relation on the underlying set of a group a subgroup of that group, in a way that respects joins (including the empty join), and also respects maps induced by group homomorphisms. Perhaps unexpectedly, there exist several constructions with these properties: Given an equivalence relation E on the underlying set of a group G, one can construct (a) the subgroup of G generated by the elements xy^{-1} for $(x,y)\in E$, (b) the subgroup generated by the elements $y^{-1}x$, as well as the subgroups generated by (c) both types of elements and (d) neither (the trivial subgroup).

On the other hand, a morphism from S to RF means a way of associating to every equivalence relation on a set X a subgroup of the free group $F(X)$, again respecting joins and morphisms. The adjointness result stated above implies that there should be such a morphism $S\to RF$ corresponding to the each of the morphisms $SU\to R$ just listed; and indeed, these can be described as associating to an equivalence relation E on X the subgroup of $F(X)$ generated by the elements xy^{-1} respectively $y^{-1}x$, respectively both, respectively neither, for $(x,y)\in E$. To get these morphisms formally from the morphisms (a)-(d) above, we look at any equivalence relation $E\in S(X)$, use it and the natural map $X\to U(F(X))$ to induce an equivalence relation on $U(F(X))$, then apply the chosen morphism $SU\to R$.

The above example is studied further in

Exercise 7.11:1. Let U, F, S and R be as in the above example. Given any set of nonzero integers, $I\subseteq \mathbf{Z}-\{0\}$, let $m_I\colon SU\to R$ associate to each equivalence relation E on the underlying set of a group G the subgroup of G generated by all the elements $x^i y^{-i}$ $((x,y)\in E$, $i\in I)$.

(i) Show that the m_I are morphisms of functors, and are all distinct.

(ii) Try to determine whether these are all the morphisms $SU\to R$. Are there any morphisms which respect finite joins (including empty joins) but not infinite joins?

Returning to the question of why adjointness is preserved not only by the construction $(–)^{\mathbf{E}}$ but also (with roles of right and left reversed) by the construction $\mathbf{B}^{(–)}$, the explanation seems to be that the definition of adjointness can be expressed as the condition that certain equations hold among given functors and morphisms in the **Cat**-enriched structure (§6.11) of **Cat**, namely those of Theorem 7.3.3(iii), and that these equations will be preserved by any functor preserving **Cat**-enriched structure – as $(–)^{\mathbf{E}}$ and $\mathbf{B}^{(–)}$ both do, one covariantly and the other contravariantly. (For an analogous but simpler situation, observe that, although conditions on a morphism a in a category such as being an epimorphism or a monomorphism are not preserved by arbitrary functors, the conditions of left, right and two-sided invertibility are preserved, because they come down to the existence of another morphism b satisfying one or both of the equations $ab=\mathrm{id}_X$, $ba=\mathrm{id}_Y$, and these conditions are clearly preserved by functors. The formulation of adjointness in terms of unit and counit morphisms in Theorem 7.3.3 is similarly ‘‘robust’’.)

To complicate things a bit further, consider next any two functors P and Q (the vertical arrows below), any adjoint pair of functors between their domain categories, and any adjoint pair of functors between their codomain categories:

(7.11.2)
$$\begin{tikzcd}
\mathbf{B} \arrow[r, shift left, "U"] \arrow[d, "P"'] & \mathbf{C} \arrow[l, shift left, "F"] \arrow[d, "Q"] \\
\mathbf{D} \arrow[r, shift left, "V"] & \mathbf{E}. \arrow[l, shift left, "G"]
\end{tikzcd}$$

(No commutativity conditions are assumed in this diagram!) Now we may apply on the one hand our isomorphisms involving composition on the right with adjoint pairs of functors, and on the other hand our isomorphisms involving composition on the left with such pairs, getting four bijections of morphism-sets

(7.11.3)
$$\begin{tikzcd}
\mathbf{E}^{\mathbf{B}}(QU,\ VP) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathbf{D}^{\mathbf{B}}(GQU,\ P) \arrow[d, leftrightarrow] \\
\mathbf{E}^{\mathbf{C}}(Q,\ VPF) \arrow[r, leftrightarrow] & \mathbf{D}^{\mathbf{C}}(GQ,\ PF)\,.
\end{tikzcd}$$

Because composition with functors on the left commutes with composition with other functors on the right, the above diagram of bijections commutes. This result and the preceding observations are summarized in

Proposition 7.11.4. *Suppose* $\mathbf{D} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{C}$ *are adjoint functors, with* F *the left adjoint and* U *the right adjoint, and with unit* $\eta\colon \mathrm{Id}_{\mathbf{C}} \to UF$ *and counit* $\varepsilon\colon FU \to \mathrm{Id}_{\mathbf{D}}$. *Then*

(i) *For any category* $\mathbf{E}$, *the functors* $\mathbf{D}^{\mathbf{E}} \underset{F\circ}{\overset{U\circ}{\rightleftarrows}} \mathbf{C}^{\mathbf{E}}$ *are adjoint, with* $F\circ$ *the left adjoint,* $U\circ$ *the right adjoint, unit* $\eta\circ\colon \mathrm{Id}_{\mathbf{C}^{\mathbf{E}}} \to UF\circ$ *and counit* $\varepsilon\circ\colon FU\circ \to \mathrm{Id}_{\mathbf{D}^{\mathbf{E}}}$.

(ii) *For any category* $\mathbf{B}$, *the functors* $\mathbf{B}^{\mathbf{D}} \underset{\circ U}{\overset{\circ F}{\rightleftarrows}} \mathbf{B}^{\mathbf{C}}$ *are adjoint, with* ${}^{\circ}U$ *the left adjoint,* ${}^{\circ}F$ *the right adjoint, unit* ${}^{\circ}\eta\colon \mathrm{Id}_{\mathbf{B}^{\mathbf{C}}} \to {}^{\circ}UF$ *and counit* ${}^{\circ}\varepsilon\colon {}^{\circ}FU \to \mathrm{Id}_{\mathbf{B}^{\mathbf{D}}}$.

(iii) *Given two pairs of adjoint functors as in* (7.11.2)*, the square of isomorphisms of bifunctors* $\mathbf{E}^{\mathbf{C}} \times \mathbf{D}^{\mathbf{B}} \to \mathbf{Set}$

(7.11.5)
$$\begin{tikzcd}
\mathbf{E}^{\mathbf{B}}(-\circ U,\ V\circ -) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\|\wr"] & \mathbf{D}^{\mathbf{B}}(G\circ - \circ U,\ -) \arrow[d, phantom, "\|\wr"] \\
\mathbf{E}^{\mathbf{C}}(-,\ V\circ - \circ F) \arrow[r, phantom, "\cong"] & \mathbf{D}^{\mathbf{C}}(G\circ -,\ -\circ F)
\end{tikzcd}$$

commutes. □

Exercise 7.11:2. Give the details of part or all of the proof of the above proposition.

My reason for setting down the above observations is to help understand a better known result, which we can get from (7.11.3) by taking $\mathbf{B} = \mathbf{D}$, $\mathbf{C} = \mathbf{E}$, and for P, Q the identity functors of these categories.

Corollary 7.11.6. *Suppose* $\mathbf{D} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{C}$ *and* $\mathbf{D} \underset{G}{\overset{V}{\rightleftarrows}} \mathbf{C}$ *are two pairs of adjoint functors between the same two categories* $\mathbf{C}$ *and* $\mathbf{D}$ (*F and G the left adjoints, U and V the right adjoints*). *Then there is a natural bijection* i: $\mathbf{D}^{\mathbf{C}}(G, F) \longleftrightarrow \mathbf{C}^{\mathbf{D}}(U, V)$ (*described below*). *In other words, morphisms in one direction between left adjoints correspond to morphisms in the other direction between right adjoints.*

Description of the bijection. Given $f \in \mathbf{D}^{\mathbf{C}}(G, F)$, one may apply $U: \mathbf{D} \to \mathbf{C}$ on the right to get $f \circ U \in \mathbf{D}^{\mathbf{D}}(GU, FU)$. Composing with the counit morphism $\varepsilon_{U,F}: FU \to \mathrm{Id}_{\mathbf{D}}$ we get $\varepsilon_{U,F}(f \circ U) \in \mathbf{D}^{\mathbf{D}}(GU, \mathrm{Id}_{\mathbf{D}})$. Finally, using the adjointness between G and V, we may turn this into the desired member of $\mathbf{C}^{\mathbf{D}}(U, V)$. This last step is equivalent to going through a process for G and V like the one we went through for F and U, so the result can be written

$$i(f) = (V \circ (\varepsilon_{U,F}(f \circ U)))\,\eta_{V,G}. \quad \square$$

As an example, let U and V both be the underlying set functor $\mathbf{Group} \to \mathbf{Set}$, so that F and G are both the free group functor $\mathbf{Set} \to \mathbf{Group}$. Then the above result says that there is a natural bijection between endomorphisms of these adjoint functors. We have already looked at endomorphisms of U; in the language of Exercise 2.3:6 they are ‘‘functorial generalized group-theoretic operations in one variable’’, which we found were just derived group-theoretic operations in one variable, i.e., the operations of exponentiation by arbitrary integers n. (Cf. also Exercises 6.9:4(iii), 7.2:10.)

As for endomorphisms of F, it is not hard to see that such an endomorphism is determined by the endomorphism it gives of the free group on one generator. That endomorphism will send the generator x to x^n for some integer n; conversely, we easily verify that for each n, an endomorphism of the whole functor F with this behavior on the free group on one generators exists; hence endomorphisms of F also correspond to exponentiation by arbitrary integers n.

In the above example, we cannot see that the direction of the morphisms has been reversed. For a less degenerate case, let $\mathbf{C} = \mathbf{Group}$ and $\mathbf{D} = \mathbf{CommRing}^1$. Let U be the functor taking each commutative ring with 1, R, to the group $\mathrm{GL}(n, R)$ of $n \times n$ invertible matrices over R, and V the functor taking R to its group of invertible elements (units). Clearly there is an important morphism $a: U \to V$, taking every invertible matrix over a ring to its *determinant*. The left adjoint F of U takes every group A to the commutative ring $F(A)$ presented by generators and relations that create a universal image of A in the group of $n \times n$ invertible matrices over $F(A)$, and likewise the left adjoint G of V will take a group A to the commutative ring $G(A)$ with a universal image of A in its group of units. (The latter is easily seen to be the *group ring* of the *abelianization* of A.) If we look at the *determinants* of the matrices over $F(A)$ comprising the universal $n \times n$ matrix representation of A, we see that these give a homomorphism of A into the group of units of $F(A)$, which by the universal property of $G(A)$ is equivalent to a ring homomorphism $G(A) \to F(A)$. This gives the morphism of functors $G \to F$ in $(\mathbf{CommRing}^1)^{\mathbf{Group}}$ corresponding to our determinant morphism $U \to V$ in $\mathbf{Group}^{\mathbf{CommRing}^1}$.

Mac Lane [**14**, p. 98, top] calls a pair of morphisms of functors related under the bijection of Corollary 7.11.6 *conjugate*. Of course, we should have proved more about this phenomenon than we have stated in Corollary 7.11.6; in particular, that the conjugate of the composite of two morphisms between three adjoint pairs of functors $\mathbf{C} \rightleftarrows \mathbf{D}$ is the composite of their conjugates in reversed order, i.e., that conjugation constitutes a contravariant equivalence between the category of all functors $\mathbf{C} \to \mathbf{D}$ having right adjoints and the category of all functors $\mathbf{D} \to \mathbf{C}$ which have left adjoints; and likewise that conjugacy behaves properly with respect to composition of adjoint functors. These results can be looked at as follows: Suppose that within the **Cat**-based category **Cat**, we form the subcategory **RightAdj**, that has the same objects as **Cat**, and the same morphisms-of-morphisms (all morphisms between functors in this subcategory), but where the intermediate-level morphisms, the functors, are restricted to those which are right adjoints (equivalently, have left adjoints) in **Cat**. Suppose we likewise form the subcategory **LeftAdj**, as above except that the functors are those that are left adjoints in **Cat**. Then we get an equivalence of **Cat**-based categories $\mathbf{RightAdj} \approx \mathbf{LeftAdj}^{\mathrm{op}}$. (Actually, one needs a notation to show that there is a ''double $^{\mathrm{op}}$'' here, applying both to composition of functors and to composition of morphisms of functors!) One might most elegantly identify these two **Cat**-categories, getting one **Cat**-category **Adj** having adjoint pairs of functors for its morphisms, and conjugate pairs of morphisms of functors for its morphisms of morphisms. For more details, see [**14**, pp. 97-102].

We could also have brought into the statement of Corollary 7.11.6 the upper right-hand and lower left-hand corners of (7.11.3). For instance, in the case involving groups and commutative rings discussed above, the reader can easily describe a morphism $\mathrm{Id}_{\mathbf{Group}} \to VF$, i.e., a functorial way of mapping each group A into the group of units of the commutative ring with a universal $n{\times}n$ representation of A, again based on the determinant function, and a morphism $GU \to \mathrm{Id}_{\mathbf{CommRing}^1}$, i.e., a functorial way of mapping the group ring on the abelianization of the group of invertible $n{\times}n$ matrices over a ring R into R, yet again based on the determinant.

7.12. Contravariant adjunctions. The concept of an adjoint pair of functors is *self-dual*, in the sense that if we write down the definition of adjointness of $\mathbf{D}\ \underset{F}{\overset{U}{\rightleftarrows}}\ \mathbf{C}$, put $\mathbf{C}^{\mathrm{op}}$ and $\mathbf{D}^{\mathrm{op}}$ in place of $\mathbf{C}$ and $\mathbf{D}$, and translate the resulting structure into language natural for our new $\mathbf{C}$ and $\mathbf{D}$, the result has the same form as the original definition, though with the roles of $\mathbf{C}$ and $\mathbf{D}$ interchanged, and likewise U and F, and η and ε.

But a concept which, like that of adjunction, involves more than one category also has ''partial dualizations''. Thus, if in the definition of adjunction we only replace $\mathbf{C}$ by $\mathbf{C}^{\mathrm{op}}$, we get a condition on a pair of functors $\mathbf{C}^{\mathrm{op}} \rightleftarrows \mathbf{D}$. Note that the one going to the right is a contravariant functor from $\mathbf{C}$ to $\mathbf{D}$, and the other corresponds to a contravariant functor from $\mathbf{D}$ to $\mathbf{C}$, i.e., a functor $\mathbf{D}^{\mathrm{op}} \to \mathbf{C}$. Writing it in the latter form, we arrive at a setup which is symmetric, that is, in which the two categories and the two functors play equivalent roles – but which is *not* self-dual. We describe this construction and its dual in the definition below.

When we defined ordinary adjunctions, we wrote the isomorphism of bifunctors ''$\mathbf{C}(-,\ U(-)) \cong \mathbf{D}(F(-),\ -)$'', with the understanding that the first argument ''–'' on the left matched the first argument on the right, and similarly for second arguments. But below, the first argument on one side of our isomorphism will represent the same variable as the second argument on the other side. To make this clear, we will use distinct place-holders, ''–'' and ''~'', for the two arguments.

Definition 7.12.1. *Let* $U\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ *and* $V\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{C}$ *be contravariant functors between categories* $\mathbf{C}$ *and* $\mathbf{D}$.

Then a contravariant right adjunction *between* U *and* V *means an isomorphism*

$$\mathbf{C}(-,\ V(\sim)) \ \cong\ \mathbf{D}(\sim,\ U(-))$$

of bifunctors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$, *where* ‘‘–’’ *denotes the* $\mathbf{C}$*-valued argument and* ‘‘~’’ *the* $\mathbf{D}$*-valued argument; equivalently, an adjunction between* U: $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ *and the functor* V^{op}: $\mathbf{D} \to \mathbf{C}^{\mathrm{op}}$ *corresponding to* V, *with* U *the right and* V^{op} *the left adjoint; equivalently, an adjunction between* V: $\mathbf{D}^{\mathrm{op}} \to \mathbf{C}$ *and* U^{op}: $\mathbf{C} \to \mathbf{D}^{\mathrm{op}}$, *with* V *the right and* U^{op} *the left adjoint.*

Likewise, a contravariant left adjunction *between* U *and* V *means an isomorphism*

$$\mathbf{C}(\,V(\sim), -) \;\cong\; \mathbf{D}(\,U(-), \sim)$$

of bifunctors $\mathbf{C} \times \mathbf{D} \to \mathbf{Set}$, *equivalently, an adjunction between* V (*left*) *and* U^{op} (*right*); *equivalently, an adjunction between* U (*left*) *and* V^{op} (*right*).

Of course, these two new kinds of adjointness also have descriptions corresponding to the other ways of describing adjoint functors noted in Theorem 7.3.3. For instance, given U: $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$, to find a contravariant right adjoint to U is equivalent to finding, for each object D of $\mathbf{D}$, a representing object for the contravariant functor $\mathbf{D}(D, U(-))$: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$; in other words, an object R_D of $\mathbf{C}$ with a map $D \to U(R_D)$, which is universal among objects of $\mathbf{C}$ with such maps.

For an example of such an adjunction, recall that in Exercise 6.6:5 we showed that for every partially ordered set P, the hom-set $h^2(P)$ has a natural lattice structure. Given an arbitrary lattice L, one can find a partially ordered set $P = V(L)$ with a universal map of L into the lattice $h^2(P)$. One finds that the universal property in question makes the given functor h^2: $\mathbf{POSet}^{\mathrm{op}} \to \mathbf{Lattice}$ and the new functor V: $\mathbf{Lattice}^{\mathrm{op}} \to \mathbf{POSet}$ mutually right adjoint. We will look more closely at this and similar examples in §9.11.

Exercise 7.12:1. Show that if P and Q are partially ordered sets, then a contravariant right adjunction between $P_{\mathbf{cat}}$ and $Q_{\mathbf{cat}}$ is equivalent to a *Galois connection* between P and Q, in the generalized sense noted at the end of Exercise 5.5:2.

Contravariant *left* adjunctions rarely come up in algebra. It is shown in [**39**] that no such nondegenerate adjunctions exist among a wide class of categories of algebras.

It may seem peculiar that we got *three* phenomena – covariant adjointness, contravariant right adjointness, and contravariant left adjointness – as the orbit of one phenomenon (the first of these) under a group of symmetries (interchanging $\mathbf{C}$ and $\mathbf{C}^{\mathrm{op}}$ and interchanging $\mathbf{D}$ and $\mathbf{D}^{\mathrm{op}}$) that seems to have the structure $\mathbf{Z}_2 \times \mathbf{Z}_2$. A closer look at the situation shows the following: The orbit of our original adjointness concept under the natural action of $\mathbf{Z}_2 \times \mathbf{Z}_2$ has four elements. However, in listing ‘‘distinct phenomena’’, we form the orbit-space of this 4-element set under the action of $\mathbf{Z}_2$ which interchanges $\mathbf{C}$ and $\mathbf{D}$, since we consider phenomena permuted by this action as the ‘‘same’’ phenomenon, just differently labeled. This action interchanges the two covariant adjointness situations, but fixes each of the contravariant situations, leading to our set of ‘‘three phenomena’’.

There is yet another sort of symmetry we might consider: that given by reversing the direction (and hence order of composition) of the functors in our statements. In general, results of category theory are *not* preserved by this symmetry, because $\mathbf{Cat}$ is not equivalent to $\mathbf{Cat}^{\mathrm{op}}$. But results and concepts which are not specific to $\mathbf{Cat}$, but can be proved or formulated for arbitrary $\mathbf{Cat}$-based categories, can be dualized in this way. We noted in the preceding section that the concept of adjointness is meaningful in an arbitrary $\mathbf{Cat}$-based category; hence we can apply this duality to it. It turns out to take each of the three kinds of adjointness to itself, leaving the roles of $\mathbf{C}$, $\mathbf{D}$, ε and η unchanged, but interchanging U and F. Indeed, the invariance of adjointness under this symmetry is the reason for the unexpected result Proposition 7.11.4(ii).

Exercise 7.12:2. Prove the claim made above that **Cat** is not equivalent to $\mathbf{Cat}^{\mathrm{op}}$. (You can do this by finding an appropriate statement which holds for **Cat** but whose dual does not.)

One might ask why, if **Cat** is not equivalent to $\mathbf{Cat}^{\mathrm{op}}$, the concept of **Cat**-based category should be invariant under reversing order of composition. Briefly, this is because in applying that reversal to a statement about a **Cat**-based category **X**, one does not replace **Cat** by $\mathbf{Cat}^{\mathrm{op}}$ in the definition of the categories occurring in the statement. Rather, one replaces composition maps $\mathbf{X}(\mathbf{D},\mathbf{E})\times\mathbf{X}(\mathbf{C},\mathbf{D})\to\mathbf{X}(\mathbf{C},\mathbf{E})$ by maps in which the order of factors in the product on the left is reversed; in other words, one uses the symmetry of the product bifunctor on **Cat**. (Replacing **Cat** by $\mathbf{Cat}^{\mathrm{op}}$ would instead redefine composition as being given by functors $\mathbf{X}(\mathbf{C},\mathbf{E})\to\mathbf{X}(\mathbf{D},\mathbf{E})\amalg\mathbf{X}(\mathbf{C},\mathbf{D})$.)

This is similar to the fact that though **Set** is non-self-dual, the symmetry of its product bifunctor allows us to define a functor $(-)^{\mathrm{op}}\colon\mathbf{Cat}\to\mathbf{Cat}$, and use this in ordinary (i.e., **Set**-based) category theory to prove the dual of any true result. (And applying this endofunctor of **Cat** to the concept of **Cat**-based category, one concludes that the latter concept is also symmetric under reversal of the order of composition of its ‘‘morphisms of morphisms’’.)

Chapter 8. Varieties of algebras.

We are at last ready to set up a general theory of algebras!

We recall our convention that a fixed universe U is assumed chosen, and that when the contrary is not stated, a ‘‘set’’ (or for emphasis, ‘‘small set’’) means a set which is a member of U, while a ‘‘category’’ means a U-legitimate category. However, the universe will almost never be referred to by name, hence we will feel free to use the symbol U in unrelated ways, in particular, for underlying-set functors.

We will begin by formalizing some of the ideas we sketched in §§1.4-1.7. (The reader who was not previously familiar with them might review those sections before beginning this formal development.)

8.1. The category Ω-Alg. In studying structures consisting of a set $|A|$ given with some operations, we will want to say that two such structures are of the same *type* if we have indexed their operations in the same way, with corresponding operations having the same arities (cf. § 1.4). Hence, below, we shall define a ‘‘type’’ to mean an index set for the operations, with an arity associated to each operation-symbol.

Without loss of generality one could index the operations by an ordinal, and take the arities to be cardinals; and indeed, one or both of these assumptions is usually made. But allowing more general index sets and arities in our definition involves no complication, so let us do so.

Definition 8.1.1. *A* type *will mean a pair* $\Omega = (|\Omega|, \mathrm{ari}_\Omega)$, *where* $|\Omega|$ *is a set, and* ari_Ω (*written* ari *when there is no danger of ambiguity*), *is a map from* $|\Omega|$ *to sets. The elements* $s\in|\Omega|$ *are called the* operation-symbols *of* Ω, *and for each such* s, *the set* $\mathrm{ari}(s)$ *is called the* arity *of the operation-symbol* s.

Ω *is called* finitary *if all of its operation-symbols have finite arity, i.e., if for all* $s\in|\Omega|$, $\mathrm{card}(\mathrm{ari}(s)) < \omega$.

We will call a type Ω conventional *if* $|\Omega|$ *is an ordinal, and for each* $s\in|\Omega|$, $\mathrm{ari}(s)$ *is a cardinal. In this situation,* Ω *may be expressed by giving the arity function as a tuple of cardinals,* $(\mathrm{ari}(0), \mathrm{ari}(1), \dots)$.

(As mentioned in §1.4, a more common notation in the literature for the arity of s is $n(s)$.)

Definition 8.1.2. *If* Ω *is a type, an* algebra of type Ω *will mean a pair* $A = (|A|, (s_A)_{s\in|\Omega|})$, *where* $|A|$ *is a set, and for each* $s\in|\Omega|$, s_A *is an* $\mathrm{ari}(s)$*-ary operation on* $|A|$, *i.e., a map* $|A|^{\mathrm{ari}(s)} \to |A|$.

For example, the type Ω which indexes the operations of *groups* has three operation-symbols, which we may write μ, ι, ε, with $\mathrm{ari}(\mu) = 2$, $\mathrm{ari}(\iota) = 1$, $\mathrm{ari}(\varepsilon) = 0$. Every group is an algebra of this type, but not every algebra of this type is a group, since there are algebras of this type not satisfying the associative, inverse and identity laws. If we replaced this by a ‘‘conventional type’’ and followed the usage that represents a type by its arity function, we would say that groups are certain ‘‘algebras of type $(2, 1, 0)$’’.

If R is a ring, then right or left *R-modules* can be described as certain algebras of type Ω, where $|\Omega| = \{+, -, 0\} \sqcup |R|$, and all these operation-symbols are unary except $+$, which is binary, and 0, which is zeroary. Here the first three operations specify an additive group structure, while the remaining generally infinite family of operations gives the scalar multiplications by all members of R. To translate this type into conventional notation, one would index $|R|$ by an ordinal α, and let $|\Omega|$ be the ordinal $3+\alpha$; here the convenience of allowing more general sets for $|\Omega|$ is clear.

For an example in which it is natural to regard some operations as having for their *arities* sets other than cardinals, let n be a fixed positive integer, and for every commutative ring R, let d denote the determinant function taking $n \times n$ matrices over R to elements of R. Suppose one wishes to construct from each commutative ring $R = (|R|, +, -, 0, \cdot, 1)$ the object $(|R|, +, -, 0, d)$, i.e., to study the set of elements of R as an additive group with an $n \times n$ ''determinant'' operation. Now one would conventionally consider d as n^2-ary, which would mean writing a typical value as $d(x_0, \ldots, x_{n^2-1})$. But it is more natural to treat d as an $(n \times n)$-ary operation, and write $d(x_{00}, x_{01}, \ldots, x_{n-1\ n-1})$, i.e., to call the typical argument of d the (i, j) argument where $0 \leq i, j < n$, rather than the mth argument where $0 \leq m < n^2$.

If there were a significant advantage in restricting ourselves to conventional algebra-types, then we might say, ''Let us use conventional types in our formal development. We can always *translate* our results into the form appropriate to a particular area of mathematics when we make our applications.'' But I see no advantage in such a restriction. At some points we will indeed find it convenient to restrict attention to cardinal-valued arities, but even there, we will put no restriction on the set of operation-symbols.

Let us note here the unfortunate ambiguity of the word ''algebra'' – there is the ring-theoretic concept of ''an algebra over a commutative ring'', and the present much broader concept used in General Algebra. It would be desirable if a new word could be coined to replace one of them; but there is a large literature in both fields, so it would be hard to get such a change accepted. Since the literature in ring theory is the more enormous of the two, I suppose it is the general-algebra definition that would have to change.

In situations where there is a danger of misunderstanding, authors generally specify ''an algebra over a commutative ring k'' on the one hand, or ''an algebra in the sense of Universal Algebra'' on the other. The Russians shorten the latter phrase to ''a universal algebra'', which is easier to say, but somewhat inappropriate, since it suggests an object with a universal property. (The term ''algebra in the sense of Universal Algebra'' should now presumably be changed to ''algebra in the sense of General Algebra'', for the reasons mentioned in §0.5.)

Incidentally, what is the original source of the word ''algebra''? It goes back to a 9th century Arabic text, *Al-jabr w’al-muqābalah*; the title is composed of two technical terms concerning the solving of equations, whose literal meanings are something like ''restoration and comparison''. This title was transliterated, rather than translated, into medieval Latin, so that the book became known as *Algebra*, which eventually became the name of the subject. Not only this work but also its author, abu-Ja‘far Muḥammed ibn-Mūsā, has entered mathematical language: He was known as *Al-Khuwārizmi*, ''the person from Khuwarizm''; this name was rendered as *algorism*, and, further distorted in English, has become the word *algorithm*.

Of course, we want to make the set of Ω-algebras into a category, so:

Definition 8.1.3. *A* homomorphism *between algebras of the same type means a map of underlying sets which respects operations.*

Precisely, if A and B are algebras of type Ω, a homomorphism $A \to B$ means a set map $f\colon |A| \to |B|$ such that for all $s \in |\Omega|$ and $(x_i)_{i \in \mathrm{ari}(s)} \in |A|^{\mathrm{ari}(s)}$, one has

$$f(s_A((x_i)_{i\in \mathrm{ari}(s)})) \ = \ s_B((f(x_i))_{i\in \mathrm{ari}(s)}).$$

For each type Ω, *the category of all* Ω*-algebras, with homomorphisms for the morphisms, will be denoted* Ω-**Alg**.

Note that when applying a set map to a tuple of elements, one generally drops one pair of parentheses, e.g., shortens $f((x_1, x_2, x_3))$ to $f(x_1, x_2, x_3)$, or $u((a_i)_{i\in I})$ to $u(a_i)$. So the above equation saying that f respects s can be simplified to $f(s(x_i)) = s(f(x_i))$. If one abbreviates the $\mathrm{ari}(s)$-tuple (x_i) to x and uses parenthesis-free notation for functions, one can still further shorten this to $f s x = s f x$, or, distinguishing between f, which acts on elements of $|A|$, and the induced map on $\mathrm{ari}(s)$-tuples of such elements, $f s x = s f^{\mathrm{ari}(s)} x$.

Definition 8.1.4. *Let* A *be an* Ω*-algebra.*

Then a subalgebra *of* A *means an* Ω*-algebra* B *such that* $|B| \subseteq |A|$, *and such that the operations of* B *are the restrictions of the corresponding operations of* A; *equivalently, such that the inclusion map* $|B| \to |A|$ *is a homomorphism* $B \to A$. *In this situation we will, by a slight abuse of notation, write ''*$B \subseteq A$*''. We shall consider the set of subalgebras of* A *to be partially ordered by inclusion* (*of underlying sets*).

A homomorphic image *of* A *means an algebra* B *given with a homomorphism* $f\colon A \to B$ *which is surjective on underlying sets.*

Another notational problem: If A is an algebra, and if we have shown that some subset $S \subseteq |A|$ is closed under the operations of A, we have no simple notation for ''the subalgebra of A whose underlying set is S''. We shall give such algebras ad hoc names when we refer to them, though it is tempting to fall back on the sloppy usage which does not distinguish between an algebra and its underlying set.

Lemma 8.1.5. *If* A *is any* Ω*-algebra, the class of subalgebras of* A *is ''closed under intersections''; i.e., for every set of subalgebras* B_i *of* A $(i\in I)$, *the intersection of the underlying sets,* $\bigcap_I |B_i|$, *is the underlying set of a subalgebra, which we may loosely call* $\bigcap_I B_i$. *Hence the subalgebras of* A *form a complete lattice, with meets given by intersections of underlying sets.*

If X *is any subset of* $|A|$, *the intersection of the underlying sets of all subalgebras of* A *containing* X *will be the underlying set of the least subalgebra containing* X, *called the subalgebra* generated *by* X. *We say that* A *is generated by a subset* $X \subseteq |A|$ *if the subalgebra of* A *generated by* X *is all of* A. □

As we observed in Chapter 1, a *zeroary* operation on a set is equivalent to a choice of a distinguished element of that set. Note that if Ω is a type with no zeroary operation-symbols, then the empty set can be made an Ω-algebra in a unique way. On the other hand, the empty set does not admit any zeroary operations, so if Ω has any operation-symbols of arity 0, all Ω-algebras are nonempty. The least element of the subalgebra lattice of an algebra A of any type Ω will be the subalgebra generated by the empty set; this can also be described as the subalgebra generated, under the operations of *positive* arity, by the values of the *zeroary* operations. So if the type has zeroary operations, this least subalgebra is nonempty, while if it does not, it is empty.

Empty algebras sometimes constitute special cases in algebraic considerations, and many general algebraists avoid this ''problem'' by requiring in their *definitions* that an algebra have a nonempty underlying set. But the problem gets back at them: For instance, they can no longer define subalgebra lattice as above, since when an algebra has no zeroary operations, an intersection of nonempty subalgebras can be empty. Thus they make definitions such as ‘‘the subalgebra lattice of an algebra A consists of all subalgebras of A, and also the empty set if A has no zeroary operations.’’ I feel strongly that it is best *not* to exclude empty algebras, but to allow them when dealing with a type without zeroary operations, and accept the need to occasionally give special arguments for them.

Let us note that in the category Ω-**Alg** we can construct products in the manner to which we have become accustomed: If $(A_i)_{i\in I}$ is a family of Ω-algebras, then the set $\prod_I |A_i|$ becomes an Ω-algebra P under componentwise operations; that is, for each $s\in|\Omega|$ and $\mathrm{ari}(s)$-tuple of elements of $\prod_I |A_i|$, say

$$(a_j)_{j\in\mathrm{ari}(s)} = ((a_{ij})_{i\in I})_{j\in\mathrm{ari}(s)} \in |P|^{\mathrm{ari}(s)} = (\prod_I |A_i|)^{\mathrm{ari}(s)},$$

we define

$$s_P(a_j) = (s_{A_i}((a_{ij})_{j\in\mathrm{ari}(s)}))_{i\in I}.$$

The resulting algebra P is easily seen to have the universal property of the product $\prod_I A_i$ in Ω-**Alg**. Products in Ω-**Alg** are often called by the traditional term, *direct products*.

Similarly, given a pair of homomorphisms of Ω-algebras $f, g: A \to B$, their difference kernel as set maps will be the underlying set of a subalgebra of A, which will constitute a difference kernel of f and g in Ω-**Alg**.

Since general *limits* can be constructed from products and difference kernels (Proposition 7.6.6), we have

Proposition 8.1.6. *Let Ω be any type. Then the category Ω-**Alg** has small limits, which can be constructed by taking the limits of the underlying sets and making them Ω-algebras under pointwise operations.*

*Explicitly, if $\mathbf{D}$ is a small category and $F: \mathbf{D} \to \Omega$-**Alg** a functor, then the set*

$$\varprojlim_{\mathbf{D}} |F(D)| = \{(a_D) \in \prod_{D\in\mathrm{Ob}(\mathbf{D})} |F(D)| \mid (\forall f\in\mathbf{D}(D_1, D_2)) \;\; a_{D_2} = F(f)(a_{D_1})\}$$

*is the underlying set of a subalgebra of $\prod_{\mathbf{D}} F(D)$, which constitutes a limit of F in Ω-**Alg**.* □

Exercise 8.1:1. Show that if empty algebras are excluded from Ω-**Alg**, the resulting category can fail to have small limits.

On the other hand, *colimits* and other *left-universal* constructions are not, in general, the same in Ω-**Alg** as in **Set**. We will construct general colimits in §8.3; but there are two cases that we can obtain now. We first need to note

Lemma 8.1.7. *Let A be an Ω-algebra and $E \subseteq |A|\times|A|$ an equivalence relation on $|A|$. Then the following conditions are equivalent:*

(i) *The set $|A|/E$ can be made an Ω-algebra A/E in such a way that the canonical map $|A| \to |A|/E$ is a homomorphism $A \to A/E$.*

(ii) *E is the equivalence relation on $|A|$ induced by a homomorphism of Ω-algebras with domain A. (I.e., there exists an Ω-algebra B and a homomorphism $f: A \to B$ such that $E =$*

$\{(x,y)\in|A|\times|A| \mid f(x) = f(y)\}$.)

(iii) *E is the underlying set of a subalgebra of $A\times A$.*

Further, if R is any subset of $|A|\times|A|$, and E the intersection of all underlying sets of subalgebras of $A\times A$ which contain R, and which form equivalence relations on $|A|$, then A/E will be universal (initial) among algebras B given with homomorphisms $f\colon A\to B$ such that for all $(r,s)\in R$, $f(r) = f(s)$. □

Definition 8.1.8. *If A is an Ω-algebra, then an equivalence relation E on $|A|$ which is the underlying set of a subalgebra of $A\times A$ will be called a* congruence *on the algebra A, and A/E will be called the* quotient algebra (*or* factor-algebra) *of A by the congruence E.*

The complete lattice of all congruences on A is called the congruence lattice *of A. The least congruence containing a given subset $R\subseteq|A|\times|A|$ is called the congruence on A* generated *by R, and the quotient of A by this congruence is often called the algebra obtained by* imposing *on A the family of relations R, or loosely, the family of relations $(x = y)_{(x,y)\in R}$.*

I say ‘‘loosely’’ in the last sentence because (as we noted in passing in §3.3), there is an abuse of notation in writing such a relation as ‘‘$x = y$’’. The symbol $x = y$ usually denotes a proposition, i.e., an assertion about elements of A, and this proposition is generally false in the case where the relation is one we wish to *impose* on A! What is true is that in our quotient algebra the *images* of x and y satisfy the corresponding relation; and when there is no danger of ambiguity, one may denote these images by the same symbols x and y as the original elements of A, so that $x = y$ becomes a true statement in that quotient algebra. But in more precise notation, the statement which is true in the latter algebra must be written $\bar{x} = \bar{y}$ or $[x] = [y]$. We will be precise about this here, but in informal algebraic use, the language of ‘‘imposing the relation $x = y$ on A’’ is very convenient.

Many workers in general algebra and logic make a convention half-way between these extremes, defining ‘‘relations’’ or ‘‘identities’’ to be symbols of the form ‘‘$x\approx y$’’. (E.g., [**15**, p.234].) These are essentially just our ordered pairs (x,y), written in a more suggestive form. A notation that allows one to avoid ambiguity while using the same symbols for elements of different algebras is that of Model Theory, where one writes $A\vDash x=y$ to mean ‘‘$x=y$ holds in A’’, so that this is distinguishable from $A/E\vDash x=y$.

Using the quotient construction, we immediately get

Lemma 8.1.9. *For any type Ω, the category Ω-**Alg** has difference* co*kernels. Namely, the difference cokernel of a pair of maps $f,g\colon A\rightrightarrows B$ may be constructed as the quotient B/E where E is the congruence on B generated by $\{(f(x),g(x)) \mid x\in|A|\}$.* □

The other left universal construction that we can get easily is that of direct limit, under appropriate restrictions on the arities of our operations:

Lemma 8.1.10. *If Ω is a* finitary *type, then Ω-**Alg** has direct limits, i.e., colimits over directed partially ordered sets. Namely, suppose J is a directed partially ordered set and $A\colon J_{\mathbf{cat}}\to\Omega$-**Alg** a functor, whose values at objects and morphisms of $J_{\mathbf{cat}}$ will be written A_j $(j\in J)$ and $A(j,j')$ $(j\le j'\in J)$ respectively. Then the Ω-algebra structures of the algebras A_j induce an Ω-algebra structure on the set-theoretic direct limit $\varinjlim_J |A_j|$ which makes it a direct limit algebra, $\varinjlim_J A_j$.*

More generally, if α is an infinite cardinal, and Ω a type in which all arities have cardinality $<\alpha$, then the category Ω-**Alg** *has direct limits over all α-directed partially ordered sets* (*Definition 7.9.7*)*, which may be constructed by giving an Ω-algebra structure to the direct limit of the underlying sets.*

Proof. We will prove the general case. Let $|L| = \underrightarrow{\mathrm{Lim}}_J\,|A_j|,$ and let $q_j: |A_j| \to |L|$ $(j\in J)$ be the coprojection maps. We wish to define an Ω-algebra structure on $|L|$. Given $s\in|\Omega|$ and an $\mathrm{ari}(s)$-tuple $(x_i)_{i\in \mathrm{ari}(s)}$ of elements of $|L|,$ let us write each x_i as $q_{j(i)}(y_i)$ for some $j(i)\in J$ and $y_i\in|A_{j(i)}|$. Because J is α-directed and $\mathrm{ari}(s)$ has cardinality $<\alpha,$ we can choose $j\in J$ majorizing all the $j(i)$. Taking such a $j,$ and letting $z_i = A(j(i),j)(y_i)\in A(j)$ for each $i,$ we have

(8.1.11) $$x_i \ = \ q_j(z_i) \qquad \text{for all } \ i\in\mathrm{ari}(s).$$

To define $s_L,$ let us say that whenever we have a family $(x_i)\in|L|^{\mathrm{ari}(s)}$ expressed as in (8.1.11) for some $j\in J,$ we will let

$$s_L(x_i) \ = \ q_j(s_{A_j}(z_i))\in|L|.$$

The verification that these operations s_L are well-defined, and that the resulting Ω-algebra L has the universal property of $\underrightarrow{\mathrm{Lim}}\ A,$ are straightforward, again by the method of ‘‘going far enough out along the α-directed set J’’. □

Exercise 8.1:2. Write out these final verifications.

As noted at the beginning of §7.9, the ‘‘reason’’ the above lemma holds is that in **Set**, direct limits respect finite products (a case of Proposition 7.9.3) and more generally, direct limits over α-directed partially ordered sets respect α-fold products (Proposition 7.9.8).

Since we shall prove in §8.3 that Ω-**Alg** has general colimits, the arity-restrictions of the above lemma are not needed for the existence statements to hold. But they are needed for the direct limits in question to have the descriptions given. Indeed

Exercise 8.1:3. Show by example that the last sentence of the first paragraph of Lemma 8.1.10 fails if the assumption that Ω is finitary is dropped. Specifically, show that there may not exist an algebra with underlying set the direct limit of the $|A_j|,$ and having the universal property of $\underrightarrow{\mathrm{Lim}}\ A_i$.

Let us note something about the definition of an α-directed partially ordered set J (Definition 7.9.7) which will be familiar to students of logic, but perhaps not to others. Where that definition requires the existence of upper bounds for all subsets of cardinality $<\alpha,$ one might feel it more natural for a condition called ‘‘α-directedness’’ to require this for all subsets of cardinality $\leq\alpha$. However, using the definition we have given, the property of having upper bounds for all subsets of cardinality $\leq\alpha$ can be described as ‘‘α'-directedness’’, where α' is the successor-cardinal to $\alpha,$ while the alternative definition would give no easy way to refer to the property of having upper bounds for all subsets of cardinality $<\alpha$ when α is a limit cardinal. So the definition as given is the more versatile one.

We note that to say a partially ordered set is *directed* is equivalent, under Definition 7.9.7, to saying that it is 3-directed, and also to saying it is ω-directed. The next-stronger condition, that of having upper bounds for all countable subsets, is ω_1-directedness, where ω_1 is the first uncountable ordinal (§4.5).

Exercise 8.1:4. Suppose that in the last sentence of Lemma 8.1.10 we drop the condition that α be infinite.

(i) Show that the resulting statement remains true (for a trivial reason) when $2 < \alpha < \omega$.

(ii) Is this statement also true for $\alpha = 2$?

8.2. Generating subalgebras from below. We want to construct other left universal objects in **Ω-Alg** – free algebras, coproducts, arbitrary small colimits, etc.. In general, these will contain new elements created by applying operations of Ω to tuples of the elements we start with, further elements obtained by applying the operations to elements we get in this way, and so on. Whatever methods we use to justify these constructions must involve showing that this iteration process ‘‘eventually ends’’.

‘‘Eventually’’ does not mean in a finite number of steps, of course – even in constructing algebras with operations of finite arity such as groups, we needed countably many iterations. When we have infinitary operations, we may have to continue the process still longer.

To see how long, let us examine the process by which a subset of an algebra generates a subalgebra. Let Ω be an arbitrary type and A an Ω-algebra. Given a subset $X \subseteq |A|$, define a sequence of subsets of $|A|$ indexed by the ordinals:

$$\begin{aligned} S^{(0)} &= X, \\ S^{(\alpha+1)} &= S^{(\alpha)} \cup \{s_A(x_i) \mid s\in|\Omega|,\ x_i \in S^{(\alpha)}\ \ (i\in \mathrm{ari}(s))\}, \qquad (8.2.1) \\ S^{(\alpha)} &= \bigcup_{\beta<\alpha} S^{(\beta)} \ \text{ for } \alpha \text{ a limit ordinal } > 0. \end{aligned}$$

We see by induction that the $S^{(\alpha)}$'s increase monotonically. Since $|A|$ is a small set, successive $S^{(\alpha)}$'s cannot all be distinct, and clearly as soon as two of them are equal, the chain will become constant. The constant value S that it assumes will contain $S^{(0)} = X$ and be closed under the operations s_A; moreover, by induction on α, each $S^{(\alpha)}$, and so in particular, S, is contained in every subalgebra of A containing X. Hence S is the underlying set of the least subalgebra of A containing X, i.e., the subalgebra generated by X.

We want to bound in terms of properties of Ω the least value of α for which $S^{(\alpha)} = S$. (Above, we implicitly bounded it in terms of $\mathrm{card}\,|A|$.)

We know how to show that if Ω is finitary, $S = S^{(\omega)}$. Namely, given a finite family of elements of $S^{(\omega)}$, all members of this family will have been reached by some finite step $S^{(n)}$, hence the value of any operation of A on this family lies in $S^{(n+1)}$, and hence is in $S^{(\omega)}$. Note that if instead of a finitary type Ω, we consider one in which all operations have arity $\leq \omega$, the above conclusion is no longer true: If $s \in |\Omega|$ is ω-ary, and we take for each nonnegative n an element x_n which first appears in $S^{(n)}$, then $S^{(\omega)}$ will not in general contain $s_A(x_0, x_1, \ldots, x_n, \ldots)$. This element will appear in $S^{(\omega+1)}$, and further elements obtained from *it* under the operations of A will in general appear at still later steps. However, I claim that this process stabilizes by the ω_1st step. Indeed, given a countable (possibly finite) family of elements $x_i \in S^{(\omega_1)}$, each occurs in some $S^{(\alpha_i)}$ for a countable ordinal $\alpha_i \in \omega_1$, hence all the x_i will occur in $S^{(\alpha)}$ where $\alpha = \sup(\alpha_i)$, and this ordinal α is still $< \omega_1$, since $\sup(\alpha_i)$ is $\leq$ the ordinal sum of the α_i (defined as in (4.5.10)), which has cardinality equal to the cardinal sum of the $\mathrm{card}(\alpha_i)$, which is a countable sum of countable cardinals, hence countable. Hence the value at $(x_1, \ldots, x_n, \ldots)$ of any operation of countable arity lies in $S^{(\alpha+1)} \subseteq S^{(\omega_1)}$, showing that $S^{(\omega_1)}$ is closed under the operations of A, and hence that (8.2.1) stabilizes by the ω_1st step. The next exercise shows that in this statement we cannot replace the estimate ω_1 by any smaller ordinal (such as ω^2 or ω^ω).

Exercise 8.2:1. Let γ be any uncountable ordinal, and let A be an algebra with underlying set γ and three operations: the zeroary operation taking the value $0\in\gamma$, the unary operation taking $\alpha\in\gamma$ to $\alpha+1$ if $\alpha+1<\gamma$, or to 0 if $\alpha+1=\gamma$, and the ω-ary operation taking $(\alpha_0,\alpha_1,\dots)$ to $\bigcup\alpha_i$ if this is $<\gamma$, to 0 otherwise. Taking $X=\varnothing\subseteq|A|$, determine explicitly the sequence of subsets $S^{(\alpha)}$, and show that this sequence does not become constant until $S^{(\omega_1)}$.

The same argument will show that if all members of $|\Omega|$ have arity $\leq\omega_1$, then we get our desired algebra as $S^{(\omega_2)}$, that if all arities are $\leq\omega_2$, we get it as $S^{(\omega_3)}$, etc.; and it might appear that the proper general statement is that if α is any infinite ordinal of cardinality *greater* than the arities of all members of $|\Omega|$, then $S^{(\alpha)}$ is closed under the operations of Ω.

But this is not quite right. The first value of α for which it fails is ω_ω. If A has operations of arities ω, ω_1, ω_2 etc. (all the infinite cardinals $<\omega_\omega$), then we have seen that the chain of subalgebras $S^{(\omega)}\subseteq S^{(\omega_1)}\subseteq S^{(\omega_2)}\subseteq\dots$ can be strictly increasing. If we now choose an element $x_i\in S^{(\omega_{i+1})}-S^{(\omega_i)}$ for each i, we get a countable family of elements of $S^{(\omega_\omega)}$, and we see that the value of an operation of (merely!) countable arity on this family cannot be expected to lie in $S^{(\omega_\omega)}$.

Exercise 8.2:2. Construct an explicit example with the properties sketched above, i.e., an algebra A all of whose operations have arities $<\omega_\omega$, and a subset $X\subseteq|A|$, such that the chain of subsets $S^{(\alpha)}$ does not reach its maximum value at $S^{(\omega_\omega)}$. (Suggestion: Adapt the idea of the preceding exercise.)

To state the right choice of α, let us recall from Definition 4.5.17 that an infinite cardinal α is called *regular* if, as a partially ordered set, α has no cofinal subset of cardinality $<\alpha$, and that a cardinal that is not regular is called *singular*. What we have run into is the first singular infinite cardinal, ω_ω. Fortunately, *regular* cardinals are quite abundant – as shown in Exercise 4.5:12, the cardinal ω is regular, and every infinite successor cardinal, i.e., every cardinal of the form $\omega_{\alpha+1}$ for α an ordinal, is also regular. We can now show

Lemma 8.2.2. *Let Ω be a type, and γ a regular cardinal $>\mathrm{card}(\mathrm{ari}(s))$ for all $s\in|\Omega|$. Then for any Ω-algebra A, and any subset $X\subseteq|A|$, if we define the chain of sets $S^{(\alpha)}$ by* (8.2.1), *then $S^{(\gamma)}$ is closed under the operations of A, hence is the underlying set of the subalgebra of A generated by X.*

Proof. Consider any $s\in|\Omega|$ and elements $x_i\in S^{(\gamma)}$ $(i\in\mathrm{ari}(s))$. Since γ is a limit ordinal, $S^{(\gamma)}=\bigcup_{\beta<\gamma}S^{(\beta)}$, hence each x_i lies in some $S^{(\beta_i)}$ $(\beta_i\in\gamma)$. Since $\mathrm{ari}(s)<\gamma$ and γ is regular, $\{\beta_i\mid i\in\mathrm{ari}(s)\}$ is not cofinal in γ, hence this set is majorized by some $\beta<\gamma$. For this choice of β, all x_i lie in $S^{(\beta)}$, hence $s(x_i)\in S^{(\beta+1)}\subseteq S^{(\gamma)}$, as required. □

In the next section we will apply the above result to the construction of left universal objects.

For later use, we record the following generalization of the familiar observation that if an algebra with finitary operations is generated by a set X, each element of the algebra can be expressed in terms of finitely many elements of X.

Lemma 8.2.3. *Let Ω be a type, and γ a regular infinite cardinal $>\mathrm{card}(\mathrm{ari}(s))$ for all $s\in|\Omega|$. Let A be any Ω-algebra, and X any generating set for A. Then each element of $|A|$ belongs to the subalgebra of A generated by a subset $X_0\subseteq X$ of cardinality $<\gamma$.*

Sketch of Proof. It is easy to verify that the set of elements of $|A|$ belonging to subalgebras generated by $<\gamma$ elements of X forms a subalgebra. As it contains X, it must be all of $|A|$. □

Exercise 8.2:3. Write out the easy verification referred to. Show that the result becomes false if the regularity assumption on γ is deleted.

It may now seem anomalous that in our results on direct limits over α-directed partially ordered sets, Proposition 7.9.8 and Lemma 8.1.10, we did *not* have to assume α regular! This is explained by

Exercise 8.2:4. Show that if α is a singular infinite cardinal and J an α-directed partially ordered set, then J is also α'-directed, where α' is the successor cardinal to α.

Thus, if J is α-directed for a cardinal α greater than the arities of all operations of Ω, it is in fact α'-directed for a *regular* cardinal α' greater than the arities of those operations.

We could have avoided using the concept of regular cardinal in this section by taking γ in the our results to be ‘‘the successor cardinal of the least infinite upper bound of the arities of the operation-symbols of Ω’’. However, in the case where Ω is finitary, this would have given $\gamma = \omega_1$, whereas the development we have used shows that ω suffices in that important case.

8.3. Terms and left universal constructions. Given a type Ω and a set X, Lemma 8.2.2 can be used to obtain a bound on the size of an Ω-algebra generated by an X-tuple of elements, and hence to establish the *solution set* hypotheses needed by the existence results for left universal constructions developed in §7.10. Now such a bound can be thought of as an estimate of the number of ‘‘Ω-algebra terms in an X-tuple of variable-symbols’’, and rather than just giving an existence proof, we can, with little additional work, construct such a set of terms, thus laying the groundwork for the more explicit approach to universal constructions that we sketched in §2.2.

Let us first define precisely the concept of a ‘‘term’’. At the beginning of this course (Definition 1.5.1) we described ‘‘the set of group-theoretic terms in the elements of X’’ as a set T given with certain structure: a map of X into it, and a family of ‘‘formal group-theoretic operations’’ satisfying some further conditions. If we make the corresponding definition for Ω-algebras, we see that the ‘‘formal operations’’ in fact make the set T into an Ω-algebra. (We could not similarly say that formal operations made the set of group-theoretic terms into a *group*, because they did not satisfy the group identities. The difference is that in the present treatment, we are studying general algebras of type Ω before introducing identities.) So we state the definition accordingly:

Definition 8.3.1. *Let Ω be any type, and X any set. Then an ‘‘Ω-term algebra on X’’ will mean a pair (F, u), where F is an Ω-algebra, and $u\colon X \to |F|$ a set map, such that*

(i) *the map $u\colon X \to |F|$, and all the maps $s_F\colon |F|^{\mathrm{ari}(s)} \to |F|$ are one-to-one,*

(ii) *the images of the above maps in $|F|$ are disjoint,*

(iii) *the union of these images is all of $|F|$, and*

(iv) *F is generated as an Ω-algebra by X.*

Note that the first three conditions of the above definition can be stated as a single condition: If we write $\sqcup$ for disjoint union of sets, and consider the map u and the operations s_F as defining a single map $X \sqcup \bigsqcup_{s\in|\Omega|} |F|^{\mathrm{ari}(s)} \to |F|$, then (i)-(iii) say that this map is *bijective*.

Since the concept of Ω-algebra involves no identities, the idea of constructing free objects by taking ‘‘terms modulo identities’’ simplifies to

Lemma 8.3.2. *Let Ω be any type, and X any set. Suppose there exists an Ω-term algebra (F, u) on X. Then (F, u) is a free Ω-algebra on X.*

Proof. To prove that (F, u) has the universal property of a free Ω-algebra on X, suppose A is an Ω-algebra and $v: X \to |A|$ any set map. We wish to construct a homomorphism $f: F \to A$ such that $v = fu$. Intuitively f should represent ‘‘substitution of the particular values $v(x)$ for the variable-symbols $u(x)$ in our terms’’.

Let us write $|F|$ as the union of a chain of subsets $S^{(\alpha)}$ as in (8.2.1), starting with the generating set $S^{(0)} = u(X)$. Assume recursively that f has been defined on all the sets $S^{(\beta)}$ with $\beta < \alpha$; we wish to define f on $S^{(\alpha)}$. If $\alpha = 0$, $S^{(\alpha)}$ consists of elements $u(x)$ $(x \in X)$, all distinct, and we let $f(u(x)) = v(x) \in |A|$. If α is a successor ordinal $\beta+1$, then an element which first appears in $S^{(\alpha)}$ will have the form $s_F(t_i)$, where $s \in |\Omega|$ and each $t_i \in |S^{(\beta)}|$. Thus the $f(t_i)$ have already been defined, and we define $f(s_F(t_i)) = s_A(f(t_i))$. If α is a nonzero limit ordinal, then $S^{(\alpha)} = \bigcup_{\beta<\alpha} S^{(\beta)}$, and having defined f consistently on $S^{(\beta)}$ for all $\beta < \alpha$, we have defined it on $S^{(\alpha)}$.

In each case, the one-one-ness condition (i) and the disjointness condition (ii) of Definition 8.3.1 insure that if an element of F occurs at some stage as $u(x)$ or $s_F(t_i)$, it cannot occur (at the same or another stage) in a different way as $u(x')$ or $s'_F(t'_i)$. Hence our definition is unambiguous. By construction, f is a homomorphism of Ω-algebras and satisfies $fu = v$; and by (iv) it is unique for this property. □

I should mention that the technique of explicit induction or recursion on the forms of elements, as in the above proof, is one that seldom has to be used. Arguments showing that if an algebra A is generated by a set X of elements having some property P, then all elements of X satisfy P, can generally be carried out simply by verifying that the set of elements of A satisfying P is closed under the algebra operations, hence forms a subalgebra containing X, hence is all of $|A|$. On the other hand, if we want to construct some *map* on the elements of the free algebra A on a set X starting from its values on elements of X, then (assuming ‘‘map’’ means homomorphism) we can do this using the universal property of A as a free object. In the case of free objects of **Ω-Alg**, we have just proved that universal property by ‘‘recursion on elements’’, but this result frees us from having to repeat that argument in similar situations.

We have not proved the converse statement, that if a *free* Ω-algebra on X exists, it will be an Ω-term algebra on X. That is what we would want if we planned to prove the existence of free algebras first, and deduce from this the existence of term algebras; but we shall be going the other way. However, this implication is not hard to prove; so let us make it

Exercise 8.3:1. Show (without assuming the existence of Ω-term algebras) that if (F, u) is a free Ω-algebra on X, then it is an Ω-term algebra on X.

(Hint: If F fails to satisfy one of conditions (i)-(iv) of Definition 8.3.1, you want to find a pair (A, v) for which the universal property of (F, u) fails. If condition (iii) or (iv) fails, make A a subalgebra of F; if (i) or (ii) fails, obtain A by replacing one element p of F by two elements p_1 and p_2, and defining the operations appropriately on $|F| - \{p\} \cup \{p_1, p_2\}$. Since the operations of Ω-algebras are not required to satisfy any identities, any definition of these operations yields an Ω-algebra.)

Let us now prove

Theorem 8.3.3. *Let Ω be any type, and X any set. Then there exists an Ω-term algebra on X; equivalently, a free Ω-algebra on X.*

Proof. Let $*$ be any element not in $|\Omega|$, and γ a regular cardinal which is $>\mathrm{card}(\mathrm{ari}(s))$ for all $s\in|\Omega|$. We define recursively a chain $(S^{(\alpha)})_{\alpha\leq\gamma}$ of sets of ordered pairs, by taking

$$\begin{aligned} S^{(0)} &= \{(*, x) \mid x\in X\}, \\ S^{(\alpha+1)} &= S^{(\alpha)} \cup \{(s,(x_i)) \mid s\in|\Omega|,\ (x_i)\in(S^{(\alpha)})^{\mathrm{ari}(s)}\}, \\ S^{(\alpha)} &= \textstyle\bigcup_{\beta<\alpha} S^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal } > 0. \end{aligned}$$

Let $|F| = S^{(\gamma)}$, and define $u\colon X\to|F|$, and maps $s_F\colon |F|^{\mathrm{ari}(s)}\to|F|$ $(s\in|\Omega|)$, by

$$\begin{aligned} u(x) &= (*, x) \qquad (x\in X), \\ s_F(x_i) &= (s,(x_i)) \qquad (s\in|\Omega|,\ (x_i)\in|F|^{\mathrm{ari}(s)}). \end{aligned}$$

That the operations s_F carry $|F| = S^{(\gamma)}$ into itself follows from our choice of γ, by the same argument we used in Lemma 8.2.2. Thus these operations make $|F|$ an Ω-algebra F. That F satisfies conditions (i)-(iii) follows from the set-theoretic fact that an ordered pair uniquely determines its first and second components. To get (iv), one verifies by induction that a subalgebra containing X must contain each $S^{(\alpha)}$. □

Since we have free Ω-algebras on all sets X, these give a left adjoint to the underlying-set functor from **Ω-Alg** to **Set**.

Exercise 8.3:2. Show how we could, alternatively, have gotten the existence of such an adjoint using Freyd's Adjoint Functor Theorem (Theorem 7.10.4) and Lemma 8.2.2.

Let us fix a notation for these functors.

Definition 8.3.4. *The underlying-set functor of* **Ω-Alg** *and its left adjoint, the free algebra functor, will be denoted U_Ω:* **Ω-Alg** → **Set** *and F_Ω:* **Set** → **Ω-Alg** *respectively.*

A symbol such as $F_\Omega(\{x_0,\dots,x_{n-1}\})$ may be abbreviated to $F_\Omega(x_0,\dots,x_{n-1})$ when there is no danger of misunderstanding.

The ''danger of misunderstanding'' referred to is that symbol $F_\Omega(X)$ for the free Ω-algebra on a set X might be misinterpreted, under the above convention, as meaning the one-generator free algebra $F_\Omega(\{X\})$. But in context, there is almost never any doubt as to whether a given entity is meant to be treated as a free generator, or as a set of free generators.

There is another sort of looseness in our usage, which we noted in Chapter 2. Although we have formally defined free algebras to be pairs (F, u), we also sometimes use the term for the first components of such pairs, thought of as algebras ''given with'' the set-maps u. (E.g., when we spoke of the free-algebra functor above, the values of the functor were algebras F, not ordered pairs (F, u); the maps u are the values of the unit of the adjunction, $\eta(X)\colon X\to U_\Omega(X)(F_\Omega(X))$.) At other times, we speak of an algebra F as being free on a given set of its elements, without specifying an indexing of this set by any external set (though we can always index it by its identity map to itself). Finally, we may speak of an object as being ''free'', meaning that there exists a generating set on which it is free, but without specifying a particular such set, as when we say that a subgroup of a free abelian group is free abelian. So we need to be sure it is always clear which version of the concept we are using.

The next exercise shows that in a category of the form Ω-**Alg**, and in certain others, the last two of the above senses of ''free algebra'' essentially coincide.

Exercise 8.3:3. (i) Show that a free Ω-algebra is free on a unique set of generators. That is, if (F, u) is a free Ω-algebra, then the *image* of the set map u, and hence also the cardinality of the domain of u, are determined by the Ω-algebra structure of F. (Hint: Definition 8.3.1.)

(ii) Is the analogous statement true for free groups? Free monoids? Free rings?

(iii) Same question for free upper (or lower) semilattices.

(iv) Same question for free lattices. (If you know the structure theorem for free lattices this is not hard. Even if you do not, a little ingenuity will yield the answer by a direct argument.)

Exercise 8.3:4. (i) Show that every subalgebra A of a free Ω-algebra F is free.

In the standard beginning graduate algebra course one learns that the same statement is true of free abelian groups, and it is a basic result of group theory that it is true for free groups. But

(ii) Is the analogous statement true for free monoids? Free rings? Free upper semilattices? Free lattices?

Exercise 8.3:5. (i) Let Ω be a finitary type without zeroary operation-symbols, and $F_\Omega(x)$ the free Ω-algebra on a single generator x. Show that the monoid of endomorphisms $\mathrm{End}(F_\Omega(x))$ (under composition) is a free monoid. If you wish, you may for simplicity assume that $|\Omega|$ consists of a single binary operation-symbol (since even in this case, the description of the free generating set for the monoid $\mathrm{End}(F_\Omega(x))$ is nontrivial).

(ii) Does the result of (i) remain true if the assumption that Ω is finitary is removed?

(iii) Show that the corresponding result is never true if Ω has zeroary operations. Can you describe the monoid in this case?

(iv) If all operation-symbols of Ω have arity 1, describe the monoid $\mathrm{End}(F_\Omega(x))$ precisely in terms of $|\Omega|$.

The next result is easily seen from the explicit description of free Ω-algebras in our proof of Theorem 8.3.3.

Corollary 8.3.5 (to proof of Theorem 8.3.3)**.** *If $a\colon X \to Y$ is an injective* (*respectively surjective*) *map of sets, then the induced map of free Ω-algebras $F_\Omega(a)\colon F_\Omega(X) \to F_\Omega(Y)$ is likewise injective* (*surjective*) *on underlying sets.* □

We can get the same result in a different way in a more general context, though we need some fussy arguments involving empty algebras:

Exercise 8.3:6. (i) Show that *every* functor A: **Set** → **Set** carries surjective maps to surjective maps, and carries injective maps with nonempty domains to injective maps. (Hint: Use right and left invertibility.)

(ii) Show that the corresponding statement is not true for maps with empty domains.

(iii) Show, however, that if A has the form UF, where U is a functor from some category to **Set**, and F is a left adjoint to U, then A carries maps with empty domain to injective maps.

(iv) Deduce Corollary 8.3.5 without calling on our explicit description of free Ω-algebras.

Using free algebras, we can obtain other left universal constructions. A basic tool will be

Definition 8.3.6. *Let* Ω *be a type,* X *a set, and* $(F_\Omega(X),\, u_X)$ *a free* Ω*-algebra on* $X.$ *An* Ω*-algebra* relation *in an* X*-tuple of variables will mean an element* $(s,t)\in|F_\Omega(X)|\times|F_\Omega(X)|$ *(often informally written ''*$s=t$*'').* *An* X*-tuple* v *of elements of an* Ω*-algebra* A *is said to* satisfy *the relation* (s,t) *if the unique homomorphism* $f\colon F_\Omega(X)\to A$ *such that* $fu_X=v$ *has the property* $f(s)=f(t).$

If $R\subseteq|F_\Omega(X)|\times|F_\Omega(X)|$ *is a set of relations, then an* Ω*-algebra* presented *by generators* X *and relations* R *will mean an initial object* (B,w) *in the category whose objects are pairs* (A,v) *with* A *an* Ω*-algebra and* v *an* X*-tuple of elements of* $|A|$ *satisfying all the relations in* $R,$ *and whose morphisms are homomorphisms of first components respecting second components; equivalently, a representing object for the functor* Ω**-Alg** $\to$ **Set** *associating to every* Ω*-algebra* A *the set of* X*-tuples* v *satisfying all the relations in* $R.$ *Such an algebra* B *will be denoted* $\langle X\mid R\rangle_{\Omega\text{-}\mathbf{Alg}}$*, or* $\langle X\mid R\rangle$ *when there is no danger of ambiguity.*

(If we wanted to be more precise, we might write our relations as $(s,\,t,\,(F_\Omega(X),\,u_X)),$ since formally, a given pair of elements s and t can belong to underlying sets of various free algebras. But to avoid messy notation, we will assume that there is no ambiguity as to which free algebra is meant. Also, strictly speaking, the representing object should be given as a pair $(\langle X\mid R\rangle,\,w),$ where w is the canonical map $X\to|\langle X\mid R\rangle|.$ But again we will generally call $\langle X\mid R\rangle$ the representing object, and leave it understood that w is there if we need to refer to it.)

Theorem 8.3.7. *Let* Ω *be a type. Then* Ω**-Alg** *has algebras* $\langle X\mid R\rangle$ *presented by arbitrary sets of generators* X *and relations* $R.$

Proof. $\langle X\mid R\rangle$ can be constructed as the quotient of $F_\Omega(X)$ by the congruence generated by $R.$ □

Exercise 8.3:7. Give an alternative proof of the above theorem using the results of §7.10.

Theorem 8.3.8. *The category* Ω**-Alg** *has all small colimits.*

Proof. By Proposition 7.6.6 (last statement), it is enough to show that Ω**-Alg** has difference cokernels of pairs of morphisms, and small coproducts. We obtained difference cokernels in Lemma 8.1.9; we shall now construct the coproduct of a small family of Ω-algebras $(A_i)_{i\in I}.$

We assume without loss of generality that the A_i have disjoint underlying sets (since we can replace them with disjoint isomorphic algebras if they do not). Let A be the algebra presented by the generating set $\bigcup|A_i|$ and, for relations, all the relations satisfied within the separate A_i's. (Precisely, we take for relations the images in $|F_\Omega(\bigcup_I|A_i|)|\times|F_\Omega(\bigcup_I|A_i|)|,$ under the canonical maps $F_\Omega(|A_j|)\to F_\Omega(\bigcup_I|A_i|),$ of all the relations $(s,t)\in|F_\Omega(|A_j|)|\times|F_\Omega(|A_j|)|$ holding in the given algebras $A_j.$) It is easy to verify that A is the desired coproduct. □

We end this section with two exercises which assume familiarity with elementary topology, concerning certain curious algebras with a single binary operation. The first exercise sets up a general construction, and obtains some basic facts to give you the feel of things. The second asks you to establish a peculiar universal property of an instance of this construction.

Exercise 8.3:8. Consider the set 2^ω of all sequences $(\iota_0,\,\iota_1,\dots)$ of 0's and 1's, topologized using the product topology induced by the discrete topology on $\{0,1\}.$ (This space can be naturally identified with the Cantor set.) Let us define two continuous maps $\alpha,\beta\colon 2^\omega\to 2^\omega,$

by letting

$$\alpha(\iota_0, \iota_1, \ldots) = (0, \iota_0, \iota_1, \ldots), \quad \text{and} \quad \beta(\iota_0, \iota_1, \ldots) = (1, \iota_0, \iota_1, \ldots).$$

Thus, 2^ω is the disjoint union of the two copies of itself, $\alpha(2^\omega)$ and $\beta(2^\omega)$.

Now let Ω be the type determined by a single binary operation $*$, and let us define a covariant functor F from the category **HausTop** of Hausdorff topological spaces to Ω-**Alg**. For every space S, the set $|F(S)|$ will be **HausTop**$(2^\omega, S)$, i.e., the space of continuous S-valued functions on 2^ω. Thus, these sets are given by the covariant hom-functor h_{2^ω}: **HausTop** → **Set**. To describe the binary operation, let $u, v \in |F(S)|$. Then we define $u*v$ to be the function $2^\omega \to S$ such that

$$(u*v)(\alpha(x)) = u(x) \qquad (u*v)(\beta(x)) = v(x) \qquad (x \in 2^\omega).$$

In other words, it is the map whose graph on the first half of the Cantor set looks like the graph of u compressed horizontally, and whose graph on the second half of the Cantor set is a similarly compressed copy of the graph of v. Let $F(S) = (|F(S)|, *)$.

(i) Show that for every S, the map $* : |F(S)| \times |F(S)| \to |F(S)|$ is bijective.

(ii) Let S be any Hausdorff topological space and X any finite subset of $|F(S)|$. Let X_0 be the set of those $x \in X$ which (as maps $2^\omega \to S$) are constant, and X_1 the set of $x \in X$ which are not constant, and such that x does not belong to the Ω-subalgebra of $|F(S)|$ generated by $X - \{x\}$. Show that the Ω-subalgebra of $F(S)$ generated by X can be presented by the generating set $X_0 \cup X_1$, and the relations $x*x = x$ for all $x \in X_0$.

(iii) Show that the set of nonconstant elements of $F(S)$ forms a subalgebra, every finitely generated subalgebra of which is free, but that this subalgebra is not itself free.

Our definition of the binary operation $*$ above involved composing elements of $F(S)$ on the right with α and β. We shall now let our construction take its tail in its mouth, by applying it with $S = 2^\omega$. Since elements of the resulting algebra also have 2^ω as *codomain*, we can compose them on the *left* with α and β; we shall use these constructions to get another map.

Exercise 8.3:9. Let α, β and F: **HausTop** → Ω-**Alg** be defined as in the preceding exercise, and let $A = F(2^\omega)$, an Ω-algebra with underlying set **HausTop**$(2^\omega, 2^\omega)$.

(i) Show that each of the Ω-algebra homomorphisms $F(\alpha), F(\beta) : A \to A$ is an embedding, and that A is the coproduct in Ω-**Alg** of the images of these homomorphisms.

This is equivalent to saying that A is a coproduct of two copies of itself, with coprojection maps $F(\alpha)$ and $F(\beta)$; or, fixing an arbitrary coproduct of two copies of A and calling it $A \amalg A$, and its coprojection maps q_0 and q_1, it is equivalent to saying that the unique map $f : A \amalg A \to A$ satisfying $fq_0 = \alpha$ and $fq_1 = \beta$ is invertible.

We now come to the peculiar universal property. Let $\mathbf{m}_A : A \to A \amalg A$ be the inverse of the above map f.

(ii) Show that if B is *any* Ω-algebra given with a homomorphism $\mathbf{m}_B : B \to B \amalg B$, there exists a unique homomorphism $\theta : B \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, "\mathbf{m}_B"] \arrow[d, "\theta"] & B \amalg B \arrow[d, "\theta \amalg \theta"] \\
A \arrow[r, "\mathbf{m}_A"] & A \amalg A.
\end{tikzcd}$$

(We will be able to make sense of the above universal property in Chapter 9.)

8.4. Identities and varieties. Here is a definition that needs no introduction!

Definition 8.4.1. *Let* Ω *be a type,* X *a set, and* $(F_\Omega(X),\ u_X)$ *a free* Ω*-algebra on* X*. An* identity *in an* X*-tuple of variables will mean an element* $(s,t)\in|F_\Omega(X)|\times|F_\Omega(X)|$*, i.e., formally the same thing as a relation, and likewise often informally written ''*$s=t$*''. However an* Ω*-algebra* A *will be said to ''satisfy'' the identity* (s,t) *if and only if* every X*-tuple* v *of elements of* $|A|$ *satisfies* (s,t) *as a relation; that is, if and only if for* every *homomorphism* $f\colon F_\Omega(X)\to A$*, one has* $f(s)=f(t)$*.*

The next lemma will relate identities in different sets of variables.

Lemma 8.4.2. *Let* Ω *be a type,* X *a set, and* $(s,t)\in|F_\Omega(X)|\times|F_\Omega(X)|$ *an identity in an* X*-tuple of variables. Then if* $f\colon X\to Y$ *is a one-to-one set map, an* Ω*-algebra* A *satisfies the identity* (s,t) *if and only if it satisfies the identity in a* Y*-tuple of variables,* $(F_\Omega(f)(s), F_\Omega(f)(t))$*.*

Hence if γ *is a regular cardinal such that* $\gamma>\mathrm{card}(\mathrm{ari}(s))$ *for all* $s\in|\Omega|$*, every identity* (s,t) *in any set* X *of variables is equivalent to an identity* (s',t') *in a* γ*-tuple of variables (i.e., there is an identity* $(s',t')\in|F_\Omega(\gamma)|\times|F_\Omega(\gamma)|$ *which is satisfied by an* Ω*-algebra* A *if and only if* A *satisfies* (s,t)*).*

Proof. First statement: It is easy to see that a Y-tuple of elements of $|A|$, $v\colon Y\to|A|$, will satisfy the *relation* $(F_\Omega(f)(s), F_\Omega(f)(t))$ if and only if the X-tuple $vf\colon X\to|A|$ satisfies the relation (s,t). Hence if A satisfies (s,t) as an *identity* it will likewise satisfy $(F_\Omega(f)(s), F_\Omega(f)(t))$ as an identity. The converse will hold if we can show that *every* map $w\colon X\to|A|$ can be written vf for some $v\colon Y\to|A|$. It is clear how to define v on elements of the one-to-one image of X in Y under f. If $|A|$ is nonempty, we can extend this map by giving v arbitrary values on other elements of Y. If $|A|$ is empty, on the other hand, then there can be no homomorphisms from the algebra $F_\Omega(X)$, which is nonempty because it contains s and t, into A, so this case is vacuous. (An empty algebra satisfies every identity (s,t), because the hypothesis of the implication defining ''satisfaction'' can never hold!)

To prove the second statement, we note that if $s,t\in|F_\Omega(X)|$, then by Lemma 8.2.3, s and t will lie in the subalgebra generated by some subset $X_0\subseteq X$ of cardinality $<\gamma$. The set X_0 can be mapped injectively into γ; hence applying the first statement of the lemma to the inclusion of X_0 in X on the one hand, and to an embedding of X_0 in γ on the other, we see that (s,t) is equivalent to some identity in a γ-tuple of variables. □

Thus, for the purpose of studying families of identities satisfied by Ω-algebras, and classes of algebras determined by identities, we can restrict ourselves to identities in a γ-tuple of variables for γ as above. (Actually, the above argument shows that every identity can be expressed using $<\gamma$ variables, hence if γ is a successor cardinal $\omega_{\alpha+1}$, we could use terms in an ω_α-tuple of variables to express our identities. However, the case where we care most about getting good bounds is $\gamma=\omega$, corresponding to finitary algebras, and ω is not a successor cardinal; so we will not worry about incorporating this refinement into our lemma.)

Exercise 8.4:1. Show that if all operation-symbols of Ω are of arity <2, then the statement of Lemma 8.2.3 holds with $\gamma=2$ (even though 2 is not a regular cardinal), and deduce that the final statement of the above lemma also holds for $\gamma=2$. On the other hand, show by example that it does not hold for $\gamma=1$.

For the remainder of this section, let us fix a type Ω and a regular cardinal γ greater than the arities of all operation-symbols of Ω, and understand ''identity'' to mean ''Ω-algebra identity in a γ-tuple of variables''. In writing identities, we shall often write x_α for the image $u(\alpha)\in|F_\Omega(\gamma)|$ of $\alpha\in\gamma$. We may also at times write x, y, etc., for x_0, x_1, etc..

Let us denote the set of all Ω-algebra identities by

$$I_\Omega \;=\; |F_\Omega(\gamma)| \times |F_\Omega(\gamma)|.$$

Thus we have a relation of *satisfaction* (Definition 8.4.1) defined between elements of the (large) set $\mathrm{Ob}(\Omega\text{-}\mathbf{Alg})$ of all Ω-algebras and elements of the (small) set I_Ω of all identities. If C is a (not necessarily small) set of Ω-algebras, we may write C^* for the set of identities satisfied by all members of C, and if J is a set of identities, we may write J^* for the (large) set of Ω-algebras that satisfy all identities in J. The theory of Galois connections (§5.5) tells us that the two composite operators ** will be closure operators, that every set J^* or C^* will be closed under the appropriate closure operator **, and that the operators * give an antiisomorphism between the complete lattice of all closed sets of algebras and the complete lattice of all closed sets of identities.

In talking about this Galois connection, it is obviously not convenient to apply to sets of algebras our convention that sets are small if the contrary is not stated, so

Convention 8.4.3. *For the remainder of this chapter we suspend for* sets of algebras (*as we have done from the start for object-sets of categories*) *the assumption that sets are small if the contrary is not stated.*

However, we still assume that any set of algebras is a subset *of our universe if the contrary is not stated; i.e., the smallness convention still applies to the underlying set of* each *algebra.*

Definition 8.4.4. *A* variety *of Ω-algebras means a full subcategory* $\mathbf{V}$ *of* Ω-**Alg** *having for object-set the set J^* of algebras determined by some set J of identities. The variety with object-set J^* will be written* $\mathbf{V}(J)$. *A category is called a* variety of algebras *if it is a variety of Ω-algebras for* some *type* Ω.

An algebra belonging to a variety $\mathbf{V}$ *will be called a* $\mathbf{V}$*-algebra. The least variety of Ω-algebras whose object-set contains a given set C of algebras, that is, the full subcategory of* Ω-**Alg** *with object-set C^{**}, is called the variety* generated *by C, written* $\mathbf{Var}(C)$.

An equational theory *for Ω-algebras means a subset of I_Ω* (*i.e., a family of identities for Ω-algebras*) *which can be written C^* for some set C of Ω-algebras; specifically, C^* is called the equational theory of the class C. If* $\mathbf{C}$ *is a full subcategory of* Ω-**Alg**, *then the equational theory of* $\mathrm{Ob}(\mathbf{C})$ *may also be called ''the equational theory of* $\mathbf{C}$''. *The least equational theory containing a set J of identities, namely, J^{**}, is called the equational theory* generated *by J.*

Examples: The categories we have named **Group**, **Ab**, **Monoid**, **Semigroup**, $\mathbf{Ring}^1$, $\mathbf{CommRing}^1$, $\vee$-**Semilattice**, $\wedge$-**Semilattice** and **Lattice** are all varieties of algebras (up to trivial notational adjustment; e.g., we originally defined an object of **Group** as a 4-tuple $(|G|,\mu,\iota,\varepsilon)$; under our present definition it is a pair $(|G|,(\mu,\iota,\varepsilon)))$. For every group G, the category G-**Set** is a variety; for every ring R the category R-**Mod** is a variety, and for every commutative ring k the category of all associative k-algebras is a variety. For every type Ω, the whole category Ω-**Alg** is a variety (the greatest element in the complete lattice of varieties of Ω-algebras, definable by the empty set of identities. Its equational theory consists of the tautological identities (s,s).) Taking for Ω the trivial type, with no operation-symbols, we see that **Set** is (up to notational adjustment) a variety.

Examples of categories which are not, as we have constructed them, varieties of algebras are **POSet**, **Top**, $\mathbf{Set}^{\mathrm{op}}$, **RelSet**, the category of *complete* lattices, the full subcategory of $\mathbf{CommRing}^1$ consisting of the *integral domains*, and the category of *torsion-free* groups (groups without elements of finite order other than e). How to determine whether any of these is or is not *equivalent to* a variety of algebras is a question we are not yet ready to tackle.

Remark 8.4.5. An algebra A satisfies the identity $x_0 = x_1$ if and only if all its elements are equal. Hence an algebra satisfying this identity satisfies all identities; i.e., $\{(x_0, x_1)\}^{**} = I_\Omega$, the greatest element of the lattice of equational theories of Ω-algebras. The corresponding variety of Ω-algebras is the least element of the lattice of such varieties, and consists of algebras with *at most one* element. If Ω has any zeroary operation-symbols, then this variety consists only of one-element algebras, which are all isomorphic; thus the variety is equivalent to the category **1** with only one object and its identity morphism. If Ω has no zeroary operations, then this least variety contains both the empty algebra and all one-element algebras, and is equivalent to the 2-object category **2**.

Let us establish some easy results about varieties.

Proposition 8.4.6. *Let* $\mathbf{V} \subseteq \Omega\text{-}\mathbf{Alg}$ *be a variety. Then:*

(i) *Any subalgebra of an algebra in* $\mathbf{V}$ *again lies in* $\mathbf{V}$.

(ii) *The limit* $\varprojlim_{\mathbf{D}} A(D)$, *taken in* $\Omega\text{-}\mathbf{Alg}$, *of any functor* A *from a small category* $\mathbf{D}$ *to* $\mathbf{V} \subseteq \Omega\text{-}\mathbf{Alg}$ *again lies in* $\mathbf{V}$.

(iii) *Any homomorphic image of an algebra in* $\mathbf{V}$ *again lies in* $\mathbf{V}$.

(iv) *The direct limit (colimit)* $\varinjlim A_j$, *taken in* $\Omega\text{-}\mathbf{Alg}$, *of any* γ*-directed system of* $\mathbf{V}$*-algebras again lies in* $\mathbf{V}$. (*See Lemma 8.1.10 for a description of this direct limit.*)

In particular, $\mathbf{V}$ *has small limits, has difference cokernels, and has colimits of* γ*-directed systems, and these are the same in* $\mathbf{V}$ *as in* $\Omega\text{-}\mathbf{Alg}$.

Proof. It is straightforward that if an algebra satisfies an identity, any subalgebra or homomorphic image satisfies the same identity, giving (i) and (iii) above, and that a direct product of algebras satisfying an identity again satisfies that identity. Since arbitrary limits can be constructed using products and difference kernels, and in $\Omega\text{-}\mathbf{Alg}$ difference kernels are certain subalgebras, we get (ii). To show (iv), let L be the direct limit in $\Omega\text{-}\mathbf{Alg}$ of a γ-directed system of algebras $(A_i)_I$ of $\mathbf{V}$, let (s, t) be an identity of $\mathbf{V}$, say involving the first $\alpha < \gamma$ variables, and hence equivalent to an identity (s', t') in an α-tuple of variables, and let v be an α-tuple of elements of L. By Lemma 8.1.10 (second paragraph) $|L|$ is the direct limit of the sets $|A_i|$, hence by γ-directedness of I, we can find an $i \in I$ such that A_i contains inverse images of all members of v. The α-tuple formed from these inverse images will satisfy the relation (s', t'), hence so does v, its image. Hence our direct limit object satisfies the identity (s', t'), and hence the equivalent identity (s, t).

The final assertion follows immediately by Lemma 7.6.7. □

Corollary 8.4.7. *Let* $\mathbf{V}$ *be a variety of* Ω*-algebras. Then*

(i) *The forgetful functor from* $\mathbf{V}$ *to* **Set** *respects limits, and also colimits over* γ*-directed partially ordered sets.*

(ii) *The inclusion functor into* $\mathbf{V}$ *of any subvariety* $\mathbf{W}$ *respects these constructions, and also* *respects difference cokernels.*

(iii) *Direct limits in* **V** *over* γ*-directed partially ordered sets respect limits in* **V** *over categories* **D** *having* $<\gamma$ *objects, and having morphism-sets generated by* $<\gamma$ *morphisms.* □

Exercise 8.4:2. Verify that the above corollary indeed follows from results we have proved.

We saw in Lemma 6.9.3 that if a category **C** is given with a concept of a *subobject* of an object, then one likewise gets a concept of a *subfunctor* of a **C**-valued functor. Let us make, for future reference

Definition 8.4.8. *If* **V** *is a variety of algebras, then unless the contrary is stated, references to* subfunctors *of* **V***-valued functors* F *are to be interpreted with ''subobject'' meaning ''subalgebra''.*

Thus, for any category **C** *and functor* $F: \mathbf{C} \to \mathbf{V}$, *a subfunctor* G *of* F *is (essentially) a construction associating to every* $X \in \mathrm{Ob}(\mathbf{C})$ *a subalgebra* $G(X) \subseteq F(X)$, *in such a way that for every morphism* $f: X \to Y$ *of* **C**, *the* **V***-algebra homomorphism* $F(f)$ *carries* $G(X) \subseteq F(X)$ *into* $G(Y) \subseteq F(Y)$.

The subfunctors of group- and vector-space-valued functors considered in the exercises following Lemma 6.9.3 are examples of this concept. (If you didn't do the last part of Exercise 6.9:11, this might be a good time to look at it again.)

Let us prove for general varieties a pair of facts that we noted earlier in many special cases.

Proposition 8.4.9. *A morphism* $f: A \to B$ *in a variety* **V** *is one-to-one if and only if it is a monomorphism, and surjective if and only if it is a difference cokernel.*

Proof. By Exercise 6.8:7, if f is one-to-one on underlying sets, it is a monomorphism. To get the converse, consider the congruence E associated to f. This is the underlying set of a subalgebra C of $A \times A$, hence it is an object of **V**, and the projections of this object onto the two factors are morphisms $C \rightrightarrows A$ having the same composite with f. Hence if f is a monomorphism, these two projections must be equal, which means that E can contain no nondiagonal elements of $|A| \times |A|$, which says that f is one-to-one. (Observe that this argument is not valid in an arbitrary full subcategory of **V**, since such a subcategory may contain A without containing C. For an example, see Exercise 6.7:5.)

We have observed that difference cokernels in **V** are difference cokernels in **Ω-Alg**, and that these are surjective. Conversely, if f is surjective, it is easy to verify that it has the universal property of the difference cokernel in **V** of the pair of maps $C \rightrightarrows A$ just defined. □

The above result does not discuss the relation between one-one-ness and the condition of being a difference *kernel* map, nor between onto-ness and being an *epimorphism*. We know from Lemma 7.6.2 that if a morphism f in a category **C** is a difference kernel map it is a monomorphism, hence if **C** is a variety of algebras, difference kernel morphisms are one-to-one. Likewise, all difference cokernel morphisms – hence in a variety of algebras, all surjective maps – are epimorphisms. Neither of the converse statements is true, but they are tied together in a curious way:

Exercise 8.4:3. (i) Show that if a variety **V** of algebras has an epimorphism which is not surjective (cf. Exercise 6.7:6(iii)) then it also has a one-to-one map which is not a difference kernel.

(ii) Is the reverse implication true?

Exercise 8.4:4. The proof of Proposition 8.4.9 used the facts that **V** is closed in Ω-**Alg** under products and subalgebras. Which of these two conditions is missing in the example from Exercise 6.7:5 mentioned in that proof? Can you also find an example in which only the other condition is missing?

We turn now to constructions which are not the same in a variety **V** and the larger category Ω-**Alg**. We will get these via the next lemma. Let us give both the proof of that result based on the ''big direct product'' idea (Freyd's Adjoint Functor Theorem), and the one based on ''terms modulo consequences of identities''.

Lemma 8.4.10. *If* **V** *is a variety of* Ω*-algebras, the inclusion functor of* **V** *into* Ω-**Alg** *has a left adjoint.*

First Proof. We have seen that **V** has small limits and that these are respected by the inclusion functor into Ω-**Alg**, so by Freyd's Adjoint Functor Theorem, it suffices to verify the solution-set condition. If $A \in \mathrm{Ob}(\Omega\text{-}\mathbf{Alg})$, then every Ω-algebra homomorphism f of A into a **V**-algebra B factors through the quotient of A by the congruence E associated to f. Since the factor-algebra A/E is isomorphic to a subalgebra of B, it belongs to **V**. Hence the set of all factor-algebras of the given Ω-algebra A which belong to **V**, with the canonical morphisms $A \to A/E$, is the desired solution-set.

Second Proof. Let $\mathbf{V} = \mathbf{V}(J)$, the variety determined by the set of identities $J \subseteq |F_\Omega(\gamma)| \times |F_\Omega(\gamma)|$. Given $A \in \mathrm{Ob}(\Omega\text{-}\mathbf{Alg})$, let $E \subseteq |A| \times |A|$ be the congruence on A generated by all relations $(f(s), f(t))$ with $(s,t) \in J$ and $f: F_\Omega(\gamma) \to A$ a homomorphism. Then it is straightforward to verify that A/E belongs to **V**, and is universal among homomorphic images of A belonging to **V**. □

We shall call the above left adjoint functor the construction of *imposing the identities of* **V** on an Ω-algebra A. Note that if we impose the identities of **V** on an algebra already in **V**, we get the same algebra.

We can now get the rest of the constructions we want:

Theorem 8.4.11. *Let* **V** *be a variety of* Ω*-algebras. Then* **V** *has small colimits, objects presented by generators and relations, and free objects on all small sets. All of these constructions can be achieved by performing the corresponding constructions in* Ω-**Alg**, *and then imposing the identities of* **V** *on the resulting algebras* (*i.e., applying the left adjoint obtained in the preceding lemma*).

Proof. The existence of these constructions in Ω-**Alg** was shown in Theorems 8.3.8, 8.3.7 and 8.3.3. That left adjoints respect such constructions was proved in Theorems 7.8.3, 7.7.1, and 7.3.5. □

Let us generalize to arbitrary varieties some notation that we had set up for categories Ω-**Alg**:

Definition 8.4.12. *The free-object functor and the underlying-set functor associated with a variety* $\mathbf{V}$ *will be denoted* $F_{\mathbf{V}}: \mathbf{Set} \to \mathbf{V}$ *and* $U_{\mathbf{V}}: \mathbf{V} \to \mathbf{Set}$. *The* $\mathbf{V}$*-algebra presented by a generating set* X *and relation set* R *will be denoted* $<X \mid R>_{\mathbf{V}}$, *or* $<X \mid R>$ *when there is no danger of confusion.*

In presenting a $\mathbf{V}$-algebra, it is often convenient to take a ''relation'' in an X-tuple of variables to mean a pair of elements of $F_{\mathbf{V}}(X)$ rather than of $F_{\Omega}(X)$. If we write $q: F_{\Omega}(X) \to F_{\mathbf{V}}(X)$ for the canonical homomorphism, it is clear that given a relation $(s,t) \in |F_{\Omega}(X)| \times |F_{\Omega}(X)|$ and an X-tuple v of elements of a $\mathbf{V}$-algebra A, the elements s and t will fall together under the homomorphism $F_{\Omega}(X) \to A$ determined by v if and only if $q(s)$ and $q(t)$ fall together under the homomorphism $F_{\mathbf{V}}(X) \to A$ determined by v; so the same condition is expressed by the original relation $(s,t) \in |F_{\Omega}(X)| \times |F_{\Omega}(X)|$, and by the induced ''relation'' $(q(s), q(t)) \in |F_{\mathbf{V}}(X)| \times |F_{\mathbf{V}}(X)|$. In particular, if R is a subset of $|F_{\mathbf{V}}(X)| \times |F_{\mathbf{V}}(X)|$, we may denote by $<X \mid R>_{\mathbf{V}}$ the quotient of $F_{\mathbf{V}}(X)$ by the congruence generated by R.

The following observation will be of importance to us in Chapter 9.

Lemma 8.4.13. *Let* $\mathbf{W}$ *be a variety of algebras, and* $U: \mathbf{W} \to \mathbf{Set}$ *a functor. Then the following conditions are equivalent:*

(i) U *is representable; i.e., there exists an object* R *of* $\mathbf{W}$ *such that* U *is isomorphic to the functor* $h_R = \mathbf{W}(R, -)$.

(ii) *There exists a set* X, *and a set of relations in an* X*-tuple of variables,* $Y \subseteq |F_{\Omega}(X)| \times |F_{\Omega}(X)|$, *such that* U *is isomorphic to the functor associating to every object* A *of* $\mathbf{W}$ *the set* $\{\xi \in |A|^X \mid (\forall\ (s,t) \in Y)\ s_A(\xi) = t_A(\xi)\}$.

Proof. If U is represented by R, take a presentation $R = <X \mid Y>$; then U will have the form shown in (ii). Conversely, if U is as in (ii), it is represented by the algebra with presentation $<X \mid Y>$. □

Thus, we immediately see that such set-valued functors on **Group** as $G \mapsto \{x \in |G| \mid x^2 = e\}$ and $G \mapsto \{(x,y) \in |G|^2 \mid xy = yx\}$ are representable. A less obvious case is the ''set of invertible elements'' functor on monoids. If we try to use the criterion of the above lemma with X a singleton, it does not work, because the condition of invertibility is not an equation in x alone. However, because inverses are *unique* when they exist, we see that this construction is isomorphic to the functor $S \mapsto \{(x,y) \in |S|^2 \mid xy = e = yx\}$, which is of the required form.

In part (ii) of the above lemma, X and/or Y may, of course, be empty. If Y is empty, then U is the Xth power of the underlying-set functor (Definition 6.9.8), and is represented by $F_{\mathbf{V}}(X)$. An example with X but not Y empty is the functor $\mathbf{Ring}^1 \to \mathbf{Set}$ represented by $\mathbf{Z}_n$ for an integer n. We recall that this ring is presented by the empty set of generators, and the one relation $n = 0$ (where ''n'' as a ring element means the n-fold sum $1 + \ldots + 1$). This ring admits no homomorphism to a ring A unless $n = 0$ in A, while when A satisfies that equation, there is a unique ring homomorphism $\mathbf{Z}_n \to A$ (namely the additive group map taking $1_{\mathbf{Z}_n}$ to 1_A). Thus, $h_{\mathbf{Z}_n}$ takes A to the empty set if the characteristic of A does not divide n, and to a one-element set if it does. In terms of point (ii) of the above lemma, this functor must be described as sending A to ''the set of 0-tuples of elements of A such that $n = 0$''. This sounds peculiar because the ''such that'' clause does not refer to anything in the preceding phrase; but it is logically correct: we get the unique 0-tuple if $n = 0$ in A, and nothing otherwise.

Exercise 8.4:5. Determine which of the following set-valued functors are representable. In each case where the answer is affirmative, give an ‘‘X’’ and ‘‘Y’’ as in Lemma 8.4.13. In (i)-(v), n is a fixed integer.

(i) The functor on $\mathbf{Ring}^1$ taking A to a singleton if n is invertible in A, and to the empty set otherwise.

(ii) The functor on $\mathbf{Ring}^1$ taking A to its underlying set if n is invertible in A, and to the empty set otherwise.

(iii) The functor on $\mathbf{Ab}$ taking A to the kernel of the endomorphism ‘‘multiplication by n’’.

(iv) The functor on $\mathbf{Ab}$ taking A to the image of this endomorphism.

(v) The functor on $\mathbf{Ab}$ taking A to the cokernel of this endomorphism.

(vi) The functor on $\mathbf{Lattice}$ taking A to the set of pairs (x, y) such that $x \leq y$.

(vii) For P a fixed partially ordered set, the functor on $\mathbf{Lattice}$ taking A to the set of isotone maps from P to the ‘‘underlying’’ partially ordered set of $|A|$.

8.5. Derived operations.

8.5. Derived operations. Having identified Ω-algebra terms s with elements of free Ω-algebras $F_{\Omega\text{-}\mathbf{Alg}}(X)$, our viewpoint in ‘‘evaluating’’ these terms has been, ‘‘a choice of an X-tuple v of elements in an Ω-algebra A induces a homomorphism $F_{\Omega\text{-}\mathbf{Alg}}(X) \to A$’’. But as noted in §1.6, we can modify which variable(s) – the X-tuple v, the term s, or both – we foreground. We do this in the next definition, again replacing Ω-$\mathbf{Alg}$ with a general variety $\mathbf{V}$.

Definition 8.5.1. *Let* $\mathbf{V}$ *be a variety of algebras,* X *a set, and* (F, u) *the free* $\mathbf{V}$*-algebra on* X*. For every element* $s \in |F|$*, every* $\mathbf{V}$*-algebra* A*, and every* X*-tuple* v *of elements of* $|A|$*, let us denote by*

$$\mathrm{eval}(s, A, v) \in |A|$$

the image of the element s *under the unique homomorphism* $f\colon F_{\mathbf{V}}(X) \to A$ *such that* $fu = v$ *(intuitively, the result of substituting into the term* s *the* X*-tuple* v *of values in* A*).*

For fixed s *and* A*, the function taking each* $v \in |A|^X$ *to* $\mathrm{eval}(s, A, v)$ *will be written*

$$s_A\colon\ |A|^X \to |A|.$$

A derived X-ary operation *on* A *will mean a map* $|A|^X \to |A|$ *which is equal to* s_A *for some* $s \in |F_{\mathbf{V}}(X)|$*.*

More generally, given s *and any full subcategory* $\mathbf{C}$ *of* $\mathbf{V}$ *(e.g., a one-object subcategory, or all of* $\mathbf{V}$*), if we write* $U_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{Set}$ *for the restriction to* $\mathbf{C}$ *of the underlying-set functor of* $\mathbf{V}$*, and* $U_{\mathbf{C}}^X\colon \mathbf{C} \to \mathbf{Set}$ *for the functor carrying an object* A *to the set* $U_{\mathbf{C}}(A)^X$*, then* $s_{\mathbf{C}}\colon U_{\mathbf{C}}^X \to U_{\mathbf{C}}$ *will denote the morphism of functors* $\mathbf{C} \to \mathbf{Set}$ *which on each object* A *of* $\mathbf{C}$ *acts by* s_A*. A morphism* $U_{\mathbf{C}}^X \to U_{\mathbf{C}}$ *which can be written* $s_{\mathbf{C}}$ *for some* $s \in |F_{\mathbf{V}}(X)|$ *will be called a derived* X*-ary operation of* $\mathbf{C}$*.*

Note that the derived operations will in particular include the *primitive operations* $s_A\colon\ |A|^{\mathrm{ari}(s)} \to |A|$ (respectively, $s_{\mathbf{C}}\colon U_{\mathbf{C}}^{\mathrm{ari}(s)} \to U_{\mathbf{C}}$), corresponding to the operation symbols $s \in |\Omega|$, and the *projection* operations $p_{X,x}\colon |A|^X \to |A|$ (respectively $U_{\mathbf{C}}^X \to U_{\mathbf{C}}$), induced by the free generators $u(x) \in F_{\mathbf{V}}(X)$.

Let us now follow up on some ideas that we toyed with at the end of §2.3. Given any full subcategory $\mathbf{C}$ of our variety $\mathbf{V}$, consider the large set of all ‘‘generalized operations on $\mathbf{C}$ in an X-tuple of variables’’, i.e., functions f associating to each object A of $\mathbf{C}$ a map f_A:

$|A|^X \to |A|$ in an arbitrary way. If we look at the set of all these generalized operations as an enormous direct product, $\prod_{A\in\mathrm{Ob}(\mathbf{C})} |A|^{|A|^X}$ (living in the next larger universe), we see that it can be made a large $\mathbf{V}$-algebra under pointwise operations; let us denote this algebra by $\mathrm{GenOp}_{\mathbf{C}}(X)$. We are not interested in this bloated monster for itself, but for the observation that the (still generally large) set of morphisms of functors, $\mathbf{Set}^{\mathbf{C}}(U^X, U)$ forms a subalgebra therein. (A description of the $\mathbf{V}$-algebra structure on this set of morphisms might have seemed unnatural without the context of the algebra structure on $\mathrm{GenOp}(\mathbf{C})$, which is why we began with the latter. Incidentally, when we first discussed this in §2.3, we were not sure it made sense to talk about large sets. Having adopted the Axiom of Universes, and the associated interpretation of large sets, we can deal with these safely!) We shall call this the algebra of *functorial X-ary operations* on $\mathbf{C}$. The *derived* X-ary operations of $\mathbf{C}$ form a subalgebra of this subalgebra:

(8.5.2) $$\mathrm{DerOp}_{\mathbf{C}}(X) \ \subseteq\ \mathbf{Set}^{\mathbf{C}}(U^X, U) \ \subseteq\ \mathrm{GenOp}_{\mathbf{C}}(X).$$

Note that the algebra of derived operations is quasi-small, i.e., isomorphic to a small algebra, since it is a homomorphic image of $F_{\mathbf{V}}(X)$. The image of each generator $x\in X$ will be the function carrying an X-tuple to its xth coordinate, thus, these ''coordinate functions'' generate $\mathrm{DerOp}_{\mathbf{C}}(X)$ as an Ω-algebra. We can describe the resulting algebra nicely, and, under appropriate hypotheses, the algebra of functorial operations as well:

Lemma 8.5.3. *Let* $\mathbf{C}$ *be a full subcategory of a variety* $\mathbf{V}$, *and* X *a* (*small*) *set. Then the* (*large*) *algebra of derived X-ary operations on* $\mathbf{C}$ *is isomorphic to the* (*small*) *algebra* $F_{\mathbf{Var}(\mathrm{Ob}(\mathbf{C}))}(X)$.

Moreover, if $\mathbf{C}$ *contains the free* $\mathbf{V}$*-algebra on* X, *then every functorial X-ary operation on* $\mathbf{C}$ *is a derived operation; i.e.,* $\mathbf{Set}^{\mathbf{C}}(U^X, U) = \mathrm{DerOp}_{\mathbf{C}}(X) \cong F_{\mathbf{V}}(X)$.

Sketch of Proof. The first assertion is straightforward. To prove the second, assume $\mathbf{C}$ contains the free algebra $F_{\mathbf{V}}(X)$, and show that a functorial X-ary operation on $\mathbf{C}$ is determined by its value on the universal X-tuple u of elements of $F_{\mathbf{V}}(X)$; equivalently, apply Yoneda's Lemma to $U^X \cong h_{F_{\mathbf{V}}(X)}$ and $U \cong h_{F_{\mathbf{V}}(1)}$. □

Exercise 8.5:1. Give the details of the above proof.

Exercise 8.5:2. (i) Show that if $\mathbf{C}$ is the full subcategory of all *finite algebras* in $\mathbf{V}$, then the algebra of functorial X-ary operations on $\mathbf{C}$ can be described as the inverse limit of all finite homomorphic images of $F_{\mathbf{V}}(X)$. (Make this statement precise.)

(ii) Show that if $\mathbf{V} = \mathbf{Group}$, $\mathbf{C}$ is as in (i), and $X = 1$, then this group of operations is uncountable. Give an explicit example of an operation in this group that is not a derived group-theoretic operation.

(iii) Interpret Exercise 6.9:6, especially part (ii) thereof, in terms of point (i) above, and if you had not yet successfully done that exercise, see whether you can make further progress on it.

In part (ii) above, the map from functorial operations on general groups to functorial operations on finite groups failed to be surjective. There are also situations where such maps fail to be one-to-one:

Exercise 8.5:3. (i) Give an example of a variety $\mathbf{V}$ not generated by its finite algebras. (If possible, get such an example in which the variety is defined by finitely many operation-symbols, all of finite arities, and finitely many identities.)

(ii) Show that the property asked for in the first sentence above is equivalent to saying that the restriction map from functorial operations in finitely many variables on **V** to such operations on the finite objects of **V** is not one-to-one.

We saw in Exercise 8.5:2(ii) above that though the variety of all groups has only countably many functorial operations of any finite arity, its full subcategory of *finite* groups has uncountably many such operations. One may ask whether, for **C** a full subcategory of a variety **V**, the class of functorial operations of **C** must even be quasi-small.

The answer depends on one's foundational assumptions; I will briefly summarize the situation. Logicians have asked the question, ''Does there exist a proper class (in our language, a non-small set) of (small) models of some first-order theory, none of which is embeddable in another?'' The answer turns out to depend on one's choice of universe. If U is the smallest universe, or a successor element in the well-ordered set of universes, the answer is yes. The negative answer, on the other hand, is called ''Vopěnka's principle''; the existence of a universe for which this holds is equivalent to the existence of a cardinal with some special properties (which force it to be enormous) but is thought likely to be consistent with ZFC.

Now the positive answer to the above question, which, as noted, is true in ''most'' universes, is known to imply the existence of a non-small set C of small algebras of some finitary type Ω such that there are no homomorphisms between distinct members of C. Given such a C, let **C** be the full subcategory of **Ω-Alg** with C as object-set. Then we see that the definition of a functorial operation f on **C** involves no conditions relating the behavior of f on *different* objects. So, for instance, for every subset $B \subseteq C$, there is a functorial binary operation on **C** which acts as the first-coordinate function on algebras in B, and as the second-coordinate function on algebras not in B. Thus, in ''most'' universes we indeed have a class of algebras with a non-small set of functorial binary operations.

Let me end this section with some interesting questions about operations on the real and rational numbers which, so far as I know, are open.

Exercise 8.5:4. (Harvey Friedman)

(i) If we make the set of real numbers an algebra under the single binary operation $a(x,y) = x^2+y^3$, does this algebra satisfy any nontrivial identities?

(ii) If we make the set of nonnegative real numbers an algebra under the single binary operation $c(x,y) = x^{1/2}+y^{1/3}$, does this algebra satisfy any nontrivial identities?

(iii) Does there exist a derived binary operation on the ring **Q** of rational numbers which is one-to-one as a map $|\mathbf{Q}|\times|\mathbf{Q}| \to |\mathbf{Q}|$?

If you cannot answer this last question, you might hand in proofs that the answer to the corresponding question for the ring of integers is ''yes'', and for the ring of real numbers, ''no''.

Another question posed by Friedman along the lines of (i) and (ii) above was whether the group of bijective maps $\mathbf{R} \to \mathbf{R}$ generated by the two maps $p(x) = x+1$ and $q(x) = x^3$ is free on those two generators. This was answered affirmatively in [**105**], with 3 replaced by any odd prime (see [**49**] for a simplified proof). The result has subsequently been generalized to show, essentially, that the group of maps generated by exponentiation by all positive rational numbers and addition of all real constants is the coproduct of the two groups generated by these two sorts of maps [**48**], and, in another direction, to show that the group generated by exponentiation by positive rationals with odd numerator and denominator, addition of real algebraic numbers, and *multiplication* by nonzero real algebraic numbers, is the coproduct of the group generated by the above addition and multiplication operations and the group generated by the multiplication and exponentiation operations, with amalgamation of the obvious common subgroup [**30**].

8.6. Characterizing varieties and equational theories. We observed at the end of §5.5 that when one obtains a Galois connection from a relation on a pair of sets, $R \subseteq S \times T$, the closure X^{**} or Y^{**} of a subset $X \subseteq S$ or $Y \subseteq T$ is constructed ‘‘from above’’, namely as the set of members of the larger set S or T that satisfy certain conditions; and that a recurring type of mathematical question is how to describe these closures ‘‘from below’’, as all elements obtainable from members of X or Y by iterating some constructions. In the case of the Galois connection between Ω-algebras and identities, these questions are: Given a set C of Ω-algebras, how can we ‘‘construct’’ from these all algebras of the variety C^{**} that they generate; and given a set J of identities, how can we construct from these the identities comprising the equational theory J^{**} that they generate? Answers to these questions should, in particular, give internal criteria for when a set of algebras is a variety, and for when a set of identities is an equational theory.

I said in §5.5 that a general approach to this kind of question is to look for operations which carry every set Y^* or X^* into itself, and having found all one can, to try to show that closure under these operations is sufficient, as well as necessary, for a set to be closed.

Now we have shown that a variety of algebras is closed under forming subalgebras, homomorphic images, products, and γ-directed direct limits for appropriate γ. (Closure under general limits need not be mentioned, since it is implied by closure under products and subalgebras. On the other hand, the existence of free objects, coproducts, etc., cannot be used in such a characterization, since they are only defined relative to the variety we are trying to construct.) The next result shows that three of the above four closure conditions suffice to characterize varieties.

In reading that result, recall that by Convention 8.4.3, sets C of algebras are not assumed small.

Theorem 8.6.1 (Birkhoff’s Theorem)**.** *Let Ω be a type. Then a set of Ω-algebras forms a variety if and only if it is closed under forming homomorphic images, subalgebras, and products* (*of small families*).

In fact, if C is a set of Ω-algebras, then any member of the variety generated by C can be written as a homomorphic image *of a* subalgebra *of a* product *of members of C.*

Proof. Clearly, it suffices to prove the final assertion. Let $\mathbf{V} = \mathbf{Var}(C)$, the variety generated by the set C. An algebra belonging to $\mathbf{V}$ can be written as a *homomorphic image* of the free $\mathbf{V}$-algebra $F_{\mathbf{V}}(X)$ for some set X, hence it suffices to show that $F_{\mathbf{V}}(X)$ can be obtained as a *subalgebra* of a *product* of objects in C. To show this, let $N \subseteq |F_{\Omega}(X)| \times |F_{\Omega}(X)|$ denote the set of all pairs (s,t) that are *not* identities of $\mathbf{V}$; equivalently, which are not identities of all members of C. For each $(s,t)\in N$, choose an X-tuple $v_{(s,t)}$ of elements of an algebra $A_{(s,t)}\in C$ such that $v_{(s,t)}$ fails to satisfy the relation (s,t). Let P be the product algebra $\prod_{(s,t)\in N} A_{(s,t)}$, and let $v\colon X \to |P|$ be the set map with (s,t)-component $v_{(s,t)}$ for each $(s,t)\in N$. It follows from its definition that this X-tuple v satisfies none of the relations in N; on the other hand, since P belongs to $\mathbf{V}$, it must satisfy all relations *not* in N. It is easily deduced that the subalgebra $F \subseteq P$ generated by this X-tuple is isomorphic to the free algebra $F_{\mathbf{V}}(X)$. □

The last sentence of Theorem 8.6.1 is often expressed in operator language:

$$\mathbf{Var}(C) \;=\; \mathbf{H\,S\,P}(C). \tag{8.6.2}$$

To make this precise, let us fix a type Ω, and let L_{Ω} denote the large lattice of all subsets $C \subseteq \mathrm{Ob}(\Omega\text{-}\mathbf{Alg})$ which are closed under going to isomorphic algebras (i.e., which satisfy $T \cong S\in C \Rightarrow T\in C$. This is essentially the power set of the set of isomorphism classes of algebras in Ω-**Alg**.)

For each $C\in|L_\Omega|$, let us define

$$\mathbf{H}(C) = \{\text{homomorphic images of algebras in } C\},$$
$$\mathbf{S}(C) = \{\text{subalgebras of algebras in } C\},$$
$$\mathbf{P}(C) = \{\text{products of algebras in } C\}.$$

Then (8.6.2) indeed expresses the last sentence of Theorem 8.6.1. (Except that $\mathbf{Var}(C)$ should, more precisely, be $\mathrm{Ob}(\mathbf{Var}(C))$. But we will ignore that distinction in this discussion, to give this statement the form in which it is usually stated.)

(The restriction to classes closed under isomorphism is not assumed in some discussions of this topic, leading to somewhat capricious behavior of the above operators: For C a class of algebras not necessarily closed under isomorphism, $\mathbf{H}(C)$ is nevertheless closed under going to isomorphic algebras, by the definition of ‘‘homomorphic image’’, though it loses this property if the definition of this operator is changed to ‘‘quotients of members of C by congruences’’. On the other hand, $\mathbf{S}(C)$ is not generally closed under going to isomorphic algebras if C is not, but it acquires that property if one changes the definition to ‘‘algebras embeddable in members of C’’. Whether $\mathbf{P}(C)$ is closed under isomorphism depends on whether one defines ‘‘product’’ to mean ‘‘any object which can be given a family of ‘projection’ maps having the appropriate universal property’’, as we do here, or as the ‘‘standard’’ set-theoretic product. Since these distinctions are irrelevant to the algebraic questions involved, it seems best to eliminate them by restricting attention to isomorphism-closed classes. These are called ‘‘abstract classes’’ by some authors, though I do not favor that term. Incidentally, while discussing this topic, we will, obviously, temporarily set aside our habit of using $\mathbf{P}$ for ‘‘power set’’.)

In view of (8.6.2), it is natural to examine the monoid of operators on $|L_\Omega|$ generated by $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$. We see from that result that the product $\mathbf{HSP}$ acts as a closure operator, and hence is *idempotent*: $(\mathbf{HSP})^2 = \mathbf{HSP}$. From this we can deduce further equalities, e.g., $\mathbf{SHSP} = \mathbf{HSP}$. This deduction is clear when we think in terms of classes of algebras; to abstract the argument, let Z denote the monoid of all operators $\mathbf{A}: |L_\Omega| \to |L_\Omega|$ satisfying

(a) $(\forall\, C\in|L_\Omega|)\ \ \mathbf{A}(C)\supseteq C$ ($\mathbf{A}$ is increasing),

(b) $(\forall\, C, D\in|L_\Omega|)\ \ C\supseteq D \Rightarrow \mathbf{A}(C)\supseteq\mathbf{A}(D)$ ($\mathbf{A}$ is isotone).

This monoid Z can be partially ordered by writing $\mathbf{A}\geq\mathbf{B}$ if and only if for all C, $\mathbf{A}(C)\supseteq\mathbf{B}(C)$. By (a), all elements of Z are $\geq$ the identity operator, which we shall denote $\mathbf{I}$; we see from (b) that $\mathbf{B}\geq\mathbf{C} \Rightarrow \mathbf{AB}\geq\mathbf{AC}$, and we see by the definition of $\geq$ that $\mathbf{B}\geq\mathbf{C} \Rightarrow \mathbf{BA}\geq\mathbf{CA}$. Hence knowing only that $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}\in Z$, we can say that $(\mathbf{HSP})^2 \geq \mathbf{SHSP} \geq \mathbf{HSP}$; hence, as claimed, the equality $(\mathbf{HSP})^2 = \mathbf{HSP}$ implies $\mathbf{SHSP} = \mathbf{HSP}$.

Having illustrated how to calculate with these operators, we pose

Exercise 8.6:1. Describe explicitly the partially ordered monoid generated by the operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ on classes of Ω-algebras for general Ω; i.e., describe the distinct products of these operators, their composition, and the order-relations among them. Are there finitely or infinitely many distinct operators? Which such operators are idempotent?

(When I say ‘‘for general Ω’’, I mean that an equality or inequality should be considered to hold if and only if it holds for *all* Ω. Special cases will be looked at in the next exercise.)

The above is a large task, but an interesting one. To carry it out fully, you need counterexamples showing that each equality or inclusion that you do *not* assert actually fails to hold for some appropriately chosen set of algebras. However, a counterexample for one relation often turns out to be a counterexample for several, so the task is not unreasonably difficult.

There are numerous modifications of this problem. For example.

Exercise 8.6:2. Suppose we restrict the operators **H**, **S**, **P** to classes of algebras in a particular variety **V**; then some additional inclusions and equalities may occur among the composites of these restricted operators. Investigate the partially ordered monoids of operators obtained when **V** is **Set**, respectively **Group**, respectively **Ab**. You may add to this list.

One could enlarge the set of operators considered above, introducing, for instance, **D** = **{**difference kernels**}** (i.e., $\mathbf{D}(C)$ = the set of difference kernels of pairs of homomorphisms among algebras of C; thus, $\mathbf{D} \leq \mathbf{S}$), $\mathbf{P}_{\mathrm{fin}}$ = **{**products of finite families**}**, and **L** = **{**direct limits of directed systems**}**. (In considering these last two we should restrict attention to finitary algebras, or else replace ''finite families'' and ''directed systems'' by ''families of $<\gamma$ objects'' and ''γ-directed systems''.) Results on the structure of the monoid generated by any subset of $\{\mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{D}, \mathbf{P}_{\mathrm{fin}}, \mathbf{L}\}$, or any other such family of natural operators, can be turned in as homework, but I will merely pose as an exercise the questions

Exercise 8.6:3. Can one in general strengthen (8.6.2) to

(i) $\mathbf{Var}(C) = \mathbf{H\,D\,P}(C)$?

(ii) $\mathbf{Var}(C) = \mathbf{H\,S\,P}_{\mathrm{fin}}(C)$?

The proof of Birkhoff's Theorem leads us to examine the class of Ω-algebras that are free in *some* variety.

Proposition 8.6.3. *Let* Ω *be a type,* F *an* Ω*-algebra,* X *a set, and* u *an* X*-tuple of elements of* $|F|$*. Then the following conditions are equivalent:*

(i) (F, u) *is a free algebra on the set* X *in* some *variety* **V** *of* Ω*-algebras.*

(ii) (F, u) *is a free algebra on the set* X *in the variety generated by* F*.*

(iii) F *is generated by the image of* X*, and there exists some full subcategory* **C** *of* Ω-**Alg** *containing* F *such that* (F, u) *is free in* **C** *on the set* X*.*

(iv) F *is generated by the image of* X*, and for every set map* $v\colon X \to |F|$*, there exists an endomorphism* e *of* F *such that* $v = eu$*.* (*If we assume* u *is an inclusion map, this latter condition can be stated, ''Every map of* X *into* $|F|$ *extends to an endomorphism of* F*.''*)

(v) *Up to isomorphism,* F *may be identified with a quotient of* $F_\Omega(X)$ *by a congruence* E *which is carried into itself by every endomorphism* f *of* $F_\Omega(X)$ (*i.e., which satisfies* $(s, t) \in |E| \Rightarrow (f(s), f(t)) \in E$), *and the map* u *corresponds to the composite of universal maps* $X \to |F_\Omega(X)| \to |F_\Omega(X)/E|$*.*

Proof. We have (i)⇒(ii) because a free algebra in a given concrete category is easily seen to remain free in any full subcategory which contains it; (ii)⇒(iii) is immediate. The universal property of a free object gives (iii)⇒(iv). To see (iv)⇒(v), identify F with the quotient of $F_\Omega(X)$ by the congruence E consisting of all relations satisfied by the X-tuple u. Then if f is an endomorphism of $F_\Omega(X)$, (iv) implies that we can extend this to an endomorphism of $F = F_\Omega(X)/E$, which says that E is carried into itself by f, which is the assertion of (v).

Finally, given (v) we see that the relations satisfied by u will be satisfied by every X-tuple of elements of F, i.e., will be identities of $\mathbf{Var}(\{F\})$ in an X-tuple of variables, and conversely the identities of $\mathbf{Var}(\{F\})$ are necessarily satisfied by u. Hence F, being generated by the image of u, which satisfies precisely those relations which are identities of $\mathbf{Var}(\{F\})$, is the free $\mathbf{Var}(\{F\})$-algebra on X, proving (i). □

Definition 8.6.4. *A pair* (F, u) *with the equivalent properties of the above proposition* (*in particular, property* (i)) *is called a* relatively free Ω-*algebra.*

Exercise 8.6:4. Suppose $\mathbf{V} = \mathbf{Monoid}$ and $\mathbf{C}$ is the class of monoids all of whose elements are invertible.

(i) Show that $\mathbf{C}$ has free algebras (i.e., that its underlying-set functor has a left adjoint), but that these are not the free algebras of $\mathbf{Var}(\mathbf{C})$.

(ii) Show using this example that the requirement that F be generated by the image of X cannot be removed from condition (iii) or (iv) of Proposition 8.6.3; specifically, that if it is removed, the resulting conditions no longer imply condition (i) of that proposition.

From Proposition 8.6.3, we can deduce the corresponding result with Ω-**Alg** replaced by an arbitrary variety $\mathbf{W}$. In particular, we record

Corollary 8.6.5. *Let* $\mathbf{V}$ *be a variety,* X *a set, and* u *an* X-*tuple of elements of an object* F *of* $\mathbf{V}$. *Then* (F, u) *is a free algebra in a* subvariety *of* $\mathbf{V}$ *if and only if it is isomorphic to a quotient of the free* $\mathbf{V}$-*algebra* $F_{\mathbf{V}}(X)$ *by a congruence invariant under all endomorphisms of* $F_{\mathbf{V}}(X)$. □

Corollary 8.6.6. *If* γ *is a regular cardinal greater than the arities of all operations of* $\mathbf{V}$, *the subvarieties of* $\mathbf{V}$ *are in bijective correspondence with congruences on* $F_{\mathbf{V}}(\gamma)$ *which are invariant under all endomorphisms of this free algebra.* □

These results solve the problem of characterizing equational theories:

Theorem 8.6.7. *Let* Ω *be a type. Then a subset* $J \subseteq |F_\Omega(\gamma)| \times |F_\Omega(\gamma)|$ *is an equational theory if and only if it is a congruence on* $F_\Omega(\gamma)$, *and is carried into itself by all endomorphisms of* $F_\Omega(\gamma)$; *in other words, if and only if it satisfies the following five conditions for all* s, t, u, *etc.*$\in |F_\Omega(\gamma)|$, $\sigma \in |\Omega|$. (*In* (iv) *and* (v), $\sigma_{F_\Omega(\gamma)}$, $s_{F_\Omega(\gamma)}$ *and* $t_{F_\Omega(\gamma)}$ *denote the derived operations on* $F_\Omega(\gamma)$ *induced by* σ, s *and* t.)

(i) $(s, s) \in J$.

(ii) $(s, t) \in J \ \Rightarrow\ (t, s) \in J$.

(iii) $(s, t) \in J,\ (t, u) \in J \ \Rightarrow\ (s, u) \in J$.

(iv) $(t_i, u_i) \in J\ \ (i \in \mathrm{ari}(\sigma)) \ \Rightarrow\ (\sigma_{F_\Omega(\gamma)}(t_i),\ \sigma_{F_\Omega(\gamma)}(u_i)) \in J$.

(v) $(s, t) \in J,\ u_i \in |F_\Omega(\gamma)|\ \ (i \in \gamma) \ \Rightarrow\ (s_{F_\Omega(\gamma)}(u_i),\ t_{F_\Omega(\gamma)}(u_i)) \in J$. □

Turning back to relatively free algebras, let us note that though by Lemma 8.4.2 an algebra relatively free on γ generators uniquely determines the corresponding variety, a relatively free algebra (F, u) on an α-tuple of generators for $\alpha < \gamma$ may be free in more than one variety! The variety $\mathbf{Var}(\{F\})$ used in the proof of Proposition 8.6.3(v)⇒(i) will necessarily be the *smallest* such variety. The *largest* variety in which (F, u) is free is the variety defined by the identities in α variables satisfied by F; equivalently, having for identities the relations satisfied by the α-tuple u in F. The details, and some examples, are indicated in

Exercise 8.6:5. (i) Let $\mathbf{V}$ be a variety, and suppose $F\in\mathrm{Ob}(\mathbf{V})$ is relatively free on an α-tuple u of indeterminates. Show that (F,u) is a free algebra in precisely those subvarieties $\mathbf{U}\subseteq\mathbf{V}$ which contain the variety $\mathbf{Var}(F)$ (defined by all identities satisfied in F), and are contained in the subvariety of $\mathbf{V}$ defined by the identities in $\leq\alpha$ variables holding in F.

(ii) Show that if $\mathbf{V}=\mathbf{Group}$, $\alpha=1$, and (F,u) is the group $\mathbf{Z}$, with u selecting 1 as free generator, then the greatest and least subvarieties of $\mathbf{V}$ in which (F,u) is free are distinct. Characterize group-theoretically those subvarieties of $\mathbf{V}$ in which (F,u) is free.

(iii) Show that if again $\mathbf{V}=\mathbf{Group}$, but we now take $\alpha=2$, and for (F,u) either the free group on 2 generators or the free abelian group on 2 generators, then in each case, the greatest and least subvarieties of $\mathbf{V}$ in which this group is free coincide.

(iv) Are there any relatively free groups F on 2 generators such that the greatest and least varieties of groups in which F is free are distinct?

(v) If Ω is the type of groups, and F is either the free group on 2 generators or the free abelian group on 2 generators, show that the greatest and the least varieties of Ω-algebras in which F is free on the given generators do not coincide, but that if F is the free group or free abelian group on 3 generators, they again coincide.

Here are some exercises on subvarieties of familiar varieties.

Exercise 8.6:6. (If you do both parts below, give the proof of one in detail, and for the other give details where the proofs differ.)

(i) Let G be a group, and G-**Set** the variety of all G-sets. Show that subvarieties of G-**Set** other than the least subvariety (characterized in Remark 8.4.5) are in one-to-one correspondence with the normal subgroups N of G, in such a way that the subvariety corresponding to N is equivalent to the variety (G/N)-**Set**, by an equivalence which respects underlying sets.

(ii) Prove the analogous result for subvarieties of R-**Mod**, where R is an arbitrary ring. (In that case, the least subvariety is not an exceptional case.)

Exercise 8.6:7. (i) Let $\mathbf{CommRing}^1$ denote the category of commutative rings. Show that if $\mathbf{V}$ is a proper subvariety of $\mathbf{CommRing}^1$ generated by an infinite integral domain, then $\mathbf{V}$ is the variety $\mathbf{V}_p$ determined by the 0-variable identity $p=0$ for some prime p, where the symbol ‘‘p’’ in this identity is an abbreviation for $1+1+\ldots+1$ with p summands.

(ii) Show that the subvariety $\mathbf{Bool}^1\subseteq\mathbf{CommRing}^1$ is a *proper* subvariety of the variety $\mathbf{V}_2$ defined as in (i).

Exercise 8.6:8. Let $F=F_{\mathbf{Ring}^1}(\omega)$, the free associative (noncommutative) ring on indeterminates x_0, x_1, ... For each positive integer n, let

$$S_n\ =\ \Sigma_\pi\,(-1)^\pi\,x_{\pi(0)}\cdots x_{\pi(n-1)}\in F,$$

where π ranges over the permutations on n elements, and $(-1)^\pi$ denotes $+1$ if π is an even permutation, -1 if π is odd.

(i) Show that any ring satisfying $S_n=0$ also satisfies $S_{n'}=0$ for all $n'\geq n$; i.e., that $(S_{n'},0)\in\{(S_n,0)\}^{**}$.

(ii) Show that for every $d>0$ there exists $n>0$ such that for every commutative ring k, the ring $M_d(k)$ of $d\times d$ matrices over k satisfies the identity $S_n=0$.

(iii) Show that for every $n>0$ there exists $d>0$ such that $M_d(k)$ does not satisfy $S_n=0$ for any nontrivial commutative ring k.

(iv) Deduce that there is an infinite chain of distinct varieties of rings of the form $\mathbf{V}(\{(S_n,0)\})$, and an infinite chain of distinct varieties of rings of the form $\mathbf{Var}(\{M_d(\mathbf{Z})\})$.

Note on the above exercise: The *least* n such that all $d\times d$ matrix rings $M_d(k)$ over commutative rings k satisfy $S_n=0$ is $2d$. The hard part of this result, namely that $M_d(k)$

satisfies $S_{2d} = 0$, is the Amitsur-Levitzki Theorem [**32**]. All known proofs are either messy (e.g., by graph theory [**95**]) or tricky (e.g., using exterior algebras [**91**]). The student is invited to attempt to find a new proof! Part (ii) of the above exercise can be done relatively easily, however, using a larger-than-optimal n.

The study of varieties of noncommutative rings is called the theory of *rings with polynomial identity*, affectionately known as *PI rings*. See [**92**, Chapter 6] for an introduction to this subject.

Here is curious variety closely related to the variety of groups.

Exercise 8.6:9. Let Ω be the type defined by a single ternary (i.e., ''3-ary'') operation-symbol, τ. Let H: **Group** → Ω**-Alg** be the functor taking a group G to the Ω-algebra with underlying set $|G|$ and operation

$$\tau(x,y,z) = xy^{-1}z. \tag{8.6.8}$$

(i) Show that the objects $H(G)$ are the nonempty algebras in a subvariety of Ω**-Alg**, and give a set J of identities defining this variety.

The algebras (empty and nonempty) in this variety are called *heaps*; we shall call the variety **Heap**.

(ii) Show that for groups G and G', one has

$$H(G) \cong H(G') \text{ in } \mathbf{Heap} \iff G \cong G' \text{ in } \mathbf{Group}.$$

(iii) Show, however, that not every isomorphism between $H(G)$ and $H(G')$ has the form $H(i)$ for i an isomorphism between G and G' !

(iv) Show that the following categories are equivalent: (a) **Group**, (b) the variety of algebras $(|A|, \tau, \iota)$ where $(|A|, \tau)$ is a heap, and ι is a zeroary operation, subject to no further identities (intuitively, ''heaps with distinguished elements ι''), (c) $\mathbf{Heap}^{\mathrm{pt}}$, where the construction $\mathbf{C}^{\mathrm{pt}}$ is defined as in Exercise 6.8:3.

(v) Show that if X, Y are two objects of any category $\mathbf{C}$, then the set of isomorphisms $X \to Y$ forms a heap under the operation $\tau(x,y,z) = xy^{-1}z$. How is the structure of this heap related to those of the groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$?

The concept of heap is not very well known, and many mathematicians have from time to time rediscovered it and given it other names (myself included). Heaps were apparently first studied by Prüfer [**89**] and Baer [**34**], under the name *Schar* meaning ''crowd'' or ''flock'', a humorous way of saying ''something like a group''. The term was rendered into Russian by Suškevič [**98**] as груда, meaning ''heap'', which gave both a loose approximation of the meaning of *Schar*, and a play on the sounds of the Russian words ''group'' = *gruppa*, ''heap'' = *gruda*. Since the concept and its generalizations have gotten most attention in Russian-language works, it has come back into Western European languages via translations of this Russian term rather than of the original German.

Part (ii) of the above exercise shows that there is no need for a separate theory of the *structure* of heaps; this is essentially contained in that of groups. However, the variety of heaps is both a taking-off point for various generalizations (''semiheaps'' etc.), and a source of examples in general algebra and category theory.

Point (iv) of the preceding exercise suggests

Exercise 8.6:10. (i) For what varieties $\mathbf{V}$ is it true that the category $\mathbf{V}^{\mathrm{pt}}$ can be identified with the variety gotten by adding to $\mathbf{V}$ one zeroary operation, and no additional identities?

(ii) What varieties $\mathbf{V}$ satisfy the conditions of Exercise 6.8:2? (Note that for varieties these conditions are all equivalent, by the last part of that exercise.)

Let us remark that in stretching the concept of ‘‘variety’’ from its classical definition as a class of algebras defined by identities to our present category-theoretic use, we have pulled it over a lot of ground, so that care is needed in using the term. For example, when should we think of two varieties as being ‘‘essentially the same’’? If they are precisely equal? If we can establish a bijection between their types such that they are defined by the corresponding identities? If they are equivalent as categories? If there is a category-theoretic equivalence which also respects the underlying-set functors of the varieties? There is no right answer, but these four conditions are all inequivalent.

Exercise 8.6:11. What implications exist among the above four conditions on a pair of varieties? Give examples showing that no two of them are equivalent.

8.7. Lie algebras. Let me digress here to introduce a variety important in algebra, geometry, and differential equations, that of *Lie algebras*. I have referred to these in previous chapters in a few comments ‘‘for the reader familiar with the concept’’. The reader who prefers to remain unfamiliar with them for the time being may skip this section, and perhaps come back to it later. In later sections, Lie algebras will be continue to be referred to in occasional exercises and remarks.

To motivate the definition, suppose we start with an associative algebra A over a field (or more generally, over a commutative ring) k, and look at the underlying set of A together with its operations of k-vector-space (or k-module), and the *commutator bracket* operation,

(8.7.1) $$[x,y] \;=\; xy-yx.$$

These operations obviously satisfy the identities saying

(8.7.2) $+,\ -,\ 0$ and the scalar multiplications by members of k make $|A|$
a k-module, and $[-,-]$ is a k-*bilinear* operation with respect to this
k-module structure.

There is a further obvious identity satisfied by $[-,-]$, and another that, though not so obvious, is straightforward to verify:

(8.7.3) $$[x,\,x] \;=\; 0 \qquad \text{(alternating identity)},$$

(8.7.4) $$[x,\,[y,\,z]\,] + [y,\,[z,\,x]\,] + [z,\,[x,\,y]\,] \;=\; 0 \qquad \text{(Jacobi identity)}.$$

Note that (8.7.3) implies (and if 2 is invertible in k, is equivalent to)

(8.7.5) $$[x,\,y] + [y,\,x] \;=\; 0 \qquad \text{(anticommutativity)}.$$

The expansion of (8.7.4) involves 12 terms, and so is not hard to check directly, but the following slightly simpler verification gives some useful insight. One first checks the following identity (which expands to only 6 terms) relating $[-,-]$ and the original multiplication of A:

(8.7.6) $$[x,\,yz] \;=\; [x,\,y]z + y[x,\,z].$$

This (in the presence of (8.7.2)) says that for any x, the operation $[x,-]$ of commutation with x acts as a *derivation* with respect to the k-algebra structure of A. We recall that a derivation on a k-algebra means a k-linear map s satisfying

(8.7.7) $$s(yz) \;=\; s(y)z + y\,s(z).$$

A map that is a derivation with respect to a given multiplication is also a derivation with respect to the opposite multiplication, $y*z = zy$. Subtracting the original and opposite multiplications gives the commutator map; so by (8.7.4) $[x,-]$ also acts as a derivation with respect to that map:

$$[x,[y,z]] = [y,[x,z]] + [[x,y],z].$$

We can use (8.7.5) to rearrange this identity so that the bracket arrangement of the last term, like that of the other two, becomes $[-,[-,-]]$, and so that the middle term has the same cyclic order of x, y and z as the other two terms. Bringing all three terms to the same side, we see that the above formula becomes precisely (8.7.4). So the Jacobi identity says that the commutator bracket operation is a derivation with respect to itself! One now makes

Definition 8.7.8. *Let* k *be a commutative ring* (*often, though not always, assumed a field*). *Then a* Lie algebra over k *means a* k*-module given with an alternating* k*-bilinear operation* $[-,-]$ *satisfying the Jacobi identity; in other words, a set* $|A|$ *with operations* $+$, $-$, 0, *''scalar multiplication'' operations corresponding to each element of* k, *and a binary operation* $[-,-]$, *satisfying* (8.7.2)-(8.7.4).

The variety of Lie algebras over k *will be denoted* $\mathbf{Lie}_k$.

For each element x *of a Lie algebra* L, *the map* $[x,-]: |L| \to |L|$ *is often denoted* ad_x. (*This stands for ''adjoint'', but for obvious reasons we will not call it by that name here.*)

In view of the way we obtained (8.7.2)-(8.7.4), we see that if we write $\mathbf{Ring}^1_k$ for the category of associative k-algebras, we have a functor

$$B: \mathbf{Ring}^1_k \to \mathbf{Lie}_k$$

taking each associative k-algebra A to the Lie algebra with the same underlying k-module, and with bracket operation given by (8.7.1). It is not hard to do

Exercise 8.7:1. Show that B has a left adjoint

$$E: \mathbf{Lie}_k \to \mathbf{Ring}^1_k.$$

This is called the *universal enveloping algebra* construction.

In a later chapter on normal forms, I hope to prove the *Poincaré-Birkhoff-Witt Theorem*, which gives a normal form for $E(L)$ when L is free as a k-module (as is automatic if k is a field), and in particular shows that the maps giving the unit of the above adjunction, $\eta(L): L \to B(E(L))$, are one-to-one. Thus, every Lie algebra over a field can be ''embedded in'' an associative algebra. An important consequence is

Exercise 8.7:2. (i) Suppose k is a field. Assuming, as asserted above, that for all Lie algebras L, the map $\eta(L)$ is one-to-one, show that the Lie algebras of the form $B(L)$ generate the variety $\mathbf{Lie}_k$. Deduce that every identity satisfied by the k-module structure and the operation $[-,-]$ in all associative k-algebras A is a consequence of the identities (8.7.2)-(8.7.4).

(ii) (For students familiar with techniques of change-of-scalars.) Deduce from (i) the same result for arbitrary $k \in \mathbf{CommRing}^1$. (Hint: First verify that every commutative ring k is a homomorphic image of a ring embeddable in a field.)

That one can argue this way is noteworthy, since it is known that for k not a field, there can be Lie algebras L over k such that $\eta(L)$ is not one-to-one.

If R is any k-algebra (which for the moment need not even be associative), and S is the associative k-algebra of all k-linear maps (i.e., k-module homomorphisms) $R \to R$, then it easy to verify that if $s, t \in S$ are both derivations (i.e., satisfy (8.7.7)), then $[s, t] \in S$ is also a derivation. Thus, the k-derivations on R form a Lie algebra $\mathrm{Der}_k(R) \subseteq B(S)$, which we will write $\mathrm{Der}(R)$ when there is no danger of ambiguity. For R a Lie algebra or an associative algebra, a derivation of the form $\mathrm{ad}_x = [x, -]$ is called an *inner derivation*.

Exercise 8.7:3. Let R be a not necessarily associative k-algebra, with multiplication denoted $*$, and for $x \in |R|$ let $\mathrm{Ad}_x : |R| \to |R|$ denote the map $y \mapsto x*y - y*x$. (Thus if R is associative, Ad_x coincides with what we have been calling ad_x, but if R is a Lie algebra, so that $*$ denotes $[-,-]$, Ad_x will be $2\,\mathrm{ad}_x$.)

Write down the identity that R must satisfy in for every such map Ad_x to be a derivation. Show that if R is anticommutative (satisfies (8.7.5)) and 2 is invertible in k, this is equivalent to the Jacobi identity, but that in general (in particular, if R is associative), it is not.

In terms of the ''ad'' notation, we can get yet another interpretation of the Jacobi identity. You will not find it hard to check that (8.7.4) is equivalent to the identity

$$\mathrm{ad}_x\,\mathrm{ad}_y - \mathrm{ad}_y\,\mathrm{ad}_x \;=\; \mathrm{ad}_{[x,y]}\,. \tag{8.7.9}$$

Thus the Jacobi identity is also equivalent to saying that $\mathrm{ad} : L \to \mathrm{Der}(L)$ is a homomorphism of Lie algebras.

If R is a commutative algebra, then the Lie algebra $B(R)$ clearly has trivial bracket operation. However, the k-algebra of k-module endomorphisms of R will in general be noncommutative, hence the Lie algebra $\mathrm{Der}(R)$ can have nontrivial bracket operation; these Lie algebras are in fact important in commutative ring theory and differential geometry.

If we take for k the field $\mathbf{R}$ of real numbers, and for R the $\mathbf{R}$-algebra of C^∞ (i.e., infinitely differentiable) functions on $\mathbf{R}^n$, appropriately topologized, then the continuous derivations on R will be given by the left R-linear combinations of the n derivations $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Geometers identify the derivation $D = \Sigma\, a_i(x)\, \partial/\partial x_i$ $(a_i(x) \in R)$ with the C^∞ *vector field* $a(x) = (a_1(x), \ldots, a_n(x))$, the idea being that $D f$ gives, at each point x, the rate of change of f that would be seen by a particle at x moving with velocity $a(x)$; thus they speak of the *Lie algebra of C^∞ vector fields* on $\mathbf{R}^n$ (and by a similar construction, on any C^∞ manifold). One can similarly look at the Lie algebra of ''polynomial vector fields on affine n-space'' over any commutative ring k; these are the derivations on the polynomial ring $R = k[x_1, \ldots, x_n]$, and have the form $\Sigma\, a_i(x)\, \partial/\partial x_i$, where the derivations $\partial/\partial x_i$ are this time defined formally (in the obvious way), and again $a_i(x) \in R$.

It turns out that Lie algebras arising in the above manner satisfy some additional identities, beyond those satisfied by all Lie algebras. The simplest case is that of part (ii) of the next exercise.

Exercise 8.7:4. (i) Let $C^\infty(\mathbf{R}^1)$ denote the ring of C^∞ functions on the real line $\mathbf{R}^1$, and $L(\mathbf{R}^1)$ the Lie algebra of vector fields $f\, d/dx$ $(f \in (C^\infty(\mathbf{R}^1)))$. Verify that the Lie bracket operation on $L(\mathbf{R}^1)$ is given by the formula

$$[f\, d/dx,\;\; g\, d/dx] \;=\; (f\, dg/dx - g\, df/dx)\, d/dx.$$

For notational convenience, let us write this as a Lie algebra structure on $|C^\infty(\mathbf{R}^1)|$:

$$[f, g] \;=\; f\, dg/dx - g\, df/dx.$$

though we shall continue to denote this Lie algebra $L(\mathbf{R}^1)$.

(ii) Show that $L(\mathbf{R}^1)$ does not generate $\mathbf{Lie}_{\mathbf{R}}$. (Note that if you find an identity satisfied in $L(\mathbf{R}^1)$, and want to show it is not satisfied in all Lie algebras, you can look for an instance of its failure in a Lie algebra $B(A)$, since you know how to compute explicitly in these.)

The above result requires some computational dirty-work. On the other hand, even if you do not do that part, a little ingenuity will allow you to do the remaining parts.

(iii) Show that for every positive integer n, $L(\mathbf{R}^1)$ contains a subalgebra which is free on n generators in $\mathbf{Var}\,(L(\mathbf{R}^1))$.

(iv) Show that $\mathbf{Var}\,(\mathrm{Der}(\mathbf{R}[x])) = \mathbf{Var}\,(L(\mathbf{R}^1))$, i.e., that polynomial vector fields satisfy no identities not satisfied by C^∞ vector fields. However, show that $\mathrm{Der}(\mathbf{R}[x])$ does *not* contain a subalgebra which is free on more than one generator in this variety.

You can carry the idea of part (ii) farther, looking at the variety generated by the Lie algebra $L(\mathbf{R}^d)$ of C^∞ vector fields on $\mathbf{R}^d$, the Lie subalgebra thereof consisting of vector fields of divergence 0, and similar constructions.

One of the most important interpretations of Lie algebras lies outside the scope of this work, and I will only sketch it briefly: their connection with Lie groups.

A *Lie group* is a topological group G whose underlying topological space is a manifold. Typical examples are the rotation group of real 3-space, which is a 3-dimensional compact Lie group, and the group of motions of 3-space generated by rotations and translations, which is 6-dimensional and noncompact. Some degenerate but important examples are the real line, which is 1-dimensional, its compact homomorphic image the circle group $\mathbf{R}/\mathbf{Z}$, and finally, all discrete groups, which are the zero-dimensional Lie groups. It is known that every Lie group admits a unique C^∞ structure respected by the group operations.

If G is a Lie group, e its identity element, and T_e the tangent space to G at e, then every tangent vector $t \in T_e$ extends by left translation to a left-translation-invariant vector field. Hence the space of left-invariant vector fields may be identified in a natural manner with T_e. The commutator bracket of two left-invariant vector fields is left-invariant, so such vector fields form a Lie algebra; hence the above identification gives us a Lie algebra structure on T_e.

That structure can also be arrived at directly, by an approach which leads to another important motivation for the concept of Lie algebra. Let us think of the additive structure of T_e as the ‘‘first order approximation to the group structure of G in the neighborhood of e’’. Such an approximation is necessarily abelian, because the commutator of two elements of G both of which are close to e is close to e ‘‘to second order’’. To measure this second-order noncommutativity of G near the identity, let us identify a neighborhood of $0 \in T_e$ with a neighborhood of $e \in G$, and on this identified neighborhood use vector-space notation for the operations of T_e, and $*$ for the multiplication of G. Then for $x, y \in T$ and real variables s and t, that second-order noncommutativity is measured by the limit

$$\lim_{s,\,t \to 0} \frac{(sx)*(ty) - (ty)*(sx)}{st}.$$

Calling this limit $[x,y] \in T_e$, and examining the properties of this operation, one arrives again at the concept of Lie algebra. Elements of this Lie algebra are thought of as ‘‘infinitesimal’’ elements of the Lie group G, and one finds that the structure of G is determined ‘‘locally’’ by that of its Lie algebra.

For a familiar case, let G be the rotation group of Euclidean 3-space. Then elements of G represent *rotations* of space through various angles about various axes, while the elements of its Lie algebra L represent various *angular velocities* about various axes. As a vector space, L may be identified with $\mathbf{R}^3$, each element being identified with a vector pointing in the direction of its axis of rotation, and with magnitude equal to its angular velocity. The Lie bracket on L then turns out to be an operation on $\mathbf{R}^3$ known to every math or engineering student: the ‘‘cross product’’ of vectors.

Exercise 8.7:5. We saw in Exercise 2.3:2 and the discussion preceding it that in the variety generated by a finite group, a free object on finitely many generators is finite. Is it similarly true that a free object on finitely many generators in the variety $\mathbf{Var}(A)$ generated by a finite-dimensional associative algebra or Lie algebra A over a field k is finite-dimensional? If not, can you prove some related condition (e.g., of ''small growth-rate'')?

Can you show that such a variety $\mathbf{Var}(A)$ must be distinct from the whole variety $\mathbf{Ring}_k^1$, respectively $\mathbf{Lie}_k$? In the Lie case, if $k = \mathbf{R}$, can you show it distinct from the subvariety $\mathbf{Var}(L(\mathbf{R}^1))$ of Exercise 8.7:4(ii)?

Some general references for the theory of Lie algebras are [**65**], [**67**], [**94**].

The relationship between Lie groups and Lie algebras on the one hand, and on the other, the fact that derivations on an algebra form a Lie algebra, leads to the heuristic principle that derivations on an algebra may be regarded as ''infinitesimal automorphisms''. This suggests that such a derivation should be determined by what it does on a generating set, and, in the case of a free algebra, that it should be possible to specify the derivation in an arbitrary way on the free generators. The next exercise gives results of these sorts.

Exercise 8.7:6. Let A be a not necessarily associative algebra over a commutative ring k.

(i) Show that the *kernel* of any derivation $d\colon A \to A$ is a subalgebra of A. (This is analogous to the *fixed subalgebra* of an automorphism.) Deduce that two derivations which agree on a generating set for A are equal.

The other result we want, about derivations on free algebras, requires a trick to turn derivations into something to which we can apply the universal properties of these algebras.

(ii) Let A' denote the algebra whose k-module structure is that of $A \times A$, and whose multiplication is given by $(a,x)(b,y) = (ab,\ ay + xb)$. Verify that this is an associative k-algebra, respectively an associative commutative k-algebra, if and only if A has the same property.

(iii) Show that a map $d\colon A \to A$ is a derivation if and only if the map $a \mapsto (a, d(a))$ is a homomorphism $A \to A'$ as k-algebras. Deduce that if A is the free nonassociative k-algebra, the free associative k-algebra, or the free associative commutative k-algebra on a set X, then every set-map $X \to |A|$ extends uniquely to a derivation $A \to A$.

(iv) Show that if A is a *field* or a *division ring*, and X a subset generating A as a field or division ring, then any derivation $A \to A$ is determined by its restriction to X. Can you generalize this result?

Remark. The concept of a derivation from a k-algebra A into itself is a case of the more general concept of a derivation from a k-algebra A into an A-module (if A is commutative and associative) or an A-bimodule (in the general associative case) B. Such a derivation is defined as a k-module homomorphism satisfying (8.7.7), and the above exercise goes over to this more general situation. Where in the exercise we put a k-algebra structure on $|A| \times |A|$, the generalized argument uses the corresponding structure on $|A| \times |B|$.

There are still other variants of the concept of derivation which we won't go into here.

Our motivation of the definition of Lie algebra starting from (8.7.1) suggests the analogous question of what identities will be satisfied by the operation

$$(x,y) \;=\; xy + yx$$

on an associative algebra. This is the starting-point of the theory of *Jordan algebras*, but the subject is not as neat as that of Lie algebras. Jordan algebras are defined using the identities of degree ≤ 4 satisfied by the above operation, but that operation also satisfies identities of higher degrees not implied by the Jordan identities; Jordan algebras satisfying these are called ‘‘semispecial’’. No analog of the connection between Lie groups and Lie algebras appears to exist for Jordan algebras. A standard reference for the theory of Jordan algebras is [**68**].

Let us now return to general algebras.

8.8. Some necessary trivialities. Suppose $g\colon S^X \to S$ is an X-ary operation on a set S, and $a\colon X \to Y$ is a set map. Then we can define a Y-ary operation $f\colon S^Y \to S$ by $f((c_y)_{y\in Y}) = g((c_{a(x)})_{x\in X})$. Let us call f the operation *induced* by g via the map a of arity-sets. The covariance of this construction in the arity-set is actually the result of two contravariances: $a\colon X \to Y$ induces a map $S^Y \to S^X$, then this gives a map $S^{(S^X)} \to S^{(S^Y)}$. If g is a (primitive or derived) operation of an algebra structure on S, say corresponding to an element $s \in |F_\Omega(X)|$, then f corresponds to the image of s under the homomorphism $F_\Omega(a)\colon F_\Omega(X) \to F_\Omega(Y)$. (In terms of this description, the covariance is straightforward.)

Given an operation $f\colon S^Y \to S$ on a set, and a subset X of the index set Y, let us say that f *depends only on the indices in* X if f takes on the same value at any two Y-tuples that (regarded as functions on Y) have the same restriction to X. If S or X is nonempty, this is equivalent to the condition that f be induced by an X-ary operation g on S, via the inclusion of X in Y.

Exercise 8.8:1. (i) Verify the equivalence mentioned in the last sentence.

(ii) Show why the condition ‘‘S or X is nonempty’’ is needed for this equivalence to hold.

That finishes the essential material of this section! But there are several interesting related points, explored in the exercises below.

Exercise 8.8:2. Let S and Y be sets and $f\colon S^Y \to S$ a Y-ary operation on S.

(i) Suppose $W, X \subseteq Y$ are sets such that f depends only on the indices in W, and f also depends only on the indices in X. Show that f depends only on the indices in $W \cap X$.

(ii) On the other hand, show that given an infinite family of subsets $X_i \subseteq Y$ such that for each i, f depends only on the indices in X_i, it may not be true that f depends only on the indices in $\bigcap X_i$.

(iii) In general, given a Y-ary operation f on S, what properties must

$$D_f \ =_{\text{def}} \ \{X \subseteq Y \mid f \text{ depends only on indices in } X\}$$

have? Specifically, try to find conditions on a family U of subsets of Y which are necessary and sufficient for there to exist a set S and a function $f\colon S^Y \to S$ such that $U = D_f$.

In some works on general algebra, there is a confusion between *zeroary* derived operations, and *constant* derived operations of *nonzero* arities. The next two exercises show some of the basis of this confusion:

Exercise 8.8:3. (Like Exercise 8.8:1, but for derived operations.)

(i) Show that if a derived Y-ary operation s of an algebra A depends only on indices in a subset $X \subseteq Y$, and X is *nonempty*, then s is in fact induced by an X-ary derived operation of A.

(ii) On the other hand, suppose the derived Y-ary operation s of A depends only on the empty set of indices in Y, i.e., is constant. If A has zeroary operations, show that, as (i) would suggest, but for a different reason, s is induced by a zeroary derived operation of A. Show, however, that if A has no zeroary primitive operations, derived operations depending on the empty set of indices can still exist, but will not be induced by derived zeroary operation.

Thus, for $m \leq n$, derived m-ary operations correspond to derived n-ary operations depending only on the first m variables, *except* for the $m = 0$ case, where this is not true unless the algebra has zeroary primitive operations.

One is still more susceptible to the confusion referred to above if one excludes empty algebras, as is shown by

Exercise 8.8:4. We have seen that the X-ary derived operations of a variety $\mathbf{V}$ can be characterized as the morphisms $U_{\mathbf{V}}^X \to U_{\mathbf{V}}$ where $U_{\mathbf{V}}$ is the underlying-set functor of $\mathbf{V}$.

Suppose now that $\mathbf{V}$ is a variety *without* zeroary operations, hence having an empty algebra I. Let $\mathbf{V}-\{I\}$ denote the full subcategory of $\mathbf{V}$ consisting of all nonempty algebras, and let $U_{\mathbf{V}-\{I\}}$ denote the restriction of $U_{\mathbf{V}}$ to this subcategory.

(i) Show that morphisms $(U_{\mathbf{V}-\{I\}})^X \to U_{\mathbf{V}-\{I\}}$ correspond to derived X-ary operations of $\mathbf{V}$ *except* in the case $X = \varnothing$, in which case they can be put in natural correspondence with the constant derived unary operations.

(ii) Show that if $\mathbf{V}$ has constant derived unary operations, then $\mathbf{V}-\{I\}$ is isomorphic in a natural way to a variety of algebras (of a different type) having zeroary operations.

As an example, suppose that (as has been proposed from time to time) one sets up a variant of the concept of ‘‘group’’, based only on the two operations of composition and inverse, axiomatizing these by the associative law for composition, and the following identities, which hold in ordinary groups as consequences of the inverse and neutral element laws:

$$x = xyy^{-1} = xy^{-1}y = y^{-1}yx = yy^{-1}x.$$

(iii) Let $\mathbf{V}$ be the variety so defined. Show that the category $\mathbf{V}-\{I\}$ is isomorphic to **Group**.

8.9. Clones and clonal theories. Given a family of *unary* operations on a set S, i.e., maps $S \to S$, the composites of these (together with the ‘‘empty composite’’, the identity map) form a *monoid* of maps of S into itself. In this section we will look at the structure of the set of derived operations of a family of *not necessarily unary* operations, under the operations analogous to composition of unary operations.

We will limit ourselves to *finitary* operations. (There is no problem with the infinitary case, but I thought the concepts would come across more clearly in the familiar finitary context. The reader interested in the infinitary case can easily make the appropriate generalizations, replacing ‘‘finite’’ by ‘‘$<\gamma$’’, for γ a regular cardinal.) We will also, in this presentation, make our arities natural numbers (in the infinitary case, read ‘‘cardinals $<\gamma$’’) rather than arbitrary finite sets, since allowing all finite sets as arities would mean that every algebra would have a *large* set of formally distinct operations.

Definition 8.9.1. *Let S be a set. Then a* clone of operations *on S will mean a set C of operations on S, of natural-number arities, which is closed under formation of derived operations. Concretely, this says that*

(i) *for every natural number n, C contains the n projection maps $p_{n,i}: S^n \to S$ $(i \in n)$, defined by*

(8.9.2) $$p_{n,i}(\xi_0, \dots, \xi_{n-1}) = \xi_i,$$

and

(ii) *given natural numbers $m, n \in \omega$, an m-ary operation $s \in C$, and m n-ary operations*

$t_0, \ldots, t_{m-1} \in C$, the set C also contains the n-ary operation

$$(8.9.3) \qquad (\xi_0, \ldots, \xi_{n-1}) \mapsto s(t_0(\xi_0, \ldots, \xi_{n-1}), \ldots, t_{m-1}(\xi_0, \ldots, \xi_{n-1}))$$

i.e., the composite

$$S^n \xrightarrow{(t_0, \ldots, t_{m-1})} S^m \xrightarrow{s} S.$$

The least clone on S containing a given set of operations will be called the clone generated *by that set. Thus, for any finitary type Ω and any Ω-algebra A, the set of derived operations of A constitutes the clone generated by the primitive operations of A.*

Let us look at an example of how this procedure of generation works. Given a binary operation f and a ternary operation g on a set S, how do we express in terms of the constructions (8.9.2) and (8.9.3) the 6-ary operation

$$(\xi_0, \ldots, \xi_5) \mapsto f(g(\xi_0, \xi_1, \xi_2), g(\xi_3, \xi_4, \xi_5))?$$

It should clearly arise as an instance of (8.9.3) with f for s, but we cannot, as we might first think, take g for t_0 and t_1. That would give the *ternary* operation $(\xi_0, \xi_1, \xi_2) \mapsto f(g(\xi_0, \xi_1, \xi_2), g(\xi_0, \xi_1, \xi_2))$. We need, rather, to use as t_0 and t_1 the two 6-ary operations $(\xi_0, \ldots, \xi_5) \mapsto g(\xi_0, \xi_1, \xi_2)$ and $(\xi_0, \ldots, \xi_5) \mapsto g(\xi_3, \xi_4, \xi_5)$. We get these, in turn, as instances of (8.9.3) with g in the role of s, and projection maps (8.9.2) in the role of the t_i's. Namely, taking for t_0, t_1, t_2 the projection maps $p_{6,0}$, $p_{6,1}$, $p_{6,2}$, we get the first of the above 6-ary operations, and using the remaining three 6-ary projection maps, we get the other. We can then, as noted, apply (8.9.3) to f and these two 6-ary operations to get the 6-ary operation first asked for. (In this example, each of our variables happened to appear exactly once in the final expression, and the occurrences were in ascending order of subscripts, but obviously, by different choices of projection maps, we can get expressions in which variables appear more than once, and in arbitrary orders.)

Above, we got a ''new'' operation by inserting into the ternary operation g the 6-ary projection maps $p_{6,0}$, $p_{6,1}$, $p_{6,2}$. It is clear that if we had instead used the ternary projections $p_{3,0}$, $p_{3,1}$, $p_{3,2}$ (in that order) we would have gotten back precisely the operation g. Note also that if we substitute any operation f into the unary projection map $p_{1,0}$, we get the operation f back. These phenomena are analogs of the *neutral element laws* in a monoid.

One also has an analog of the associative law: If m, n and p are nonnegative integers, then given an m-ary operation s, any m n-ary operations t_i $(i \in m)$, and any n p-ary operations u_j $(j \in n)$, one can either substitute the t's into s, and the u's into the resulting operation, or first substitute the u's into the t's, and then the resulting operations into s. In each case one gets the p-ary operation which is the composite of the set maps

$$S^p \xrightarrow{(u_0, \ldots, u_{n-1})} S^n \xrightarrow{(t_0, \ldots, t_{m-1})} S^m \xrightarrow{s} S.$$

Hence the results of these two substitution-procedures are equal.

It looks as though we ought to abstract these properties, and use them as the definition of a new sort of algebraic object, which we might call a ''formal substitution algebra'' or a ''clonal algebra''. We would then have a new way of looking at varieties of algebras: Given a type Ω and a family J of identities, we would be able to construct a ''clonal algebra'' $\langle \Omega \mid J \rangle$ presented by these operation-symbols and identities. We could then define a ''representation'' of this clonal algebra on a set $|A|$ to mean a homomorphism of $<\Omega \mid J>$ into the clone of *all* finitary operations on that set. Such representations could be identified with Ω-algebras satisfying the identities of J; thus, each variety of algebras could be looked at as the category of representations of a clonal algebra.

Unfortunately, these ‘‘clonal algebras’’ would not be algebras of the sort we have developed in this chapter. The algebras $|A|$ we have been studying have an underlying set $|A|$; but ‘‘clonal algebras’’ would have an underlying *family* of sets, one set for each arity of the operations symbolized, with various composition operations associated to various combinations of these.

Now there is in fact a concept of *many-sorted algebra* (algebra having different ‘‘sorts’’ of elements), and in an as-yet-unwritten chapter, I hope to show that the adaptation of the ideas of this chapter to that context is fairly straightforward. If I were going to develop the ideas sketched above in that form, I would wait for that chapter.

But in fact, we don’t need a new kind of mathematical object to do what we have been discussing! After all, we introduced the concept of a *category* to formalize the properties of composition of maps, which is what we are dealing with here. The apparent difficulty with looking at the members of a clone of operations as morphisms in a category is that an n-ary operation in a clone can be composed on the right, not with a single operation, but with a family of n operations. The solution is to define our category so that a general morphism therein is not a single n-ary operation $|A|^n \to |A|$, but an m-tuple of n-ary operations, corresponding to a map $|A|^n \to |A|^m$.

Now everything falls into place! The category should have objects X_n in one-to-one correspondence with the natural numbers n, and a morphism between X_n and X_m should correspond to an m-tuple of n-ary operations in our clone.

In saying ‘‘a morphism between X_n and X_m’’, I have skirted the question of which of these objects is to be the domain, and which the codomain! This is simply a notational choice: whether we want to encode our structure as a certain category, or as its opposite. The development we have just seen suggests that the morphisms corresponding to m-tuples of n-ary operations should go from X_n to X_m, since an m-tuple of n-ary operations of an algebra A gives a set map $|A|^n \to |A|^m$. More globally, an m-tuple of derived n-ary operations of the whole variety $\mathbf{V}$ is equivalent to a morphism $U_{\mathbf{V}}^n \to U_{\mathbf{V}}^m$, so the ‘‘clone of derived operations’’ of $\mathbf{V}$ is encoded by the full subcategory of $\mathbf{Set}^{\mathbf{V}}$ having the functors $U_{\mathbf{V}}^n$ as objects.

But there is also motivation for the opposite choice. Recall that the n-ary operations of a variety $\mathbf{V}$ correspond to the elements of the free algebra $F_{\mathbf{V}}(n)$. An m-tuple of such elements is picked out by a homomorphism $F_{\mathbf{V}}(m) \to F_{\mathbf{V}}(n)$; so the full subcategory of $\mathbf{V}$ consisting of the free objects $F_{\mathbf{V}}(n)$ also embodies the structure of the operations of $\mathbf{V}$, in the manner opposite to way it is embodied in morphisms $U_{\mathbf{V}}^n \to U_{\mathbf{V}}^m$. This is, of course, a case of the contravariance of the Yoneda equivalence between the covariant functors $U_{\mathbf{V}}^n$ and their representing objects $F_{\mathbf{V}}(n)$.

Postponing the above question for a moment, let us note that, whichever choice we make, we will want to know *which* categories with object-set of the form $\{X_n \mid n\in\omega\}$ correspond in this way to clones of operations. Clearly, such a category should be given with a distinguished family of n morphisms $p_{n,i}$ $(i\in n)$ between X_1 and X_n for each n (corresponding, in one description to the n projection maps $|A|^n \to |A|$, and in the other to the n obvious morphisms $F_{\mathbf{V}}(1) \to F_{\mathbf{V}}(n)$). It must also have the property that morphisms between X_n and X_m (in the appropriate direction) correspond, via composition with the $p_{m,i}$, to m-tuples of morphisms between X_n and X_1.

These conditions together say that in the category, each object X_n is the *product* (respectively *coproduct*) of n copies of X_1, with the given morphisms $p_{n,i}$ as (co)projection maps. As to the choice of direction of the morphisms, Lawvere, in his doctoral thesis [**11**] where he introduced this approach, made X_n a *product* of copies of X_1, but in later published work switched to the definition under which it would be a coproduct, in other words, under which the category would look like the category of free algebras $F_{\mathbf{V}}(n)$. An attractive feature of the latter choice for Lawvere is that the category having *only* the maps $p_{n,i}$ for morphisms (corresponding to the variety with no primitive operations) is the full subcategory $\mathbf{N} \subseteq \mathbf{Set}$ having the natural numbers for objects; hence the category corresponding to a general variety can be characterized as a certain kind of extension of $\mathbf{N}$. This fit with his project of creating a category-theoretic foundation for set theory and for mathematics, with the category $\mathbf{N}$ as a basic building-block. I prefer the other choice of variance because it leads to a covariant relationship between this category of formal operations, and the actual operations in the variety. I will include both versions in the definition below, calling them the ''contravariant'' and ''covariant'' versions, but from that point on we will generally work with the covariant formulation.

Lawvere calls the category of formal operations of a variety $\mathbf{V}$ the ''theory of $\mathbf{V}$'', and any category of this form an ''algebraic theory''. For us this would be awkward, for though these categories carry approximately the same information as equational theories, the two concepts are different enough that we cannot identify them. So let us introduce a different term.

Definition 8.9.4. *A* covariant clonal category *will mean a category* $\mathbf{X}$ *given with a bijective indexing of its object-set by the natural numbers,* $\mathrm{Ob}(\mathbf{X}) = \{X_n \mid n \in \omega\}$, *and with morphisms* $p_{n,i}: X_n \to X_1$ $(i \in n)$ *which make each* X_n *the* product *of* n *copies of* X_1, *and such that* $p_{1,0}$ *is the identity map of* X_1*; equivalently, given with a functor* $\mathbf{N}^{\mathrm{op}} \to \mathbf{X}$ *which is bijective on object-sets, and turns coproducts in* $\mathbf{N}$ *to products in* $\mathbf{X}$ (*where* $\mathbf{N}$ *denotes the full subcategory of* $\mathbf{Set}$ *whose objects are the natural numbers*).

A contravariant clonal category *will mean a category* $\mathbf{X}$ *given with the dual sort of structure, i.e., with a covariant clonal category structure on* $\mathbf{X}^{\mathrm{op}}$, *equivalently, with a functor* $\mathbf{N} \to \mathbf{X}$ *which is bijective on object-sets and respects finite coproducts.*

(*More generally, for any regular cardinal* γ *one may define concepts of covariant and contravariant* γ-clonal *category, using in place of* $\mathbf{N}$ *the full subcategory of* $\mathbf{Set}$ *having for object-set the cardinal* γ, *and in place of finite* (*co*)*products,* (*co*)*products of* $<\gamma$ *factors.*)

Exercise 8.9:1. Establish the equivalence of the structures described in the first paragraph of the above definition.

A clonal category is a sort of algebraic object, so we make

Definition 8.9.5. *By* **Clone** *we shall denote the category whose objects are the covariant clonal categories, and where a morphism* $\mathbf{X} \to \mathbf{Y}$ *is a functor which carries* X_n *to* Y_n *for each* n, *and respects the morphisms* $p_{n,i}$. *In other words,* **Clone** *will denote the full subcategory of the comma category* $(\mathbf{N}^{\mathrm{op}} \downarrow \mathbf{Cat})$ *whose objects are the clonal categories.*

Incidentally, when one forms the category of *contravariant* clonal categories, this is isomorphic to our present category **Clone**, *not* opposite thereto, since the direction of morphisms within clonal categories does not affect the direction of functors among them.

We now wish to establish the relation between clonal categories and varieties of algebras. First, given a variety $\mathbf{V}$, we want to define the associated clonal category. The most convenient choice from the formal point of view would be to use n-ary derived operations of $\mathbf{V}$ as the morphisms from X_n to X_1. Unfortunately, these derived operations are not small as sets (though the set of them is quasi-small). So we will use in their stead the corresponding elements of the free $\mathbf{V}$-algebra $F_{\mathbf{V}}(n)$. Of course, we will define the composition operation of the clone so as to correspond to composition of derived operations. (This construction was in fact sketched in §6.2, with $\mathbf{V} = \mathbf{Group}$, when we were noting examples of ''nonprototypical'' ways of defining categories. We described it as a category $\mathbf{C}$ such that $\mathbf{C}(m,n)$ consisted of all n-tuples of m-ary derived group-theoretic operations.)

Definition 8.9.6. *If* $\mathbf{V}$ *is a variety of finitary algebras, the covariant* clonal theory *of* $\mathbf{V}$ *will mean the clonal category* $\mathbf{X_V}$ *in which a morphism from* X_n *to* X_m *means an m-tuple of elements of* $|F_{\mathbf{V}}(n)|$, *where composition*

$$|F_{\mathbf{V}}(n)|^m \times |F_{\mathbf{V}}(p)|^n \to |F_{\mathbf{V}}(p)|^m$$

is defined by substitution of n-tuples of expressions in p *indeterminates into expressions in* n *indeterminates, and where each* $p_{n,i}$ *is given by* i*th member of the universal n-tuple of generators of* $F_{\mathbf{V}}(n)$. *We note that this is equivalent* (*up to natural isomorphism*) *to the full subcategory of the large category* $\mathbf{Set}^{\mathbf{V}}$ *having for objects the functors* $U_{\mathbf{V}}^n$ $(n\in\omega)$, *and also to the opposite of the* (*small*) *full subcategory of* $\mathbf{V}$ *having for objects the free* $\mathbf{V}$*-algebras* $F_{\mathbf{V}}(n)$.

Given a clonal category $\mathbf{X}$, *an* $\mathbf{X}$-algebra *will mean a functor* $\mathbf{X} \to \mathbf{Set}$ *respecting the product structures defined on the objects* X_n *by the projection maps* $p_{n,i}$. *The category of all* $\mathbf{X}$*-algebras will be written* $\mathbf{X}$-$\mathbf{Alg}$. *The functor* $\mathbf{X}$-$\mathbf{Alg} \to \mathbf{Set}$ *taking each* $\mathbf{X}$*-algebra* A *to the set* $A(X_1)$ *will be written* $U_{\mathbf{X}\text{-}\mathbf{Alg}}$, *or* U *when there is no danger of ambiguity, and called the ''underlying-set functor'' of* $\mathbf{X}$-$\mathbf{Alg}$.

Note that by our general convention, unless the contrary is stated a clonal category $\mathbf{X}$ is legitimate, hence, as it has a small set of objects, it is small. Thus, the corresponding category $\mathbf{X}$-$\mathbf{Alg}$ is legitimate.

We have designed these concepts so that the categories $\mathbf{X}$-$\mathbf{Alg}$ are essentially the same as classical varieties of algebras. Let us state this property as

Lemma 8.9.7. *If* $\mathbf{X}$ *is a clonal category, then* $\mathbf{X}$-$\mathbf{Alg}$ (*defined in the second paragraph of Definition 8.9.6*) *is equivalent to a variety* $\mathbf{V}$ *of finitary algebras, by an equivalence respecting underlying-set functors.*

Moreover, if $\mathbf{V}$ *is any variety of finitary algebras, then* $\mathbf{X_V}$-$\mathbf{Alg}$ (*where* $\mathbf{X_V}$ *is defined as in the first paragraph of Definition 8.9.6*) *is equivalent in this way to* $\mathbf{V}$. *Inversely, if* $\mathbf{X}$ *is a clonal category, then* $\mathbf{X}_{\mathbf{X}\text{-}\mathbf{Alg}}$ *is naturally isomorphic in* $\mathbf{Clone}$ *to* $\mathbf{X}$. □

Exercise 8.9:2. Prove Lemma 8.9.7.

For $\mathbf{X}$ a clonal category, an $\mathbf{X}$-algebra can be thought of as a ''representation'' of the clone $\mathbf{X}$ by sets and set maps. This suggests the following more general definition, which we will find useful in the next chapter:

Definition 8.9.8. *If* $\mathbf{X}$ *is a clonal category and* $\mathbf{C}$ *any category with finite products, a* representation *of* $\mathbf{X}$ *in* $\mathbf{C}$ *will mean a covariant functor* $A: \mathbf{X} \to \mathbf{C}$ *respecting the product structures defined on the objects* X_n *by the projection maps* $p_{n,i}$.

Let us note that the information given by a clonal category is not quite the same as that given by a variety, since the clonal category does not distinguish between *primitive* and *derived* operations, while under our definition, a variety does.

Lawvere *defines* a variety of algebras (in his language, an ''algebraic category'') to mean a category of the form $\mathbf{X}\text{-}\mathbf{Alg}$, where $\mathbf{X}$ is what we call a clonal category (and he calls a theory). This is a reasonable and elegant definition, but since we began with the classical concepts of variety and theory, and it is pedagogically desirable to hold to one definition, we shall keep to our previous language, and study this construction as a closely related concept.

Exercise 8.9:3. Let $2\mathbf{N}$ be the full subcategory of $\mathbf{Set}$ having for objects the nonnegative *even* integers. For each integer n, the object $2n$ of $2\mathbf{N}$ is a coproduct of n copies of the object 2, hence the opposite category $(2\mathbf{N})^{\mathrm{op}}$ can be made a covariant clonal category by an appropriate choice of maps $p_{n,i}$. Write down such a system of maps $p_{n,i}$, and obtain an explicit description of $(2\mathbf{N})^{\mathrm{op}}\text{-}\mathbf{Alg}$ as a variety $\mathbf{V}$ determined by finitely many operations and finitely many identities. Your answer should show what it means to put a $(2\mathbf{N})^{\mathrm{op}}$-algebra structure on a set.

Exercise 8.9:4. In defining an $\mathbf{X}$-algebra as a certain kind of functor in Definition 8.9.6, we required that this functor respect the given structures of the objects X_n as n-fold products of X_1.

(i) Show that under the conditions of that definition, to respect these distinguished products is equivalent to respecting *all* finite products that may exist in $\mathbf{X}$.

(ii) Show on the other hand that an $\mathbf{X}$-algebra may fail to respect infinite products in $\mathbf{X}$. (To do this, you must start by finding a clonal category $\mathbf{X}$ having a nontrivial infinite product of objects!)

Exercise 8.9:5. Show that, up to isomorphism, there are just two clonal categories $\mathbf{X}$ such that the functor $\mathbf{N}^{\mathrm{op}} \to \mathbf{X}$ is not faithful. What are the corresponding varieties?

The next exercise does not involve the concept of clonal category, and could have come after the definition of a clone of operations on a set, but I didn't want to break the flow of the discussion. It requires familiarity with a bit of elementary electronics.

Exercise 8.9:6. (Inspired by a question of F. E. J. Linton.)

If n is a positive integer, let us understand an ''n-labeled circuit graph'' to mean a finite connected graph Γ (which may have more than one edge between two given vertices), with two distinguished vertices v_0 and v_1, and given with a function sending the edges of Γ to the set $n = \{0, \ldots, n-1\}$. To each such graph let us associate the n-ary operation on the nonnegative real numbers that takes each n-tuple $(r_0, \ldots, r_{n-1})$ of numbers to the *resistance* that would be measured between v_0 and v_1 if each edge of Γ labeled i were a resistor with resistance r_i.

(i) Explain (briefly) why the set of operations on nonnegative real numbers arising in this way from labeled circuit graphs forms a *clone*.

(ii) Let s denote the binary operation in this clone corresponding to putting two resistors in *series*, p the binary operation corresponding to putting two resistors in *parallel*, and w the 5-ary operation corresponding to a *Wheatstone Bridge*; i.e., determined by the graph , with v_0 and v_1 the top and bottom vertices, and distinct labels on all five edges. Show that none of these three operations is in the subclone generated by the other two. (Suggestion: Look at order-properties of these three operations.)

A much more difficult question is

(iii) Is the clone of (i) generated by the three operations listed in (ii)?

I do not know answers to the next two questions.

(iv) Can one *characterize* the set of operations belonging to the clone of part (i), i.e., describe some test that can be applied to an operation to determine whether it belongs to the clone?

(v) Can one find a generating set for the identities satisfied by the two binary operations s and p? (This was the question of Fred Linton's which inspired this exercise.)

Generating sets for identities of other families of operations in this clone would likewise be of interest.

(vi) Suppose one is interested in more general electrical circuits; e.g., circuits containing resistors, capacitors and inductances, and possibly other elements. Can one somehow extend the ‘‘clonal’’ viewpoint to such circuits?

(vii) If you succeed in extending the clonal approach to circuits composed of resistors, capacitors and inductances, is the clone you get isomorphic to the clone of part (i) (the clone one obtains assuming all elements are resistors)?

You might also try to answer this question for other sets of circuit elements.

We have defined the concept of a *morphism* between clonal categories. What does this mean from the viewpoint of the corresponding varieties of algebras? If $\mathbf{V}$ is a variety of Ω-algebras and $\mathbf{W}$ is a variety of Ω'-algebras, we see that to specify a morphism $f\in\mathbf{Clone}(\mathbf{X_V}, \mathbf{X_W})$ one must associate to every primitive operation s of $\mathbf{V}$ a derived operation $f(s)$ of $\mathbf{W}$ of the same arity, so that the defining identities for $\mathbf{V}$ in those primitive operations are satisfied by the operations $f(s)$ in $\mathbf{W}$. We find that such a morphism f determines a functor in the opposite direction, $\mathbf{W}\to\mathbf{V}$; namely, given a $\mathbf{W}$-algebra A, we get a $\mathbf{V}$-algebra A_f with the same underlying set by using for each primitive $\mathbf{V}$-operation s_{A_f} the derived operation $f(s)_A$ of the $\mathbf{W}$-structure on $|A|$. In fact we have

Lemma 8.9.9. (Lawvere) *Functors between varieties of algebras which preserve underlying sets correspond bijectively to functors in the opposite direction between the clonal theories of these varieties, via the construction described above.* □

Exercise 8.9:7. Prove Lemma 8.9.9.

Easy examples of such functors among varieties are the *forgetful* functors $\mathbf{Group}\to\mathbf{Monoid}$, $\mathbf{Ring}^1\to\mathbf{Monoid}$, $\mathbf{Ring}^1\to\mathbf{Ab}$, $\mathbf{Lattice}\to\vee\textbf{-Semilattice}$, and similar constructions, including the underlying-set functor of every variety, and the inclusion functor of any subvariety in a larger variety, e.g., $\mathbf{Ab}\to\mathbf{Group}$. In the above list of cases, each primitive operation of the codomain variety happens to be mapped to a primitive operation of the domain variety. Some examples in which primitive operations are mapped to proper derived operations are the functor $\mathbf{Bool}^1\to\vee\textbf{-Semilattice}$ under which the semilattice operation $x\vee y$ is mapped to (i.e., given by) the Boolean operation $(x,y)\mapsto x+y+xy$; the functor H: $\mathbf{Group}\to\mathbf{Heap}$ of Exercise 8.6:9, under which the ternary heap operation τ is mapped to the group operation $xy^{-1}z$, and the functor B: $\mathbf{Ring}^1_k\to\mathbf{Lie}_k$ of §8.7, under which the primitive k-module operations of $\mathbf{Lie}_k$ are mapped to the corresponding primitive operations of $\mathbf{Ring}^1_k$, but the Lie bracket is mapped to the commutator operation $xy-yx$.

We have seen most of the above constructions before as examples of functors having left adjoints. In fact, one can prove that any functor between varieties induced by a morphism of their clonal theories – in other words, every functor between varieties that preserves underlying sets – has a left adjoint! We will not stop to do this here, because it will be an immediate consequence of a *necessary and sufficient* criterion for a functor between varieties to have a left adjoint that we will obtain in the next chapter. But you can prove this case now as an exercise:

Exercise 8.9:8. (Lawvere) Show that any functor between varieties of finitary algebras which preserves underlying sets has a left adjoint.

You may drop the ‘‘finitary’’ condition if you wish, either using generalized versions of the results of this section, or proving the result without relying on the ideas of this section.

Here are a few more exercises on underlying-set-preserving functors and their adjoints:

Exercise 8.9:9. Let U: $\mathbf{Group} \to \mathbf{Monoid}$ denote the forgetful functor, and F: $\mathbf{Monoid} \to \mathbf{Group}$ its left adjoint (called in §3.11 the ‘‘universal enveloping monoid’’ construction).

(i) Show that there exist proper subvarieties $\mathbf{V} \subseteq \mathbf{Group}$ such that $U(\mathbf{V})$ does not lie in a proper subvariety of $\mathbf{Monoid}$.

A much harder problem is

(ii) If $\mathbf{V}$ is a proper subvariety of $\mathbf{Monoid}$, must $F(\mathbf{V})$ be contained in a proper subvariety of $\mathbf{Group}$? Must one in fact have $UF(\mathbf{V}) \subseteq \mathbf{V}$?

Exercise 8.9:10. Let H: $\mathbf{Group} \to \mathbf{Heap}$ be the functor described by (8.6.8) in Exercise 8.6:9, and F: $\mathbf{Heap} \to \mathbf{Group}$ its left adjoint. Let A be a nonempty heap. We recall that $A \cong H(G)$ for some group G.

(i) Characterize the structure of the group $F(A)$ in terms of that of G.

(ii) It follows from Exercise 8.6:9(ii) that in general, A has automorphisms i not arising from automorphisms of the group G. Give an example of such an automorphism (or better, a complete characterization of automorphisms of any nonempty heap $A = H(G)$), and describe the induced automorphism $F(i)$ of the group $F(H(G))$.

Exercise 8.9:11. Show that there exist exactly two underlying-set-preserving functors $\mathbf{Set} \to \mathbf{Semigroup}$. (Hint: What derived operations does $\mathbf{Set}$ have?) Find their left adjoints.

The next exercise looks at clonal categories as algebraic objects:

Exercise 8.9:12. (i) Show that the category $\mathbf{Clone}$ has small limits and colimits.

(ii) Is $\mathbf{Clone}$ equivalent to a variety of algebras?

The discussion leading up to Lemma 8.9.9 shows more than was stated in that lemma. So let us also record

Lemma 8.9.10. *Let $\Omega = (|\Omega|, \mathrm{ari})$ be any type. Then the functor $\mathbf{Clone} \to \mathbf{Set}$ associating to each clonal category $\mathbf{X}$ the set of maps $|\Omega| \to \bigsqcup_n \mathbf{X}(X_n, X_1)$ taking each $s \in |\Omega|$ to an element of $\mathbf{X}(X_{\mathrm{ari}(s)}, X_1)$ is representable, with representing object $\mathbf{X}_{\Omega\text{-}\mathbf{Alg}}$. Thus, $\mathbf{X}_{\Omega\text{-}\mathbf{Alg}}$ may be regarded as a ‘‘free clonal category on an $|\Omega|$-tuple of formal operations of arities given by the function* ari*’’.*

Now suppose also that J is a set of identities for Ω-algebras, which we will here express, not as pairs of elements of $|F_\Omega(\omega)|$, but as pairs of elements of $|F_\Omega(n)|$ for various $n \in \omega$; and let us identify these sets $|F_\Omega(n)|$ with the sets $\mathbf{X}_{\Omega\text{-}\mathbf{Alg}}(n, 1)$. Then the functor $\mathbf{Clone} \to \mathbf{Set}$ associating to each clonal category $\mathbf{X}$ the set of those maps $|\Omega| \to \bigsqcup_n \mathbf{X}(X_n, X_1)$ as in the preceding paragraph, which satisfy the additional condition that for each $(s, t) \in J$, the induced map $\mathbf{X}_{\Omega\text{-}\mathbf{Alg}} \to \mathbf{X}$ carries s and t to the same element, is representable, with representing object $\mathbf{X}_{\mathbf{V}(J)}$. Thus, $\mathbf{X}_{\mathbf{V}(J)}$ may be written $<\Omega \mid J>_{\mathbf{Clone}}$, i.e., regarded as the clonal category ‘‘presented by the family Ω of formal operations, and the family J of relations among the derived operations of this family’’. □

Let me end this section by mentioning a few concepts on which there is considerable literature, though we will not study them further here.

One is often interested in properties of a variety $\mathbf{V}$ of algebras that do not depend on which operations are considered primitive. Not surprisingly, these can often be expressed as elementary statements about the clone $\mathbf{X}_{\mathbf{V}}$ of derived operations of $\mathbf{V}$.

The formally simplest properties to state about a clone of operations are universally and existentially quantified equations in families of operations of specified arities. Universally quantified equations of this sort are called *hyperidentities* [**99**]. An example, and its interpretation in terms of ordinary identities, is noted in

Exercise 8.9:13. Show that a variety $\mathbf{V}$ satisfies the hyperidentity saying that all derived unary operations are equal if and only if all primitive operations s of $\mathbf{V}$ (of all arities) satisfy the identity of idempotence: $s(x, \dots, x) = x$.

The above hyperidentity is satisfied, for instance, by the varieties of lattices, semilattices, and heaps. On the other hand, there are many varieties that satisfy no nontrivial hyperidentities; e.g., it is shown in [**99**] that this is true of the variety of commutative rings. A class of varieties determined by a family of hyperidentities is called a *hypervariety*.

(The term ‘‘hyperidentity’’ is used by some authors, e.g., in [**86**], with the related but different meaning of an identity holding for all families of *primitive* operations of given arities in a variety. But this seems to me a less useful concept.)

Exercise 8.9:14. (i) Show that for every monoid identity $s = t$ there is a hyperidentity $s' = t'$ such that for S a monoid, the variety S-**Set** satisfies $s' = t'$ if and only if the monoid S satisfies $s = t$.

(ii) Is the analogous statement true for ring identities and hyperidentities of varieties R-**Mod**? If not, how much can be said about the relation between hyperidentities satisfied by varieties R-**Mod**, and identities or other conditions, satisfied by R?

Because hyperidentities involve both universal quantification over derived operations and universal quantification over the elements to which these operations are applied, they tend to be very strong, and hence somewhat ‘‘crude’’ conditions, as illustrated by the fact that the variety of commutative rings satisfies no hyperidentities. *Existentially* quantified equations in derived operations, on the other hand, have proved a delicate tool in General Algebra. An example of this sort of condition on a variety $\mathbf{V}$ is the statement that there exists a derived ternary operation M of $\mathbf{V}$ satisfying the identities

$$M(x,x,y) \;=\; M(x,y,x) \;=\; M(y,x,x) \;=\; x.$$

This is satisfied, for instance, by the variety of lattices, where one can take $M(x,y,z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Another example is the condition of the same form, but with ‘‘$= y$’’ on the right in place of ‘‘$= x$’’; this is satisfied by the variety of abelian groups of exponent 2, with $M(x,y,z) = x+y+z$. Many important technical conditions on a variety $\mathbf{V}$ (for instance, the condition that for any two congruences E and E' on an object A of $\mathbf{V}$, one has $E \circ E' = E' \circ E$ under composition of binary relations on $|A|$; or the condition that each subalgebra B of a direct product algebra $A_1 \times \dots \times A_n$ in $\mathbf{V}$ be determined by its images in the pairwise products $A_i \times A_j$) turn out to be equivalent to the statement that $\mathbf{V}$ belongs to the union of some chain of classes, each determined by an existentially quantified equation in derived operations. The condition that a variety belong to such a union is called a *Mal’cev condition*; see [**3** §II.12] and [**9**, §60] for examples and applications.

Finally, let me indicate a concept similar to that of clonal category, but designed to apply to a wider class of situations, called an *operad*. To motivate the idea, suppose that we wish to think of an algebra over a field k, not as a set with operations $+$, 0, $-$, $\cdot$, etc., but as a k-vector-space with a single additional k-bilinear operation ''$\cdot$'', and that we want to look at this in the context of general systems consisting of k-vector-spaces V given with k-multilinear operations satisfying various multilinear identities. To study such entities, we would like to set up an abstract model, analogous to a clonal category, but modeling, not an unstructured set with set-theoretic operations, but a vector space with multilinear operations. As in the situation that motivated clonal categories, one can form *derived* multilinear operations from given multilinear operations; but there are things one can do in a clone of set-theoretic operations but not in this context: The projection maps $V^n \to V$ are not multilinear, so they will not appear in our structure, nor, for the same reason, will any derived operations in which some variable occurs more than once. On the other hand, one has some structure in this multilinear context which one does not have for ordinary clones, namely a k-vector-space structure on the set of multilinear operations of each arity, under which composition of multilinear operations is given by multilinear maps. The analog of a clonal category that one gets on taking these features into account is called a *k-linear operad*.

Now let the role that was held by direct products of sets in our development of the concept of clonal category, and by tensor products of vector spaces in that of k-linear operad, be filled by a general bifunctor ''$\square$'' on some category $\mathbf{C}$, satisfying appropriate associativity conditions. One can write down a description of the sort of composition of operations that is possible without any more specific assumptions on $\square$. The structure one obtains in this way is called an *operad*. For more details, see [**59**].

8.10. Structure and Semantics. The results of this section will not be essential to what follows, and our presentation will be sketchy. These results give, however, a useful perspective on what we have been doing, and the ideas below will be referred to several times in the next chapter.

Let us look back at the way we associated a clonal theory to a variety $\mathbf{V}$ in Definition 8.9.6. I claim that the various equivalent forms of that construction all reduce to an observation that is applicable in much broader contexts, namely

Lemma 8.10.1. *Let $\mathbf{C}$ be a category, and A an object of $\mathbf{C}$ such that all finite products $A\times\ldots\times A$ exist in $\mathbf{C}$, and let a product $A^n = \prod_{i\in n} A$ $(n\in\omega)$ be chosen for each n, so that the objects A^n are distinct in* $\mathrm{Ob}(\mathbf{C})$. *Then the full subcategory of $\mathbf{C}$ whose objects are the A^n, given with the projection maps $p_{n,i}\colon A^n \to A$, is a clonal category.* □

To see that this was essentially what we were using in Definition 8.9.6, note on the one hand that each free object $F_{\mathbf{V}}(n)$ is a coproduct of n copies of $F_{\mathbf{V}}(1)$, hence in $\mathbf{C}^{\mathrm{op}}$, the corresponding objects are products of n copies of one object, and the full subcategory of $\mathbf{C}^{\mathrm{op}}$ with these objects is one of our descriptions of the clonal theory of $\mathbf{V}$. The description based on looking at the products $U_{\mathbf{V}}^n$ of copies of the object $U_{\mathbf{V}}$ used the same idea in the large category $\mathbf{Set}^{\mathbf{V}}$.

We may generalize this latter example by considering any category $\mathbf{C}$ given with a functor U: $\mathbf{C} \to \mathbf{Set}$. The full subcategory of $\mathbf{Set}^{\mathbf{C}}$ having for objects the functors U^n will in general be large; however, in many cases it will be *quasi-small*, i.e., isomorphic to a small category $\mathbf{X}$. (This is true whenever the U^n are representable, or more generally, if the solution-set condition for representability holds, even though the other conditions may not.) To formalize this class of examples, let us make

Definition 8.10.2. *For the remainder of this section,* **Conc** *will denote the large category having for objects all pairs* $(\mathbf{C}, U)$, *where* $\mathbf{C}$ *is a category, and* U *a functor* $\mathbf{C} \to \mathbf{Set}$, *such that for every integer* n, $\mathbf{Set}^{\mathbf{C}}(U^n, U)$ *is quasi-small, and where a morphism* $(\mathbf{C}, U) \to (\mathbf{D}, V)$ *means a functor* $F: \mathbf{C} \to \mathbf{D}$ *such that* $VF = U$.

(I've chosen the symbol **Conc** as an abbreviation for ''concrete'', though that term is only an approximation, since we are not assuming that the functors to **Set** are faithful, while we *are* assuming a quasi-smallness hypothesis not in the definition of ''concrete category''. The point of this terminology is to make us think of U (at least at the beginning) as like an ''underlying-set functor'', so that we can picture the morphisms of **Conc** as the *underlying-set-preserving* functors.)

If we associate to each object of **Conc** the clonal category having for object-set the powers of U (Definition 6.9.8), this gives a contravariant construction (because of the way morphisms are defined in **Conc**) of clonal categories from these objects. Unfortunately, this cannot be regarded as a functor to **Clone** because the values assumed, though quasi-small, are not in general small. Hence, for each $(\mathbf{C}, U) \in \mathrm{Ob}(\mathbf{Conc})$ let us choose a small category isomorphic to the category of natural-number powers of U, and regard this as an object of **Clone**. In this way we get a functor $\mathbf{Conc}^{\mathrm{op}} \to \mathbf{Clone}$. Since the morphisms $X_n \to X_1$ in the category constructed in this way from $(\mathbf{C}, U)$ correspond to the functorial *n-ary operations* we can put on the sets $U(C)$ $(C \in \mathrm{Ob}(\mathbf{C}))$, the category can be thought of as describing the *algebraic structure* we can put on the values the functor U; hence Lawvere has named this functor ''Structure''.

Exercise 8.10:1. Describe precisely how to make Structure a functor. (Cf. Lemma 7.2.8.)

On the other hand, Lawvere calls the construction taking a clonal category $\mathbf{X}$ to the variety $\mathbf{X}$**-Alg** given with its underlying set functor, i.e., the clonal category $(\mathbf{X}\text{-}\mathbf{Alg}, U_{\mathbf{X}\text{-}\mathbf{Alg}})$ (which we have seen is also a contravariant construction) ''Semantics'', because it takes a category of *symbolic* operations, and interprets these in all possible ways as *actual* operations on sets.

Consider now an arbitrary $(\mathbf{C}, U) \in \mathrm{Ob}(\mathbf{Conc})$, and let $\mathbf{X} = \mathrm{Structure}(\mathbf{C}, U)$. By construction of $\mathbf{X}$, the sets $U(C)$ $(C \in \mathrm{Ob}(\mathbf{C}))$ have structures of $\mathbf{X}$-algebra, and these are functorial, in the sense that for f a morphism of $\mathbf{C}$, the set-map $U(f)$ is a homomorphism of $\mathbf{X}$-algebras. This is equivalent to saying that we have an underlying-set-preserving functor $(\mathbf{C}, U) \to (\mathbf{X}\text{-}\mathbf{Alg}, U_{\mathbf{X}\text{-}\mathbf{Alg}})$. Of course, there are other clonal categories $\mathbf{Y}$ for which one can put functorial $\mathbf{Y}$-algebra structures on these sets (e.g., clonal subcategories of $\mathbf{X}$), but it is not hard to verify that $\mathbf{X}$ is universal for this property, i.e., that every functorial $\mathbf{Y}$-algebra structure arises from a morphism of clones, $\mathbf{Y} \to \mathbf{X}$. This universal property is expressed in Lawvere's celebrated theorem, ''Structure is adjoint to Semantics''.

Since in the universal property, the general clonal category $\mathbf{Y}$ is mapped to the universal clonal category $\mathbf{X}$, the latter is *right universal*. So the precise statement is:

Theorem 8.10.3 (Lawvere)**.** *The functors*

$$\mathrm{Structure}: \mathbf{Conc}^{\mathrm{op}} \to \mathbf{Clone} \qquad \textit{and} \qquad \mathrm{Semantics}: \mathbf{Clone}^{\mathrm{op}} \to \mathbf{Conc}$$

are mutually right adjoint contravariant functors. □

Exercise 8.10:2. Prove the above theorem.

As with any adjunction, we have a pair of *universal morphisms* connecting the two *composites* of these functors with the identity functors of the given categories. In the more familiar case of a *covariant* adjunction, one of these morphisms, the unit, goes from the identity functor to the composite (e.g., the map from each set X to the underlying set of the free group on X), and the other, the counit, from the composite to the identity (e.g., from the free group on the underlying set of a group G to G itself). But in the case of a contravariant adjunction, they both go in the same direction; in the right-adjoint case, which we have here, from the identity functor to the composite functor. In the present example, one of these universal maps, namely

$$\mathrm{Id}_{\mathbf{Clone}} \to \mathrm{Structure} \circ \mathrm{Semantics} \tag{8.10.4}$$

is an isomorphism; this is essentially the last assertion of Lemma 8.5.3. Looking at the other composite,

$$\mathrm{Id}_{\mathbf{Conc}} \to \mathrm{Semantics} \circ \mathrm{Structure}, \tag{8.10.5}$$

it is not hard to see that it will give an equivalence when applied to an object of **Conc** if and only if that object is (up to equivalence) of the form $(\mathbf{V}, U_{\mathbf{V}})$ where $\mathbf{V}$ is a variety and $U_{\mathbf{V}}$ its underlying-set functor. When we apply $\mathrm{Semantics} \circ \mathrm{Structure}$ to a general object $(\mathbf{C}, U)$ of **Conc**, it can be thought of as giving a best approximation of that category by a variety and its underlying-set functor. Thus, for every given pair $(\mathbf{C}, U)$, (8.10.5) gives a ‘‘comparison functor’’

$$(\mathbf{C}, U) \to \mathrm{Semantics} \circ \mathrm{Structure}\,(\mathbf{C}, U),$$

between the given object of **Conc** and that ‘‘approximation’’.

Exercise 8.10:3. Describe $\mathrm{Structure}(\mathbf{C}, U)$ in each of the following cases (e.g., by choosing a set of ‘‘primitive operations’’ and identities), and determine whether the comparison functor is an equivalence.

(i) $\mathbf{C} = \mathbf{Set}$, $U(X) = X \times X$.

(ii) $\mathbf{C} = \mathbf{Set} \times \mathbf{Set}$, $U(X, Y) = X \times Y$.

(iii) $\mathbf{C} = \mathbf{Ab}$, $U(X) = U_{\mathbf{Ab}}(X \times X)$.

(iv) $\mathbf{C} = \mathbf{Ab} \times \mathbf{Ab}$, $U(X, Y) = U_{\mathbf{Ab}}(X \times Y)$.

(v) $\mathbf{C} = \mathbf{POSet}$, U = the underlying-set functor.

In cases (iii) and (iv), show that the clone $\mathrm{Structure}(\mathbf{C}, U)$ can be naturally identified with the clonal theory of modules over a certain ring.

Exercise 8.10:4. (i) Same task as in the above exercise, for $\mathbf{C} = \mathbf{Set}^{\mathrm{op}}$, and U the power-set functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$.

(ii) If you are comfortable generalizing the concepts of this and the preceding section to algebras with operations of possibly infinite arities, getting in particular a functor γ-Structure: $\mathbf{Conc}^{\mathrm{op}} \to \gamma$-**Clone** for γ a regular cardinal, investigate $\gamma\text{-Structure}(\mathbf{C}, U)$ for the above case.

Exercise 8.10:5. Let **CpLattice** denote the category of complete lattices, and $\mathbf{CpSemilattice}^{0}$ the category of complete upper semilattices with least element (regarded as a zeroary operation). We recall that the objects of these two categories are essentially the same, but the morphisms are not (cf. Proposition 5.2.3).

(i) Show that the underlying-set functor on one of these categories satisfies the smallness condition in the definition of **Conc**, but that of the other does not.

(ii) In the case that does give an object $(\mathbf{C}, U)$ of **Conc**, describe the variety $\mathrm{Semantics} \circ \mathrm{Structure}(\mathbf{C}, U)$.

Let us end this section with a few observations on the question, ‘‘Given a category, how can one tell whether it is equivalent to a variety of algebras?'' (Birkhoff's Theorem tells us which full subcategories of a category **Ω-Alg** are varieties, but the above question, about abstract categories, is of a different sort.) By our preceding observations, a necessary and sufficient condition is that there exist a functor $U\colon \mathbf{C} \to \mathbf{Set}$ such that $(\mathbf{C}, U)$ lies in **Conc**, and the comparison functor

$$(\mathbf{C}, U) \to \mathrm{Semantics} \circ \mathrm{Structure}(\mathbf{C}, U)$$

is an equivalence. Note also that the underlying-set functor of any variety is *representable* (by the free object on one generator), so if the above condition holds, U can be taken to have the form h_G for some object G of $\mathbf{C}$. In this situation (since by our general convention, $\mathbf{C}$ is assumed legitimate), the quasi-smallness condition on the powers of U automatically holds by Yoneda's Lemma. In summary:

Lemma 8.10.6. *Let* Deconc: $\mathbf{Conc} \to \mathbf{Cat}$ *be the ''deconcretization'' functor* $(\mathbf{C}, U) \mapsto \mathbf{C}$. *Then a category* $\mathbf{C}$ *is equivalent to a variety of finitary algebras if and only if there exists some* $G \in \mathrm{Ob}(\mathbf{C})$ *such that the functor*

$$\mathrm{Deconc} \circ \mathrm{Semantics} \circ \mathrm{Structure}(\mathbf{C}, h_G) \xrightarrow{\mathrm{Deconc} \circ \mathrm{Comparison}} \mathbf{C} \qquad (8.10.7)$$

is an equivalence of categories.

(*The analogous result holds with ''finitary'' replaced by ''having all operations of arity* $<\gamma$*'' for any fixed regular cardinal* γ, *if we use corresponding modified functors* γ-Structure *and* γ-Semantics.) □

Though this does not say very much, it gives a useful heuristic pointer: If we want to determine whether a category $\mathbf{C}$ is equivalent to a variety of algebras, we should look for possible candidates for the free object on one generator. Parts (i)-(v) of the next exercise are cases where you can show that no such object exists. I do not advise trying to use the above lemma in this exercise, but only the ''heuristic pointer''.

Exercise 8.10:6. Show that none of the categories named in (i)-(v) below are equivalent to varieties of algebras. (Here a ''variety'' is *not* required to have finitary operations, though as always, it is assumed to have a small set of operations.)

(i) **POSet**. (Suggestion: For each of the situations (a) $\mathbf{C}$ a variety of algebras, and A a free algebra in $\mathbf{C}$ on a nonempty set, (b) $\mathbf{C} = \mathbf{POSet}$, and A a discrete partially ordered set, and (c) $\mathbf{C} = \mathbf{POSet}$, and A a nondiscrete partially ordered set, investigate the relationship between the set of difference cokernel maps in $\mathbf{C}$, and the set of morphisms in $\mathbf{C}$ that h_A takes to surjective set maps.)

(ii) **Clone**. (Suggestion: Show that (a) an object corresponding to a free object on one generator would have to be a finitely generated clonal category, (b) if it were generated by elements of arities $\leq n$, this would be true of all clonal categories, and (c) this is not the case.)

(iii) **Compact**, the category of compact Hausdorff spaces and continuous maps. (Suggestion: If $\mathbf{V}$ is a variety with all operations having arities $<\gamma$, what does this imply about the closure operator ''subalgebra generated by –'' on the underlying sets of algebras in $\mathbf{V}$? (Cf. Definition 5.3.7 for the case $\gamma = \omega$.) Translate this into a statement involving the free object on one generator in $\mathbf{V}$, and show that no object has this property in **Compact**.)

(iv) The full subcategory of **Ab** whose objects are the torsion-free abelian groups.

(v) The full subcategory of **Ab** whose objects are the divisible abelian groups (groups such that for every group element x and nonzero integer n, the equation $ny = x$ has a solution y in A.)

(vi) On the other hand, show that the full subcategory of **Ab** whose objects are the divisible torsion-free abelian groups *is* equivalent to a variety of algebras.

In contrast to point (iii) above, it is proved in [**82**] that **Compact** *can* be identified with a ‘‘variety’’ if we generalize that concept to allow a *large* set of operations – as we would also have to do, for instance, to speak of the ‘‘variety’’ of complete lattices or semilattices. Under this construction of **Compact**, the operations of each cardinality α correspond to the points of the Stone-Čech compactification of the discrete space α. Note that this means that, in contrast to the case of complete lattices (but as for complete upper or lower semilattices), *each* of these sets of operations is small; i.e., the corresponding generalized clonal category, though not generated by a small set, is legitimate. A consequence is that compact Hausdorff spaces actually behave more like ordinary algebras than do complete lattices! In particular, there is a ‘‘free compact Hausdorff space’’ on every small set X, namely, its Stone-Čech compactification.

Lemma 8.10.6(ii) does *not* say that an object G with the indicated properties is unique if it exists. Let us examine the extent to which we can vary G in a couple of familiar varieties, and what happens when we do.

Exercise 8.10:7. (i) When $\mathbf{C} = \mathbf{Ab}$, determine for what objects G the functor (8.10.7) is an equivalence. Show that for every such G, Structure$(\mathbf{Ab}, h_G)$ can be identified with the theory of modules over some ring R.

(ii) Similarly, for $\mathbf{C} = \mathbf{Set}$ determine what objects make (8.10.7) an equivalence, and try to describe in these cases the theory Structure$(\mathbf{Set}, h_G)$.

Thus, in case (i) we find that **Ab** is equivalent to several different varieties R-**Mod**, and in the second we similarly discover that **Set** is equivalent to several nontrivial varieties of algebras.

Lawvere gives in his thesis [**11**, §III.2] a version of Lemma 8.10.6 which is less trivial than ours, but also more complicated to formulate; we will not present it here.

We remark that despite the technical term given the word ‘‘structure’’ in this section, we will also continue to use it as a versatile meta-term in our mathematical discussions.

Part III. More on adjunctions.

Chapter 9 (the only chapter of this part yet written) represents the culmination of the course. In it we obtain Freyd's beautiful characterization of functors among varieties of algebras that have left adjoints, and study several classes of examples and related results.

Chapter 9. Algebra and coalgebra objects in categories, and functors having adjoints.

One of our long-range goals, since we took our ‘‘Cook’s tour’’ of universal constructions in Chapter 3, has been to obtain general results on when algebras with given universal properties exist. We have gotten several existence results holding in any variety **V**, namely, for free objects, limits and colimits, and objects presented by generators and relations. The result for free objects can be restated as the existence of a left adjoint to the forgetful functor **V** → **Set**, and we have also shown that the inclusion **V** → Ω-**Alg** has a left adjoint, where Ω is the type of **V**. In the first three sections of this chapter, we shall develop a result of a much more sweeping sort: a characterization of *all* functors between varieties of algebras **V** and **W** which have left adjoints.

To get an idea what this characterization should be, we should look at some representative examples. Most of the functors with left adjoints among varieties of algebras that we have seen so far have been cut from a fairly uniform mold: underlying-set-preserving constructions that forget some of the operations, and things close to these. We shall begin by looking at an example of a different sort, which will give us some insight into the features that make the construction of a left adjoint possible. We will then formalize these features, arriving at a pair of concepts (those of algebra and coalgebra objects in a general category) of great beauty in their own right, in terms of which we shall establish the desired condition in §9.3. In the remaining sections of this chapter we will work out in detail some classes of examples, and note various related results.

9.1. An example: SL(*n*). Let n be a positive integer. Then for any commutative ring A, the $n\times n$ matrices over A having determinant 1 form a group, called the *special linear* group $\mathrm{SL}(n, A)$. Clearly, $\mathrm{SL}(n, -)$ is a functor $\mathbf{CommRing}^1 \to \mathbf{Group}$. Let us simplify our name for this functor to $\mathrm{SL}(n)$, but continue to write its value at A as $\mathrm{SL}(n, A)$.

Does $\mathrm{SL}(n)$ have a left adjoint? In concrete terms, this asks: Given a group G, can we find a universal example of a commutative ring A_G with a homomorphism $G \to \mathrm{SL}(n, A_G)$?

Let us approach this question in our standard way (noted in comment 2.2.10), namely, by considering an arbitrary commutative ring A with a homomorphism

$$h:\ G\ \to\ \mathrm{SL}(n, A),$$

and asking what elements of A, and what relations among these, are determined by this situation.

Clearly, we can get n^2 elements of A from each element g of G, to wit, the components of the matrix $h(g)$:

$$h(g)_{ij} \qquad (g\in|G|,\ \ i, j = 1, \ldots, n). \tag{9.1.1}$$

By definition of $\mathrm{SL}(n, A)$, these satisfy the relation saying that the determinant of the matrix they form is 1:

$$\det(h(g)_{ij})\ =\ 1 \qquad (g\in|G|). \tag{9.1.2}$$

The condition that h be a group homomorphism says that for every two elements $g, g'\in|G|$, the matrix $(h(gg')_{ij})$ is the product of the matrices $(h(g)_{ij})$ and $(h(g')_{ij})$. Each such matrix equation is equivalent to n^2 equations in the ring A:

(9.1.3) $$h(gg')_{ik} \ = \ \Sigma_j \ h(g)_{ij} \ h(g')_{jk} \qquad (g, g' \in |G|, \ \ i, k = 1, \ldots, n).$$

Clearly, a system of elements (9.1.1) satisfying (9.1.2) and (9.1.3) is equivalent to a homomorphism $G \to \mathrm{SL}(n, A)$. Hence, if we let A_G be the object of $\mathbf{CommRing}^1$ presented by *generators* (9.1.1) and *relations* (9.1.2) and (9.1.3), and denote by $h\colon G \to \mathrm{SL}(n, A_G)$ the resulting group homomorphism, then the pair (A_G, h) will be initial among commutative rings A given with such homomorphisms, and the construction $G \mapsto A_G$ will be the desired left adjoint to $\mathrm{SL}(n)$.

What properties of the functor $\mathrm{SL}(n)$ have we used here? First, the fact that for every commutative ring A, the elements of $\mathrm{SL}(n, A)$ could be described as all families of elements of A indexed by a certain fixed set (in this case the set $n \times n$) which satisfied certain equations (in this case, the single equation saying that the matrix they formed had determinant 1). It was this that allowed us to write down the generators (9.1.1) and relations (9.1.2) in the definition of A_G. Secondly, we used the fact that the multiplication of the group $\mathrm{SL}(n, A)$ takes a pair of matrices s, t to a matrix st whose entries are given by certain fixed polynomials (i.e., derived operations) in the $2n^2$ entries of the two given matrices. This allowed us to express the condition that h be a homomorphism by the equations (9.1.3).

We also used, implicitly, a fact special to the variety of groups, namely that for a map of underlying sets to be a homomorphism, it suffices that it respect multiplication. If we want to put this example into a form that generalizes to arbitrary varieties, we should note that the unary ‘‘inverse’’ operation and the zeroary ‘‘neutral element’’ operation of $\mathrm{SL}(n, A)$ also have the property that their entries are given by polynomials in the entries of their arguments: The inverse of a matrix of determinant 1 is a matrix of determinants of minors (with certain ± signs); the identity matrix consists of 0’s and 1’s in certain positions, and these 0’s and 1’s can be regarded as polynomials in the empty set of variables. Hence if we do not wish to call on the special property of group homomorphisms mentioned, we can still guarantee the universal property of A_G, by supplementing (9.1.3) with relations saying that for all $g \in |G|$, the entries of $h(g^{-1})$ are given by the appropriate signed minors in the entries of $h(g)$, and that the (i,j) entry of $h(e)$ has the value δ_{ij}.

To abstract the conditions noted above, let us now consider arbitrary varieties $\mathbf{V}$ and $\mathbf{W}$ (in general of different types), and a functor

$$V\colon \ \mathbf{W} \ \to \ \mathbf{V}$$

for which we hope to find a left adjoint. The analog of the first property noted for $\mathrm{SL}(n)$ above should be that for $A \in \mathrm{Ob}(\mathbf{W})$, the underlying set $|V(A)|$ is describable as the set of X-tuples of elements of $|A|$, for some fixed set X, which satisfy a fixed set Y of relations. We recall from Lemma 8.4.13 that this is equivalent to saying that the set-valued functor $A \mapsto |V(A)|$, i.e., the functor $U_{\mathbf{V}} V$ (where $U_{\mathbf{V}}\colon \mathbf{V} \to \mathbf{Set}$ is the underlying-set functor of $\mathbf{V}$) is *representable*, with representing object the $\mathbf{W}$-algebra defined using X and Y as generators and relations:

(9.1.4) $$R \ = \ < X \mid Y >_{\mathbf{W}}.$$

The object (9.1.4) thus ‘‘encodes’’ the functor V at the set level! Is there a way to extend these observations so as to encode also the $\mathbf{V}$-algebra structures on the sets $|V(A)|$?

Let us look at this question in the case $V = \mathrm{SL}(n)$. We see that the functor $U_{\mathbf{Group}} \circ \mathrm{SL}(n)$ is represented by the commutative ring R presented by n^2 generators r_{ij} and one relation $\det(r_{ij}) = 1$; in other words, the commutative ring having a universal $n \times n$ matrix r of determinant 1. Can we find a universal instance of *multiplication* of such matrices? Since multiplication is a binary operation, we should multiply a universal *pair* of matrices of determinant 1. The ring with such a universal pair is the coproduct of two copies of R. If we denote these two matrices $r_0, r_1 \in |\mathrm{SL}(n, R \amalg R)|$, then the n^2 entries of the product matrix $r_0 r_1 \in |\mathrm{SL}(n, R \amalg R)|$ can, like any elements of $R \amalg R$, be expressed as polynomials in our generators for that ring, the entries of r_0 and r_1. Using the universality of r_0, $r_1 \in |\mathrm{SL}(n, R \amalg R)|$, it is not hard to show that those same polynomials, when applied to the entries of two *arbitrary* elements of $\mathrm{SL}(n, A)$ for an *arbitrary* commutative ring A, must also give the entries of their product. So it appears that $r_0 r_1$ does in some sense encode the multiplication operation of $\mathrm{SL}(n)$.

There is a more abstract way of looking at this. By the universal property of R, the element $r_0 r_1 \in |\mathrm{SL}(n, R \amalg R)|$ corresponds to some morphism

$$\mathbf{m}: R \to R \amalg R \tag{9.1.5}$$

(the unique morphism taking the entries of r to those of $r_0 r_1$). Now given a commutative ring A, any two elements $x, y \in |\mathrm{SL}(n, A)|$ arise as images of the universal element r under unique homomorphisms $f, g: R \to A$. Such a pair of morphisms corresponds, by the universal property of the coproduct, to a single morphism $(f, g): R \amalg R \to A$ (the morphism carrying the entries of r_0 to those of x and the entries of r_1 to those of y). Composing with (9.1.5), we get a morphism

$$R \xrightarrow{\mathbf{m}} R \amalg R \xrightarrow{(f,\, g)} A, \tag{9.1.6}$$

which corresponds to an element of $\mathrm{SL}(n, A)$. From the facts that $\mathbf{m}$ corresponds to (i.e., sends r to) the *product* of r_0 and r_1, and that $\mathrm{SL}(n)$, applied to the map (f, g) gives a *group homomorphism* $\mathrm{SL}(n, R \amalg R) \to \mathrm{SL}(n, A)$, we can deduce that the matrix given by (9.1.6) (i.e., the result of applying the ring-homomorphism (9.1.6) entrywise to r) is the product of x and y. So the ring homomorphism $\mathbf{m}$ indeed ‘‘encodes’’ our multiplication.

We note similarly that $r^{-1} \in |\mathrm{SL}(n, R)|$ will be the image of the universal element r under a certain morphism

$$\mathbf{i}: R \to R \tag{9.1.7}$$

and we find that this morphism $\mathbf{i}$ encodes the *inverse* operation on $\mathrm{SL}(n)$.

If we are going to treat the zeroary neutral-element operation similarly, it should correspond to a morphism from R to the coproduct of zero copies of itself. This vacuous coproduct is the *initial object* of $\mathbf{CommRing}^1$, namely the ring $\mathbf{Z}$ of integers. And indeed, if we let

$$\mathbf{e}: R \to \mathbf{Z} \tag{9.1.8}$$

be the map sending the universal element $r \in |\mathrm{SL}(n, R)|$ to the identity matrix in $\mathrm{SL}(n, \mathbf{Z})$, we find that for every commutative ring A, the unique homomorphism $\mathbf{Z} \to A$, when composed with (9.1.8), gives the morphism $R \to A$ that specifies the identity matrix of A.

The structure $(R, \mathbf{m}, \mathbf{i}, \mathbf{e})$ that we have sketched above is what we shall see in subsequent sections is called a *cogroup* in the category $\mathbf{CommRing}^1$. The maps (9.1.5), (9.1.7), (9.1.8) are called its *comultiplication*, its *coinverse*, and its *co-neutral-element*, and the cogroup $(R, \mathbf{m}, \mathbf{i}, \mathbf{e})$ is said to *represent* the functor $\mathrm{SL}(n): \mathbf{CommRing}^1 \to \mathbf{Group}$, just as R alone is said to represent the functor $U_{\mathbf{Group}} \circ \mathrm{SL}(n): \mathbf{CommRing}^1 \to \mathbf{Set}$.

In the next two sections, we shall develop general definitions and results, of which the case sketched above is an example. We shall see that given a functor $V: \mathbf{W} \to \mathbf{V}$, if the first of the two properties we called on above holds, namely that the set-valued functor $U_{\mathbf{V}} V$ is representable, then the other condition, that the operations of the algebras $V(A)$ arise from a co-

V-structure on the representing object, follows automatically. (Indeed, our development of (9.1.5) above did not use our knowledge that the group operations of $\mathrm{SL}(n)$ had this form, but deduced that fact from their functoriality.) This does not mean that we will ignore the co-**V** structure, however! Rather, since it encodes the **V**-algebra structure of our otherwise merely set-valued functors, it will be the key to the investigation of these constructions.

9.2. Algebra objects in a category. We shall approach the concept of a coalgebra object in a category $\mathbf{C}$ by starting with the dual concept, that of an algebra object. Let us make:

Convention 9.2.1. *Throughout this section,* γ *will be a regular cardinal,* $\mathbf{C}$ *will be a category admitting products indexed by all families of cardinality* $<\gamma$ (*which we will abbreviate to ''*$<\gamma$*-fold products''*), *and* Ω *will be a type all of whose operations have arities* $<\gamma$.

(If you are most comfortable with finitary algebras, you may assume $\gamma = \omega$ without missing any of the ideas of this chapter.)

Definition 9.2.2. *For* $\beta<\gamma$, *a* β-ary operation *on an object* R *of* $\mathbf{C}$ *will mean a morphism* $s = s_R : R^\beta \to R$.

By Yoneda's Lemma, such operations correspond bijectively to morphisms of the induced contravariant hom-functors, $h^{R^\beta} \to h^R$; and by the universal property of the product object R^β, we can identify h^{R^β} with $(h^R)^\beta$, so such a map corresponds to a morphism $(h^R)^\beta \to h^R$, i.e., a β-ary operation on h^R. In concrete terms, if s_R is a β-ary operation of R, then given an object A of $\mathbf{C}$ and a β-tuple of elements $(\xi_\alpha)_{\alpha<\beta} \in \mathbf{C}(A,\,R)^\beta$, we first combine these into a single element of $\mathbf{C}(A,\,R^\beta)$, then compose this with $s_R : R^\beta \to R$ to get an element of $\mathbf{C}(A,\,R)$, which we may denote $s_{\mathbf{C}(A,R)}((\xi_\alpha)_{\alpha\in\beta})$. This is the category-theoretic abstraction of the familiar technique of making the set of functions from a space A to an algebra R an algebra under *pointwise* application of the operations of R. These observations are summarized in the next lemma (in which the equivalence of the last two descriptions holds by the definition of morphism of functors).

Lemma 9.2.3. *Let* β *be a cardinal* $<\gamma$, *and* R *an object of* $\mathbf{C}$. *Then the following data are equivalent* (*via the construction just described*)*:*

(i) *A* β*-ary operation* $s_R : R^\beta \to R$.

(ii) *A morphism* $s_{\mathbf{C}(-,\ R)}$: $\mathbf{C}(-,\ R)^\beta \to \mathbf{C}(-,\ R)$ *as functors* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, *i.e., as contravariant set-valued functors on* $\mathbf{C}$.

(iii) *A way of defining on each set* $\mathbf{C}(A,\ R)$ $(A\in\mathrm{Ob}(\mathbf{C}))$ *a* β*-ary operation* $s_{\mathbf{C}(A,R)}$: $\mathbf{C}(A,\ R)^\beta \to \mathbf{C}(A,\ R)$, *so that for every morphism* $f\in\mathbf{C}(A,\ B)$, *the induced map* $\mathbf{C}(B,\ R)\to \mathbf{C}(A,\ R)$ *respects these operations.* □

Recalling that Ω denotes a type all of whose operation-symbols have arities $<\gamma$, we now make

Definition 9.2.4. *An* Ω-algebra object R *in the category* $\mathbf{C}$ (*or a* $\mathbf{C}$-based Ω-algebra) *will mean a pair* $(|R|,\,(s_R)_{s\in|\Omega|})$, *where* $|R|\in\mathrm{Ob}(\mathbf{C})$, *and each* s_R *is an operation*

$$s_R\colon\ |R|^{\mathrm{ari}(s)} \to |R| \qquad (s\in|\Omega|).$$

A morphism between Ω*-algebra objects of* **C** *will mean a morphism between their underlying* **C***-objects which forms commuting squares with these operations.*

If R *is an* Ω*-algebra object of* **C**, *and* A *any object of* **C**, *then* $\mathbf{C}(A, R)$ *denotes the ordinary* (*i.e., set-based*) Ω*-algebra with underlying set* $\mathbf{C}(A, |R|)$, *and operations induced by those of* R *as in Lemma 9.2.3.*

Below, the word ''algebra'' will continue to mean ''set-based algebra'' except when the contrary is indicated by writing ''algebra object'', ''**C**-based algebra'', etc.. Occasionally, when referring to set-based algebras, I may add the words ''set-based'' for emphasis.

Observe that the | |-notation introduced above is relative. E.g., if **C** is itself a category of algebras, and R a **C**-based algebra, then $|R|$ denotes the underlying **C**-object of R, and if S is this **C**-object, then $|S| = \|R\|$ denotes its underlying *set*. I shall, in fact, sometimes use the letter R and its alphabetical neighbors for algebra-objects in categories **C**, and other times for the underlying **C**-objects of such objects. Of course, in any given context I shall be consistent about which meaning I am giving a symbol.

Finally, the reader should note the new use of the symbol $\mathbf{C}(A, R)$ introduced in the above definition: Though A denotes an object of **C**, R does not; rather, it is a **C**-*based* Ω*-algebra*, and the whole symbol denotes, not a set, but a (set-based) Ω-algebra. Of course, a **C**-based Ω-algebra is intuitively ''an object of **C** with additional structure'', and an Ω-algebra is likewise a set with additional structure; and modulo this additional structure, we have the old meaning of $\mathbf{C}(A, R)$. So this extended notation is ''reasonable''. But we need to remember when we are considering algebra objects of categories that in order to interpret a symbol $\mathbf{C}(A, R)$, we have to check whether R is an object of **C**, or a **C**-based algebra-object of some sort.

The above definition also introduced the concept of a *morphism* of **C**-based Ω-algebras. Combining this with Yoneda's Lemma, we easily get

Lemma 9.2.5. *Let* R *and* S *be* Ω*-algebra objects in* **C**. *Then the following data are equivalent*:

(i) *A morphism of* **C***-based algebras* $R \to S$.

(ii) *A morphism* $f \in \mathbf{C}(|R|, |S|)$ *such that for every object* A *of* **C**, *the induced set map* $\mathbf{C}(A, |R|) \to \mathbf{C}(A, |S|)$ *is a homomorphism of* Ω*-algebras* $\mathbf{C}(A, R) \to \mathbf{C}(A, S)$.

(iii) *A morphism* $\mathbf{C}(-, R) \to \mathbf{C}(-, S)$ *of functors* $\mathbf{C} \to \Omega$-**Alg**. □

We next want to define, for an Ω-algebra object R of a category **C**, the *derived operations* of R corresponding to the various derived operations of set-based Ω-algebras. This will allow us to say what it means for such an object to satisfy a given *identity*; namely, that the derived operations specified by the two sides of the identity are equal.

One cannot, of course, describe a derived operation of R by giving a formula for its value on a tuple of ''elements of $|R|$'', since **C** is a general category. An approach that is often used is to express operations and identities by diagrams. For example, observe that if m is a binary operation on a *set* $|R|$, the condition that m be associative can be expressed as the condition that the diagram

(9.2.6)
$$\begin{tikzcd} {|R| \times |R| \times |R|} \arrow[r, "m \times \mathrm{id}_{|R|}"] \arrow[d, "\mathrm{id}_{|R|} \times m"] & {|R| \times |R|} \arrow[d, "m"] \\ {|R| \times |R|} \arrow[r, "m"] & {|R|} \end{tikzcd}$$

commute, since the path that goes through the upper right-hand corner gives the ternary derived operation $(x,y,z) \mapsto m(m(x,y),z)$, and the one through the lower left-hand corner gives $(x,y,z) \mapsto m(x,\ m(y,z))$. Analogously, for any object $|R|$ of our category $\mathbf{C}$ and any binary operation $m\colon |R| \times |R| \to |R|$, the same diagram can be used to define two ternary ''derived operations'' on $|R|$, and their equality (the commutativity of the diagram) can be made the definition of associativity of the $\mathbf{C}$-based algebra $R = (|R|,\ m)$.

The above approach is nice in simple cases, but has the disadvantage of requiring us to figure out the diagram appropriate to every derived operation we want to consider. Another approach, which is equivalent to the above but avoids this dependence on diagrams, is based on considering the algebra $\mathbf{C}(A,\ R)$ for an appropriate *universal* choice of A. If we want to consider derived operations in β variables, let us look at $\mathbf{C}(|R|^{\beta},\ R)$. Since this is a set-based algebra, we know how to construct its derived β-ary operations from its primitive operations. Applying such a derived operation u to the β projections $p_{\alpha}\colon |R|^{\beta} \to |R|$ $(\alpha \in \beta)$, we get an element $u(p_{\alpha}) \in \mathbf{C}(|R|^{\beta},\ |R|)$ which we *define* to be the derived operation u_R of the $\mathbf{C}$-based algebra R. Identities are then defined as equalities among such derived operations.

Incidentally, although in §8.4 we found it convenient to reduce all identities for Ω-algebras to identities (pairs of terms) in a γ-tuple of variables, we shall here revert to expressing them as identities in β-tuples of variables for various ordinals $\beta < \gamma$. (So, for instance, the diagram (9.2.6) expresses associativity using three variables, rather than countably many.) The advantage will be that we only need to assume that $\mathbf{C}$ has these β-fold products, rather than making the unnecessary stronger assumption that it has γ-fold products.

As we observed in §8.9, the operations on $|R|$ (equivalently, on $h^{|R|}$) form a clone; and we see that a $\mathbf{C}$-based Ω-algebra structure on $|R|$ as defined above is equivalent to a representation of the γ-clonal category $\mathbf{X}_{\mathbf{\Omega\text{-}Alg}}$ in $\mathbf{C}$ which takes $X_1 \in \mathrm{Ob}(\mathbf{X}_{\mathbf{\Omega\text{-}Alg}})$ to $|R| \in \mathrm{Ob}(\mathbf{C})$. The condition that this $\mathbf{C}$-based algebra R satisfy the identities of a given variety $\mathbf{V}$ is equivalent to saying that this representation of $\mathbf{X}_{\mathbf{\Omega\text{-}Alg}}$ arises from (i.e., factors through) a representation of $\mathbf{X}_{\mathbf{V}}$:

$$\mathbf{X}_{\mathbf{\Omega\text{-}Alg}} \to \mathbf{X}_{\mathbf{V}} \to \mathbf{C}.$$

In the next lemma and definition we set down the observations of the preceding paragraphs, and prove the one nontrivial implication.

Lemma 9.2.7. *Let $R = (|R|,\ (s_R)_{s \in |\Omega|})$ be an Ω-algebra object of $\mathbf{C}$, and let $u,\ v$ be two derived β-ary operations $(\beta < \gamma)$ for ordinary (i.e., set-based) algebras of type Ω. Then the following conditions are equivalent:*

(i) *For all $A \in \mathrm{Ob}(\mathbf{C})$, the algebra $\mathbf{C}(A,\ R)$ satisfies the identity $u = v$.*

(ii) *In the algebra $\mathbf{C}(|R|^{\beta},\ R)$, one has $u((p_{\alpha})_{\alpha \in \beta}) = v((p_{\alpha})_{\alpha \in \beta})$, where the p_{α} $(\alpha \in \beta)$ are the projection maps.*

(iii) *The morphisms $u,\ v\colon X_{\beta} \rightrightarrows X_1$ in the γ-clonal category $\mathbf{X}_{\mathbf{\Omega\text{-}Alg}}$ fall together under the morphism from $\mathbf{X}_{\mathbf{\Omega\text{-}Alg}}$ to the clone of operations on $|R|$ induced by the $|\Omega|$-tuple of operations*

(s_R). (*See Lemma 8.9.10 for the universal property of* $\mathbf{X}_{\Omega\text{-}\mathbf{Alg}}$ *which allows one to define this morphism.*)

(iv) *The algebra object* R *satisfies the ''diagrammatic translation'' of the identity* $u = v$.

Proof. (ii)-(iv) are simply different ways of stating the same condition. Clearly, (i)⇒(ii). The converse can be gotten by Yoneda's Lemma; to see it directly, consider any object A of $\mathbf{C}$ and any β-tuple $(\xi_\alpha)_{\alpha\in\beta}$ of elements of $\mathbf{C}(A, |R|)$. By the universal property of the product object $|R|^\beta$, these morphisms correspond to a single morphism $\xi\colon A \to |R|^\beta$, and applying $\mathbf{C}(-, R)$ we get an Ω-algebra homomorphism $\mathbf{C}(|R|^\beta, R) \to \mathbf{C}(A, R)$ carrying each p_α to ξ_α. Hence, any equation satisfied by the former β-tuple is also satisfied by the latter. □

Definition 9.2.8. *If the equivalent conditions of Lemma 9.2.7 hold, the* Ω*-algebra object* R *of* $\mathbf{C}$ *will be said to* satisfy *the identity* $u = v$.

If $\mathbf{V}$ *is a variety of* Ω*-algebras, defined by a family* J *of identities, then a* $\mathbf{V}$-object *of* $\mathbf{C}$ *will mean an* Ω*-algebra object* R *of* $\mathbf{C}$ *satisfying the identities in* J *in this sense; equivalently, such that the induced functor* $\mathbf{C}(-, R)$ *carries* $\mathbf{C}$ *into* $\mathbf{V}$*; equivalently, such that the corresponding representation of* $\mathbf{X}_\Omega\text{-}\mathbf{Alg}$ *in* $\mathbf{C}$ *arises from a representation of* $\mathbf{X}_\mathbf{V}$ *in* $\mathbf{C}$.

(Since the same variety $\mathbf{V}$ can be determined by more than one set of identities J, we need to be sure that the condition of being a $\mathbf{V}$-object of $\mathbf{C}$ is well-defined. The second equivalent formulation of the above definition makes this clear.)

Our point of view so far has been, ''Given a representable functor $\mathbf{C}(-, |R|)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ $(|R|\in\mathrm{Ob}(\mathbf{C}))$, how can we put operations on $\mathbf{C}(-, |R|)$, and when will such operations satisfy the identities of a variety $\mathbf{V}$?'' But note that the concept of ''a representable set-valued functor given with operations that make it $\mathbf{V}$-valued'' can also be looked at as ''a $\mathbf{V}$-valued functor whose composite with the forgetful functor $\mathbf{V} \to \mathbf{Set}$ is representable''. This observation shows the equivalence of the two formulations of the next definition, in which we extend the term ''representable functor'' to include algebra-valued constructions.

Definition 9.2.9. *If* $\mathbf{V}$ *is a variety of* Ω*-algebras, a functor* $\mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ *will be called* representable *if it is isomorphic to a functor of the form* $\mathbf{C}(-, R)$*, for* R *a* $\mathbf{V}$*-object of* $\mathbf{C}$*, equivalently, if its composite with the underlying-set functor* $\mathbf{V} \to \mathbf{Set}$ *is representable in the sense of Definition 7.2.3.*

9.3. Coalgebras, and Freyd's criterion for the existence of left adjoints. In the next few sections we shall study *coalgebra* objects, and the functors these represent. A $\mathbf{V}$-coalgebra object in a category $\mathbf{C}$ will be defined simply as a $\mathbf{V}$-algebra object in $\mathbf{C}^{\mathrm{op}}$. But pedagogically, the relationship between these two concepts is tricky. The definition of algebra object is easier to think about (to begin with) because it generalizes the familiar concept of a set-based algebra. But in varieties of algebras, *coalgebra* objects and the covariant functors they represent will turn out to be more diverse and interesting than algebra objects and their associated contravariant representable functors, and, as suggested by our example of $\mathrm{SL}(n)$, they will be the main object of study in this chapter. Hence our flip-flop approach of using the algebra concept to introduce the definitions, but then moving immediately to coalgebras. However, in §§9.11-9.12 we will return briefly to algebra objects, and note some examples and results on these.

In this section we continue to assume that γ is a regular cardinal, and Ω a type all of whose operations have arity $<\gamma$; however, we drop here the assumption of the preceding section that $\mathbf{C}$

has $<\gamma$-fold products; what we will want is the dual hypothesis, and we will state that explicitly when it is needed.

Definition 9.3.1. *Let* $\mathbf{C}$ *be a category having* coproducts *of all families of* $<\gamma$ *objects. Then for* $\beta<\gamma$, *a* β*-ary* co-operation *on an object* $|R|$ *of* $\mathbf{C}$ *will mean a morphism of* $|R|$ *into the coproduct of* β *copies of* $|R|$; *in other words, a* β*-ary operation on* $|R|$ *in* $\mathbf{C}^{\mathrm{op}}$. *A pair* $R = (|R|, (\mathbf{s}_R)_{s\in|\Omega|})$ *such that* $|R|\in\mathrm{Ob}(\mathbf{C})$, *and for each* $s\in|\Omega|$, $\mathbf{s}_R$ *is an* $\mathrm{ari}(s)$*-ary co-operation on* $|R|$, *will be called an* Ω-coalgebra *object in* $\mathbf{C}$. *A morphism of* Ω*-coalgebra objects of* $\mathbf{C}$ *will mean a morphism of underlying* $\mathbf{C}$*-objects which respects co-operations.*

For any Ω*-coalgebra object* R *and object* A *of* $\mathbf{C}$, *we shall write* $\mathbf{C}(R, A)$ *for the set-based algebra whose underlying set is* $\mathbf{C}(|R|, A)$, *and whose operations are induced by the co-operations of* R *under the dual of the construction of the preceding section. Explicitly, for* $s\in|\Omega|$, *the operation* $s_{\mathbf{C}(R,\,A)}$ *induced by* $\mathbf{s}_R$ on $\mathbf{C}(|R|, A)$ *is defined to take each* $\mathrm{ari}(s)$*-tuple* $(\xi_\alpha)\in\mathbf{C}(|R|, A)^{\mathrm{ari}(s)}$ *to the composite morphism*

$$|R| \xrightarrow{\mathbf{s}_R} \coprod_{\mathrm{ari}(s)} |R| \xrightarrow{(\xi_\alpha)_{\alpha\in\mathrm{ari}(s)}} A,$$

where the second arrow denotes the map whose composite with the α*th coprojection* $|R| \to \coprod_{\mathrm{ari}(s)} |R|$ *is* ξ_α *for each* $\alpha\in\mathrm{ari}(s)$.

I will in general, as above, use lower-case boldface letters $\mathbf{s}$ etc. to denote co-operations corresponding to operations denoted by the corresponding lower-case italic letters, s etc..

Note that (as in the parallel definition in the preceding section), the R in the above definition of $\mathbf{C}(R, A)$ is not an object of $\mathbf{C}$; here it is a $\mathbf{C}$-based *coalgebra* with underlying $\mathbf{C}$-object $|R|$, and $\mathbf{C}(R, A)$ is likewise not a set, but an algebra with underlying set $\mathbf{C}(|R|, A)$.

Let us recall from Lemma 8.4.13 what the general covariant representable set-valued functor $\mathbf{C}(|R|,-)$ ''looks like'' when its domain category $\mathbf{C}$ is a *variety of algebras* $\mathbf{W}$. Taking a presentation $|R| = \langle X \mid Y\rangle_{\mathbf{W}}$ for the representing object, the functor can be described as carrying each object A to the set of all X-tuples of elements of A that satisfy the family of relations Y. We now want to describe the form that a β-ary operation s on such a functor takes.

We know that s will be induced by a co-operation $\mathbf{s}_R\colon |R| \to \coprod_\beta |R|$ of the representing object $|R| = \langle X \mid Y\rangle_{\mathbf{W}}$. The homomorphism $\mathbf{s}_R$ will correspond to some X-tuple of elements of $\coprod_\beta |R|$ which satisfies the relations Y. For each $x\in X$, the xth entry of this X-tuple, being an element of $\coprod_\beta |R|$, may be expressed in terms of the β images of X generating that algebra, using some derived operation, which we may name

$$s_x\in|F_{\mathbf{W}}(\beta\times X)|.$$

Now using the universality of $\coprod_\beta |R|$ as a $\mathbf{W}$-algebra with a β-tuple of elements of $\mathbf{C}(|R|, -)$, we can deduce that if A is an arbitrary $\mathbf{W}$-algebra, and we regard elements of $\mathbf{W}(|R|, A)$ as X-tuples ξ of elements of A which satisfy the relations Y, then for each β-tuple $(\xi_\alpha)_{\alpha\in\beta}$ of such X-tuples, the xth coordinate of the element $s_{\mathbf{W}(R,\;A)}(\xi_\alpha)_{\alpha\in\beta} \in \mathbf{W}(|R|, A)$ will be expressed in terms of the coordinates of the β X-tuples ξ_α by the same derived operation s_x. In summary:

Lemma 9.3.2. *Let* $\mathbf{W}$ *be a variety of algebras,* $|R|$ *an object of* $\mathbf{W}$, *and* $\langle X \mid Y\rangle_{\mathbf{W}}$ *a presentation of* $|R|$ *by generators and relations. For any* $\mathbf{W}$*-algebra* A, *element* $\xi\in\mathbf{W}(|R|, A)$, *and* $x\in X$, *let us call the image in* A *of the generator* x *of* $|R|$ *under* ξ *''the* x*th coordinate* *of* ξ''.

Then if $\mathbf{s}: |R| \to \amalg_\beta\, |R|$ *is a* β*-ary co-operation on* $|R|$, *there exists an* X*-tuple of derived* $\beta\times X$*-ary operations* $(s_x)_{x\in X}$ *of* $\mathbf{W}$, *such that for every object* A *of* $\mathbf{W}$, *if we write* s *for the operation on the set* $\mathbf{W}(|R|, A)$ *induced by the co-operation* $\mathbf{s}$ *on* $|R|$, *then* s *can be described as follows: For every* β*-tuple* $(\xi_\alpha)_{\alpha\in\beta}$ *of elements of* $\mathbf{W}(|R|, A)$ *and each* $x\in X$, *the* x*th coordinate of* $s(\xi_\alpha)$ *is computed from the coordinates of the given elements* ξ_α *by the derived operation* s_x.

Conversely, given an X*-tuple of derived* $\beta\times X$*-ary operations* s_x *of* $\mathbf{W}$ $(x\in X)$, *if the identities of* $\mathbf{W}$ *imply that, when applied to any* β X*-tuples all of which satisfy the relations* Y, *the* s_x *give an* X*-tuple of elements which also satisfies* Y, *then* $(s_x)_{x\in X}$ *determines a morphism of functors* $s: \mathbf{W}(|R|, -)^\beta \to \mathbf{W}(|R|, -)$, *equivalently, a* β*-ary co-operation* $\mathbf{s}: |R| \to \amalg_\beta\, |R|$. □

So, for instance, if $\mathbf{W}$ is the variety of commutative rings, and $|R|$ the commutative ring with a universal $n\times n$ matrix of determinant 1, we can take for X a family of n^2 symbols $(x_{ij})_{i,j\leq n}$, and for Y the set consisting of the single relation $\det(x_{ij}) = 1$. To describe from the above point of view the comultiplication $\mathbf{m}$ on $|R|$ sketched in §9.1, take $\beta = 2$ and for each $i,j\leq n$ let m_{ij} be the polynomial in $2n^2$ indeterminates by which one computes the (i,j)th entry of the product of two matrices. The multiplicativity of the determinant function implies that these operations, when applied to the entries of two matrices of determinant 1, give the entries of a third matrix of determinant 1, so the condition of the last sentence of the above lemma is satisfied. Thus, these n^2 derived operations yield a binary co-operation on $|R|$, which induces, in a manner described abstractly in Definition 9.3.1 and concretely in Lemma 9.3.2, a binary operation on the sets $\mathbf{CommRing}^1(|R|, A) = |\mathrm{SL}(n, A)|$, namely, multiplication of matrices of determinant 1.

Back, now, to dualizing the concepts and results of the preceding section for a general category $\mathbf{C}$ (not necessarily a variety of algebras). Dualizing Definitions 9.2.8 and 9.2.9 respectively, we get

Definition 9.3.3. *Let* $\mathbf{C}$ *be a category with* $<\gamma$*-fold coproducts, and* $\mathbf{V}$ *a variety of* Ω*-algebras defined by a set* J *of identities. Then a* co-$\mathbf{V}$ object (*or* $\mathbf{V}$-coalgebra) *of* $\mathbf{C}$ *will mean an* Ω*-coalgebra* R *satisfying the following equivalent conditions:*

(i) *For all objects* A *of* $\mathbf{C}$, *the algebra* $\mathbf{C}(R, A)$ (*Definition 9.3.1*) *lies in* $\mathbf{V}$.

(ii) *For each identity* $(u, v)\in J$, *say in* β *variables, if we form the* β*-fold coproduct* $\amalg_\beta\, |R|$ *with its canonical coprojections* q_α $(\alpha\in\beta)$, *then in the algebra* $\mathbf{C}(R, \amalg_\beta\, |R|)$, *one has* $u(q_\alpha) = v(q_\alpha)$. (*This equality of morphisms* $|R| \to \amalg_\beta\, |R|$ *may be called the ''coidentity'' corresponding to the identity* $u = v$.)

(iii) *Writing* $\mathbf{X}_{|R|^{\mathrm{op}}}$ *for the clone of all* co-operations *on* $|R|$ (*i.e., operations on* $|R|$ *in* $\mathbf{C}^{\mathrm{op}}$), *the morphism of clones* $\mathbf{X}_{\Omega\textbf{-Alg}} \to \mathbf{X}_{|R|^{\mathrm{op}}}$ *induced by the* Ω*-coalgebra structure of* $|R|$ *factors through the canonical map* $\mathbf{X}_{\Omega\textbf{-Alg}} \to \mathbf{X}_{\mathbf{V}}$,

$$\mathbf{X}_{\Omega\textbf{-Alg}} \to \mathbf{X}_{\mathbf{V}} \to \mathbf{X}_{|R|^{\mathrm{op}}}.$$

(iv) R *satisfies the dualized diagrammatic condition corresponding to each identity in* J.

(v) *Interpreted as an algebra object of* $\mathbf{C}^{\mathrm{op}}$, R *is a* $\mathbf{V}$*-object.*

Definition 9.3.4. *Let* $\mathbf{C}$ *be a category with* $<\gamma$*-fold coproducts, and* $\mathbf{V}$ *a variety of* Ω*-algebras. Then a covariant functor* $\mathbf{C}\to\mathbf{V}$ *will be called* representable *if it is isomorphic to a functor of the form* $\mathbf{C}(R,-)$, *for* R *a co-*$\mathbf{V}$ *object of* $\mathbf{C}$*; equivalently, if its composite with the forgetful functor* $\mathbf{V}\to\mathbf{Set}$ *is representable in the sense of Definition 7.2.3; equivalently, in the case where* $\mathbf{C}$ *is a variety* $\mathbf{V}$ *of algebras, if there is some set* Y *of relations in a family* X *of variables such that this composite is the functor associating to every object* A *of* $\mathbf{C}$ *the set of all* X*-tuples of elements of* A *satisfying* Y.

The full subcategory of $\mathbf{V}^{\mathbf{C}}$ *consisting of the representable covariant functors will be denoted* $\mathbf{Rep}(\mathbf{C},\mathbf{V})$.

For example, $\mathrm{SL}(n)$ is an object of $\mathbf{Rep}(\mathbf{CommRing}^1, \mathbf{Group})$. The next few sections will study further classes of representable functors among varieties of algebras. For students with some knowledge of topology, I insert here a nonalgebraic example.

Exercise 9.3:1. Let $\mathbf{HtpTop}^{(\mathrm{pt})}$ be the category whose objects are topological spaces with basepoint, and whose morphisms are homotopy classes of basepoint-preserving maps.

(i) Show that $\mathbf{HtpTop}^{(\mathrm{pt})}$ has finite coproducts.

(ii) We noted at the end of §6.5 that the functor $\mathbf{HtpTop}^{(\mathrm{pt})}\to\mathbf{Set}$ taking an object (X, x_0) to $|\pi_1(X, x_0)|$ (the underlying set of its fundamental group) was representable, with representing object $(S^1, 0)$. By the above results, the structure of group on these sets must be induced by a cogroup structure on $(S^1, 0)$. Describe the co-operations, and verify the cogroup identities.

(iii) Describe likewise the structure of *group object* on $(S^1, 0)$ which represents the contravariant *first cohomotopy group* functor π^1.

Note that $\mathbf{W}$-*algebra* objects of a category $\mathbf{C}$ represent *contra*variant functors $\mathbf{C}^{\mathrm{op}}\to\mathbf{W}$, while *co*variant functors $\mathbf{C}\to\mathbf{W}$ are represented by *coalgebra* objects. This is a consequence of the behavior of the covariant and contravariant Yoneda embeddings, discussed in Remark 7.2.7. For the same reason, *morphisms* among covariant representable functors correspond *contravariantly* to morphisms among their representing coalgebras:

Corollary 9.3.5 (to Lemma 9.2.5). *If* $\mathbf{C}$ *is a category with* $<\gamma$*-fold coproducts, and* $\mathbf{V}$ *a variety of* Ω*-algebras, then the category* $\mathbf{Rep}(\mathbf{C},\mathbf{V})$ *of covariant representable functors* $\mathbf{C}\to\mathbf{V}$ *is equivalent to the* opposite *of the category of co-*$\mathbf{V}$ *objects of* $\mathbf{C}$. □

We shall now establish the relationship between representability and the existence of adjoints! Our proof that a representable functor V has a left adjoint will be, in essence, the same as our construction at the beginning of §9.1 of a left adjoint for $\mathrm{SL}(n)$; but the generators-and-relations presentation of the rings A_G used there will be replaced by a colimit construction, allowing us to get a left adjoint to our functor V without requiring the domain of V to be a variety of algebras. We have already stated in Definition 9.3.4 the equivalence of the conditions labeled (ii) and (iii) below. (We proved it in the dual context in the preceding section.) We include both formulations here for completeness.

Theorem 9.3.6 (after Freyd [**8**]). *Let* $\mathbf{C}$ *be a category with small colimits,* $\mathbf{V}$ *a variety of* Ω*-algebras, and*

$$V:\ \mathbf{C}\ \to\ \mathbf{V}$$

a (*covariant*) *functor. Then the following conditions are equivalent:*

(i) *V has a left adjoint G: $\mathbf{V} \to \mathbf{C}$.*

(ii) *V is representable, i.e., is isomorphic to the* $\mathbf{V}$*-valued functor represented by a co-*$\mathbf{V}$ *object R of* $\mathbf{C}$ *(Definition 9.3.3).*

(iii) *The composite $U_{\mathbf{V}} V$ of V with the underlying set functor $U_{\mathbf{V}}$:* $\mathbf{V} \to \mathbf{Set}$ *is representable, i.e., is isomorphic to the set-valued functor represented by an object of* $\mathbf{C}$.

Proof. As noted, we already know that (ii) ⇔ (iii). (The forward implication follows from the definition of (ii); the backward implication holds because all operations on a representable set-valued functor $\mathbf{C}(|R|, -)$ are induced by co-operations on $|R|$, i.e., a coalgebra structure.) It is also easy to show (i) ⇒ (iii); namely, assuming V has left adjoint G, we have

$$U_{\mathbf{V}}\, V(-) \;\cong\; \mathbf{Set}(1,\, U_{\mathbf{V}}\, V(-)) \;\cong\; \mathbf{V}(F_{\mathbf{V}}(1),\, V(-)) \;\cong\; \mathbf{C}(G F_{\mathbf{V}}(1),\, -),$$

so $U_{\mathbf{V}}\, V$ is represented by $G F_{\mathbf{V}}(1)$. We shall complete the proof by showing (ii) ⇒ (i).

To get (i), we need to show that for each $A \in \mathrm{Ob}(\mathbf{V})$, there exists $G(A) \in \mathrm{Ob}(\mathbf{C})$ such that $\mathbf{C}(G(A), -) \cong \mathbf{V}(A, V(-))$ (Theorem 7.3.3(ii)). Now though $\mathbf{C}$ need not be a variety, $\mathbf{V}$ is; so let us take a presentation of A in $\mathbf{V}$:

$$A \;=\; \langle Z \mid S \rangle_{\mathbf{V}}. \qquad (9.3.7)$$

(We use symbols Z and S here rather than X and Y so that if one looks at this proof in the case where $\mathbf{C}$ is a variety $\mathbf{W}$ of algebras, there will be no confusion between this presentation for A in $\mathbf{V}$, and the presentation in $\mathbf{W}$ for the representing object $|R|$, which was written $\langle X \mid Y \rangle_{\mathbf{W}}$ in Lemma 9.3.2.) Thus, $\mathbf{V}(A, V(-))$ can be described as associating to each $B \in \mathrm{Ob}(\mathbf{C})$ the set of all Z-tuples of elements of the $\mathbf{V}$-algebra $V(B)$ that satisfy the relations given by S.

Let us form a coproduct $\coprod_{z\in Z} |R|^{(z)} \in \mathrm{Ob}(\mathbf{C})$ of a Z-tuple of copies, $|R|^{(z)}$ $(z \in Z)$, of the underlying $\mathbf{C}$-object $|R|$ of our representing coalgebra. Then for any object B of $\mathbf{C}$, the set $\mathbf{C}(\coprod_Z |R|^{(z)}, B)$ can be naturally identified with $\mathbf{C}(|R|, B)^Z \cong |V(B)|^Z$, the set of *all* Z-tuples of elements of $V(B)$. To obtain the subset of Z-tuples satisfying the family of relations S, we want to formally ''impose'' these relations on $\coprod_Z |R|^{(z)}$. Hence, for each relation $(s, t) \in S$ let us form the two morphisms $|R| \rightrightarrows \coprod_Z |R|^{(z)}$ corresponding to s and t, namely $s(q_z)$ and $t(q_z)$, where $(q_z)_{z\in Z}$ is the Z-tuple of coprojection morphisms $|R| \to \coprod_Z |R|^{(z)}$, and let $G(A)$ be the colimit of the diagram formed out of all these pairs of arrows:

$$\begin{tikzcd}
\vdots & & \\
{|R|} \arrow[dr, shift left] \arrow[dr, shift right] & & \\
{|R|} \arrow[r, shift left] \arrow[r, shift right] & \coprod_Z |R|^{(z)} \arrow[r] & G(A). \\
{|R|} \arrow[ur, shift left] \arrow[ur, shift right] & & \\
\vdots & &
\end{tikzcd}$$

It is easy to verify that this object has the desired universal property $\mathbf{C}(G(A), -) \cong \mathbf{V}(A, V(-))$. □

(Note that the above theorem required that $\mathbf{C}$ have arbitrary small colimits, so that we could construct $G(A)$ for arbitrary small $\mathbf{V}$-algebras A. This subsumes the condition of having $<\gamma$-fold coproducts assumed earlier. Incidentally, if $\mathbf{V} = \mathbf{Set}$, then every object of $\mathbf{V}$ has a presentation with no relations, and the above proof becomes much simpler, and only needs $\mathbf{C}$ to have coproducts, not general colimits. This was Exercise 7.3:3.)

Exercise 9.3:2. Establish the universal property of $G(A)$ asserted in the last line of the above proof.

Exercise 9.3:3. Describe the construction used in proving (ii) ⇒ (i) above in the particular case $\mathbf{C} = \mathbf{CommRing}^1$, $\mathbf{V} = \mathbf{Group}$, $V = \mathrm{SL}(n)$, $A = \mathbf{Z}_2$. (You are not asked to find a normal form for the ring obtained; simply show the generators-and-relations description that the construction gives in this case.) Show directly from your description that the result is a ring with a universal determinant-1 $n\times n$ matrix of exponent 2.

An alternative way to complete the proof of the above theorem, by showing (iii)⇒(i) rather than (ii)⇒(i), is indicated in

Exercise 9.3:4. Assuming condition (iii) of the above theorem, let $\mathbf{A}$ denote the full subcategory of $\mathbf{V}$ consisting of those objects A such that the functor $\mathbf{V}(A,\ V(-))\colon \mathbf{C} \to \mathbf{Set}$ is representable, and let $G_{\mathbf{A}}: \mathbf{A} \to \mathbf{C}$ be the resulting ‘‘partial adjoint’’ to V. Show that $F_{\mathbf{V}}(1)$ belongs to $\mathbf{A}$, that $\mathbf{A}$ is closed under small colimits, and that every object of $\mathbf{V}$ can be obtained from the free object on one generator by iterated small colimits. Deduce that $\mathbf{A} = \mathbf{V}$.

9.4. Some corollaries and examples.

Since composites of adjunctions are adjunctions (Theorem 7.3.5), the above result yields

Corollary 9.4.1. *A composite of representable functors among varieties of algebras is representable.* □

Actually, this reasoning shows that a composite of representable functors $\mathbf{C} \to \mathbf{V} \to \mathbf{W}$, where $\mathbf{V}$ and $\mathbf{W}$ are varieties and $\mathbf{C}$ any category with small colimits, is representable, but I have given the above more limited statement because of its simplicity.

What does the representing object for a composite of representable functors among varieties look like? Suppose we have

$$\begin{tikzcd}
\text{representing objects:} & & R & & S & \\
\text{right adjoints:} & \mathbf{U} \arrow[rr, shift left, "V"] & & \mathbf{V} \arrow[ll, shift left, "D"] \arrow[rr, shift left, "W"] & & \mathbf{W}, \arrow[ll, shift left, "E"] \\
\text{left adjoints:} & & & & &
\end{tikzcd}$$

so that the composite functor WV has left adjoint DE. To obtain the underlying $\mathbf{U}$-object of the $\mathbf{W}$-coalgebra representing WV, we note that this object will represent the functor $U_{\mathbf{W}}WV$. The factor $U_{\mathbf{W}}W$ is represented by $|S|$, so by Theorem 7.7.1, the object representing $U_{\mathbf{W}}WV$ can be obtained by applying to $|S|$ the left adjoint of V. Thus, the underlying $\mathbf{U}$-object of our desired representing object is $D(|S|)$.

We can combine this observation with the description of D that the proof of Theorem 9.3.6 gives us; namely, that D takes a $\mathbf{V}$-algebra A to a $\mathbf{U}$-algebra obtained by ‘‘pasting together’’ a family of copies of $|R|$ indexed by the generators in any presentation of A, using ‘‘pasting instructions’’ obtained from the relations in that presentation. Hence the representing object $D(|S|)$ for $U_{\mathbf{W}}WV$ can be obtained by ‘‘pasting together’’ a family of copies of $|R|$ in a way prescribed by any presentation of $|S|$. From this one can deduce that if $|R| = \langle X \mid Y \rangle_{\mathbf{U}}$ and $|S| = \langle X' \mid Y' \rangle_{\mathbf{V}}$, then the representing object for $U_{\mathbf{W}}WV$ can be presented in $\mathbf{U}$ by a generating set indexed by $X\times X'$ and a set of relations indexed by $X\times Y' \sqcup X'\times Y$.

Of course, we also want to know the co-$\mathbf{W}$ structure on this object. Not unexpectedly, this arises from the co-$\mathbf{W}$ structure on the object $|S|$. We shall see some examples of representing objects of composite functors in §9.8. I won’t work out the details of the general description of such objects, but if you are interested, you can do this, as

Exercise 9.4:1. Describe precisely how to construct a presentation of the object representing WV and its co-**W** structure, in terms of presentations of $|R|$ and $|S|$, and their co-**V** and co-**W** structures.

Theorem 9.3.6 has the following consequence (noted as Exercise 8.9:8 in the last chapter); though it is unfortunate that the consequence is better known than the theorem, and is thought by many to be the ‘‘last word’’ on the subject!

Corollary 9.4.2. *Any functor* $V\colon \mathbf{W} \to \mathbf{V}$ *between varieties of algebras which respects underlying sets has a left adjoint.*

Proof. By Theorem 9.3.6 (iii)⇒(i), to show V has a left adjoint it suffices to show that $U_{\mathbf{V}}V\colon \mathbf{W} \to \mathbf{Set}$ is representable. But by hypothesis, $U_{\mathbf{V}}V = U_{\mathbf{W}}$, which is clearly representable, by any of our three criteria (representing object: $F_{\mathbf{W}}(1)$; description: sends each object A to the set of 1-tuples of elements of A satisfying the empty set of relations; left adjoint: $F_{\mathbf{W}}$). □

This corollary applies to such constructions as (i) the underlying-set functor $U_{\mathbf{W}}: \mathbf{W} \to \mathbf{Set}$ of any variety $\mathbf{W}$, the left adjoint of which is, we already know, the *free algebra* construction; (ii) the inclusion of any variety $\mathbf{W}$ in a larger variety $\mathbf{V}$ of algebras of the same type, the left adjoint of which is the construction of ‘‘imposing the identities of $\mathbf{W}$’’ on algebras in $\mathbf{V}$; (iii) the functor $\mathbf{Set} \to G\text{-}\mathbf{Set}$ (for any group G) which takes a set A and regards it as a G-set with trivial action; this has for left adjoint the *orbit-set* functor $G\text{-}\mathbf{Set} \to \mathbf{Set}$ (cf. Exercise 7.6:1); (iv) the functor taking an associative ring A to its underlying additive group, whose left adjoint is the *tensor ring* construction, and similarly (v) the functor taking an associative ring A to its underlying multiplicative monoid, whose left adjoint is the *monoid-ring* construction (both these left adjoint constructions were described as constructions of rings with universal properties in §3.12), and (vi) the ‘‘commutator brackets’’ functor from associative algebras over a commutative ring k to Lie algebras over k, whose left adjoint is the *universal enveloping algebra* construction (§8.7).

On the other hand, the functor $\mathrm{SL}(n)\colon \mathbf{CommRing}^1 \to \mathbf{Group}$ with which we began this chapter certainly does not preserve underlying sets. That was a good example for getting away from functors represented by free algebras on one generator, because the representing algebra both requires more than one generator, and requires nontrivial relations, i.e., is nonfree. There are also important examples where a representing algebra is free on more than one generator (equivalently, where the functor has the property that the underlying set $|V(A)|$ of the constructed algebra is a fixed power $|A|^X$ of the underlying set of the given algebra A), or can be generated by one element but subject to some relations (equivalently, where $|V(A)|$ can be described as the subset of $|A|$ consisting of those elements which satisfy certain equations). Among constructions of the first type are the $n\times n$ matrix ring functor $M_n: \mathbf{Ring}^1 \to \mathbf{Ring}^1$, the representing object for which is free on n^2 generators, and the formal power series functor (either $\mathbf{Ring}^1 \to \mathbf{Ring}^1$ or $\mathbf{CommRing}^1 \to \mathbf{CommRing}^1$) taking a ring A to the ring $A[[t]]$, whose representing object is free on countably many generators. The left adjoints of these have no standard names, but can be described as taking a ring B to the ring ‘‘with a universal $n\times n$ matrix representation of B’’, respectively ‘‘with a universal representation of B by formal power series’’. A functor with representing algebra presented by one generator and a nonempty set of relations is the construction $\mathbf{CommRing}^1 \to \mathbf{Bool}^1$ taking a ring A to the set of its idempotent elements, made a Boolean ring as described in Exercise 3.14:3. The underlying ring of its representing coalgebra is presented by a generator x and the relation $x^2 = x$, and can be described as $\mathbf{Z}\times\mathbf{Z}$, with $x = (1, 0)$. Another example with one generator and a nonempty relation-set is the functor $\mathbf{Ab} \to \mathbf{Ab}$ taking any abelian group to its subgroup of elements of exponent n (for any fixed $n>0$), represented by the cyclic group of order n. Still another is the functor $G\text{-}\mathbf{Set} \to \mathbf{Set}$ (for G any nontrivial group) represented by the one-element G-set. This takes a G-set A to the set of fixed points of the action of G; its left adjoint is the functor $\mathbf{Set} \to G\text{-}\mathbf{Set}$ mentioned in point (iii) of the preceding paragraph, which thus has both a left and a right adjoint!

We saw in Chapter 3 that every monoid had both a universal map into a group, and a universal map of a group into it. This says that the forgetful functor

$$U\colon\ \mathbf{Group} \to \mathbf{Monoid}$$

also has both a left and a right adjoint. That it has a left adjoint is now clear from that fact that it preserves underlying sets. Our present results do not say anything about why it should have a right adjoint, but they do say that that right adjoint must be a representable functor. Let us find its representing cogroup.

We recall that that right adjoint is the functor

$$V\colon\ \mathbf{Monoid} \to \mathbf{Group}$$

taking every monoid A to its group of invertible elements. Since the invertible elements of a monoid A form a subset of $|A|$, one would at first glance expect $U_{\mathbf{Group}}V$, when expressed in the form described in Lemma 8.4.13(ii), to have X a singleton, i.e., to be represented by a monoid presented by one generator and some relations. But at second glance, we see that this cannot be so: the condition of invertibility on an element of a monoid is not an equation in that element alone. We can find the representing monoid for V by applying its left adjoint U to the free group on one generator. The result is this same group, regarded as a monoid, and as such, it has presentation

(9.4.3) $$R\ =\ \langle x, y \mid xy = e = yx\rangle.$$

Thus for any monoid A, the description of $|V(A)|$ in the form described in Lemma 8.4.13(ii) is

(9.4.4) $$\{(\xi,\eta)\in|A|\times|A| \mid \xi\eta = e = \eta\xi\}.$$

Since two-sided inverses to monoid elements are unique when they exist, every element (ξ,η) of $|V(A)|$ is determined by its first component, subject to the condition that this have an inverse. So up to functorial isomorphism, (9.4.4) is indeed the set of invertible elements of A. (We noted this example briefly in the paragraph following Lemma 8.4.13.)

Let us write down the cogroup structure on the representing monoid (9.4.3). If we write the coproduct of two copies of this monoid as

$$R \amalg R\ =\ \langle x_0, y_0, x_1, y_1 \mid x_0y_0 = e = y_0x_0,\ x_1y_1 = e = y_1x_1\rangle,$$

then we find that the comultiplication is given by

$$\mathbf{m}(x)\ =\ x_0x_1, \qquad \mathbf{m}(y)\ =\ y_1y_0.$$

(If you are uncertain how I got these formulas, stop here and think it out. If you are still not sure, ask in class! Note the reversed multiplication of the y's, a consequence of the fact that when one multiplies two invertible monoid elements, their inverses multiply in the reverse order.) It is also easy to see that the coinverse operation $\mathbf{i}\colon R \to R$ is given by

$$\mathbf{i}(x) = y, \qquad \mathbf{i}(y) = x,$$

and, finally, that the co-neutral-element map, from R to the initial object of **Monoid**, namely $\{e\}$, is the unique element of $\mathbf{Monoid}(R, \{e\})$, characterized by

$$\mathbf{e}(x) = e = \mathbf{e}(y).$$

Exercise 9.4:2. Describe explicitly the co-operations of the coalgebras representing two of the other examples discussed above, as we have done for the group-of-units functor $\mathbf{Monoid} \to \mathbf{Group}$.

Exercise 9.4:3. We noted above that we might naively have expected the group-of-invertible-elements functor $\mathbf{Monoid} \to \mathbf{Group}$ to be represented by a 1-generator monoid, but that it was not. Let us look more closely at this type of situation. Suppose $W: \mathbf{V} \to \mathbf{W}$ is a representable functor among varieties of algebras, with representing $\mathbf{W}$-coalgebra R.

(i) Show that $U_{\mathbf{W}} W: \mathbf{V} \to \mathbf{Set}$ is isomorphic to a subfunctor of $U_{\mathbf{V}}$ if and only if there exists a map $F_{\mathbf{V}}(1) \to |R|$ which is an *epimorphism* in $\mathbf{V}$ (but not necessarily surjective).

(ii) Describe the epimorphism implicit in our discussion of the group-of-invertible-elements functor.

(iii) Generalize the result of (i) in one way or another.

We can get other examples of representable functors by composing some of those we have described. For instance, if we start with the $n \times n$ matrix ring functor $\mathbf{Ring}^1 \to \mathbf{Ring}^1$, follow it by the underlying multiplicative monoid functor $\mathbf{Ring}^1 \to \mathbf{Monoid}$, and this by the group-of-units functor $\mathbf{Monoid} \to \mathbf{Group}$, we get a functor $\mathbf{Ring}^1 \to \mathbf{Group}$ which takes every ring A to the group of all invertible $n \times n$ matrices over A, known as $\mathrm{GL}(n, A)$.

Let us record a couple of other general results on representability of functors, equivalently, on existence of adjoints. As we noted in example (ii) following Corollary 9.4.2, that corollary implies

Corollary 9.4.5. *The inclusion of any subvariety* $\mathbf{U}$ *in a variety* $\mathbf{V}$ *has a left adjoint.* □

Combining this with Corollary 9.4.1 (composites of representable functors are representable) , we get

Corollary 9.4.6. *If a functor* $W: \mathbf{V} \to \mathbf{W}$ *between varieties of algebras is representable, then so is its restriction to any subvariety of* $\mathbf{U} \subseteq \mathbf{V}$. □

For instance, having observed that $\mathrm{GL}(n)$ is a representable functor on $\mathbf{Ring}^1$, we know automatically that it gives a representable functor on $\mathbf{CommRing}^1$. (What is the relation between the representing objects for these two functors?)

When a functor between varieties of algebras $W: \mathbf{V} \to \mathbf{W}$ is representable, this representability is usually easy to see and to prove – the construction of the underlying set of $W(A)$ is easily expressed in the form described in Lemma 8.4.13(ii). On the other hand, when we want to prove that a functor V is *not* representable, this criterion is clearly not as helpful; the more useful criterion here is Proposition 7.10.3, which says that W is representable if and only if it respects limits and satisfies a ‘‘solution-set condition’’. As we noted in §§7.7-7.10, most cases of nonrepresentability reveal themselves through failure of the functor to respect limits of one sort or another. For example:

Exercise 9.4:4. Verify that *none* of the following covariant functors from abelian groups to abelian groups is representable:

(i) $F(A) = A \otimes A$.

(ii) $G(A)$ = the torsion subgroup of A (the subgroup of all elements of finite order).

(iii) $H(A) = A/nA$ (n a fixed integer).

(iv) $J(A) = nA$ (n a fixed integer).

In Exercises 7.10:4-7.10:5, we saw examples of the rarer situation in which some left universal construction was impossible only because the solution-set condition was not satisfied. Those examples were of nonexistence of *initial objects* and of *free objects*, so by Theorem 8.4.11, the domain categories were, necessarily, not varieties (though the domain in one of the examples, the category of complete lattices, failed to be a variety only in that it had a large set of operations). The following exercise shows that in the case of the criterion for *representability*, there are counterexamples where the domain *is* a variety.

Exercise 9.4:5. Let us call an object S of a variety $\mathbf{V}$ *simple* if the only congruences on S are the trivial congruence and the total congruence (the least and the greatest equivalence relations on $|S|$).

(i) Find a variety $\mathbf{V}$ having the properties that (a) for every cardinal α there exists a simple algebra S_α in $\mathbf{V}$ of cardinality $\geq \alpha$, and (b) every algebra in $\mathbf{V}$ contains a unique one-element subalgebra. (Suggestion: Show either that there are simple groups of arbitrarily large cardinalities, or that there are fields of arbitrarily large cardinalities; in the latter case you must also say how to regard fields as simple objects of a variety satisfying (b).)

Now assume we have chosen such a $\mathbf{V}$, and for each α some S_α, as above. For every object A of $\mathbf{V}$, define $V(A) = \mathbf{V}(\coprod_{\alpha \leq |A|} S_\alpha,\ A) \in \mathrm{Ob}(\mathbf{Set})$; equivalently (up to natural isomorphism) $V(A) = \prod_{\alpha \leq |A|} \mathbf{V}(S_\alpha,\ A)$.

(ii) Show how to make V a functor, and show that this functor respects small limits, but is not representable. (You may either get these results directly, or with the help of part (iii) below.)

(iii) Recall that the variety we are writing $\mathbf{V}$ could be more precisely written as $\mathbf{V}_{(U)}$, the category of *U-small* objects of a certain type that satisfy a certain system of identities. Letting U' be any universe properly larger than U, show that $\mathbf{V}_{(U')}$ contains an object S such that the restriction to $\mathbf{V}_{(U)}$ of the functor $h_S\colon \mathbf{V}_{(U')} \to \mathbf{Set}_{(U')}$ is isomorphic to the functor V of (ii) above.

Thus, intuitively, this example is based on a functor which is representable, but by an object outside our universe. What was tricky was to find such a functor which nevertheless took U-small algebras to U-small sets.

Curiously, in the condition from Chapter 7 for the existence of *right* adjoint functors, one *can* drop the solution-set condition when the domain category is a variety:

Exercise 9.4:6. Show that if $\mathbf{V}$ is a variety of algebras and $\mathbf{C}$ a category with small colimits, then every functor $F\colon \mathbf{V} \to \mathbf{C}$ which respects small colimits has a right adjoint; i.e., is the left adjoint to a representable functor.

Knowing that representable functors from a variety $\mathbf{W}$ to a variety $\mathbf{V}$ correspond to $\mathbf{V}$-coalgebra objects of $\mathbf{W}$, it is natural to try, for various choices of $\mathbf{V}$ and $\mathbf{W}$, to find *all* such coalgebras, and hence all such functors. How difficult this task is depends on the varieties in question. At the easy extreme are certain large classes of cases for which we shall see in §9.9 that there can be no nontrivial representable functors. At the other end are cases such as that of representable functors from the variety of commutative rings (or commutative algebras over a fixed commutative ring k) to $\mathbf{Group}$. Such functors are called ‘‘affine algebraic groups’’, and are an important area of research in algebraic geometry.

In the next three sections, we shall tackle some cases of an intermediate level of difficulty, for which the problem is nontrivial, but where with a reasonable amount of work we can get a complete classification.

9.5. Representable endofunctors of Monoid. Let us consider representable functors from the variety **Monoid** into itself.

A representable functor from an arbitrary category **C** with finite coproducts to **Monoid** is represented by a comonoid, which we shall for convenience write as a 3-tuple $(R, \mathbf{m}, \mathbf{e})$, (rather than as a pair $(R, (\mathbf{m}, \mathbf{e})))$, where R is an object of **C**, and the other two components are a binary *comultiplication*

$$\mathbf{m}:\ R \to R \amalg R$$

and a zeroary *co-neutral-element*

$$\mathbf{e}:\ R \to I.$$

Here I denotes the initial object of **C**, that is, the coproduct of the empty family. These co-operations must satisfy the coassociative law, and the right and left coneutral laws. The coassociative law can be shown diagrammatically as the dual to (9.2.6); thus, it says that the diagram

(9.5.1)
$$\begin{tikzcd} R \arrow[r, "\mathbf{m}"] \arrow[d, "\mathbf{m}"] & R \amalg R \arrow[d, "\mathbf{m} \amalg \mathrm{id}_R"] \\ R \amalg R \arrow[r, "1_R \amalg \mathbf{m}"] & R \amalg R \amalg R \end{tikzcd}$$

commutes. The two coneutral laws likewise say that the composite maps

(9.5.2)
$$R \xrightarrow{\mathbf{m}} R \amalg R \xrightarrow{(\mathbf{e},\, \mathrm{id}_R)} R, \qquad R \xrightarrow{\mathbf{m}} R \amalg R \xrightarrow{(\mathrm{id}_R,\, \mathbf{e})} R$$

are both the identity morphism of R, where in each of these latter diagrams, the parenthesized pair shown above the second arrow is an abbreviation for the morphism obtained from the two entries of that pair via the universal property of the coproduct $R \amalg R$.

Let us now specialize to the case **C** = **Monoid**. Then the initial object I is the trivial monoid $\{e\}$; hence the homomorphism $\mathbf{e}$ can only be the map taking every element of R to e. (Contrast this with the case of $\mathrm{SL}(n)$ discussed in §9.1, where $\mathbf{e}$ had for codomain the initial object $\mathbf{Z}$ of $\mathbf{CommRing}^1$, and the specification of the identity matrix was nontrivial.) Nonetheless, the fact that this unique zeroary co-operation satisfies the coneutral laws (9.5.2) will be a nontrivial condition.

To study (9.5.1) and (9.5.2), we need to recall the structure of a coproduct of monoids. We noted in §3.10 that such a coproduct

(9.5.3)
$$\coprod_{\alpha \in I} R^{\alpha}$$

could be described in essentially the same way as for groups; namely, assuming for notational convenience that the sets $|R^{\alpha}| - \{e\}$ are disjoint, each element of (9.5.3) can be written uniquely as a product

(9.5.4) $$r_0 r_1 \ldots r_{h-1}, \qquad \text{with } h \geq 0, \text{ each } r_i \text{ in some } |R^{\alpha_i}| - \{e\}, \text{ and } \alpha_i \neq \alpha_{i+1} \text{ for } 0 \leq i < h-1.$$

(Here the neutral element e of (9.5.3) is understood to be the case $h = 0$ of (9.5.4).)

However, in the case of the coproduct $R \amalg R$ we are interested in now, the two monoids being put together are *not* disjoint. Let us therefore distinguish our two canonical images of R in $R \amalg R$ as R^λ and R^ρ (the superscripts corresponding to the ''left'' and ''right'' arguments of the comultiplication we want to study). We shall thus write $R \amalg R$ as $R^\lambda \amalg R^\rho$, i.e., as the coproduct of these two copies of R, and write the images of an element $x \in |R|$ under the two coprojections $R \rightrightarrows R^\lambda \amalg R^\rho$ as x^λ and x^ρ respectively.

The coassociative law involves three variables, hence in (9.5.1), R is ultimately mapped into a three-fold coproduct of copies of itself; let us write this object $R^\lambda \amalg R^\mu \amalg R^\rho$, the μ standing for the ''middle'' variable in the associativity identity.

An obvious invariant of an element (9.5.4) is the sequence of indices $(\alpha_0, \ldots, \alpha_{h-1})$. So let us define an *index-string* to mean a finite (possibly empty) sequence of members of $\{\lambda, \mu, \rho\}$, with no two successive terms equal. We shall call h the *length* of the index-string $(\alpha_0, \ldots, \alpha_{h-1})$. For every index-string $\sigma = (\alpha_0, \ldots, \alpha_{h-1})$, we shall denote by $|R|^\sigma$ the set of all products (9.5.4) with that sequence of superscripts, i.e.,

$$|R|^\sigma = (|R^{\alpha_0}| - \{e\}) \ldots (|R^{\alpha_{h-1}}| - \{e\}).$$

The underlying set of each of the monoids $R^\lambda \amalg R^\rho$ and $R^\lambda \amalg R^\mu \amalg R^\rho$ is thus the disjoint union of its subsets $|R|^\sigma$. We define the *height* $\mathrm{ht}(s)$ of $s \in |R^\lambda \amalg R^\rho|$ as the length of the unique σ such that $s \in |R|^\sigma$. Finally, to study our comultiplication $\mathbf{m}$, let us define the *degree* of an element of R by

$$\deg(x) = \mathrm{ht}(\mathbf{m}(x)).$$

We note that for each $h > 0$, there are precisely two index-strings of length h consisting only of ρ's and λ's: one beginning with ρ and the other beginning with λ. Thus, if $x \in |R|$ is an element of positive degree h, then $\mathbf{m}(x)$ either belongs to $|R|^{(\lambda, \rho, \lambda, \ldots)}$ (h entries in the superscript) i.e., has the form $y_0^\lambda z_1^\rho y_2^\lambda \ldots$, or it belongs to $|R|^{(\rho, \lambda, \rho, \ldots)}$, and has the form $z_0^\rho y_1^\lambda z_2^\rho \ldots$.

It is easy to see that the coneutral laws (9.5.2) say

(9.5.5) $$\text{If } \mathbf{m}(x) = \ldots y_i^\lambda z_{i+1}^\rho y_{i+2}^\lambda z_{i+3}^\rho \ldots, \quad \text{then} \quad x = \ldots y_i y_{i+2} \ldots = \ldots z_{i+1} z_{i+3} \ldots .$$

(Note that the way we have written $\mathbf{m}(x)$ here covers both the cases $x \in |R|^{(\lambda, \rho, \lambda, \ldots)}$ and $x \in |R|^{(\rho, \lambda, \rho, \ldots)}$.) In particular, (9.5.5) implies

(9.5.6) $$\text{If } x \neq e, \text{ then } \deg(x) \geq 2.$$

On the two sorts of elements of degree exactly 2, we see that (9.5.5) precisely determines the action of $\mathbf{m}$:

(9.5.7) $$\begin{cases} \text{If } \mathbf{m}(x) \in |R|^{(\lambda, \rho)}, \text{ then } \mathbf{m}(x) = x^\lambda x^\rho. \\ \text{If } \mathbf{m}(x) \in |R|^{(\rho, \lambda)}, \text{ then } \mathbf{m}(x) = x^\rho x^\lambda. \end{cases}$$

Let us also record what (9.5.5) tells us about the degree 3 case:

$$(9.5.8)\qquad \begin{cases} \text{If } \mathbf{m}(x)\in|R|^{(\lambda,\,\rho,\,\lambda)}, \text{ then } \mathbf{m}(x) = y_0^{\lambda}\, x^{\rho} y_2^{\lambda} \text{ where } y_0 y_2 = x. \\ \text{If } \mathbf{m}(x)\in|R|^{(\rho,\,\lambda,\,\rho)}, \text{ then } \mathbf{m}(x) = z_0^{\rho}\, x^{\lambda} z_2^{\rho} \text{ where } z_0 z_2 = x. \end{cases}$$

We now turn to the coassociative law. This says that for any $x\in|R|$,

$$(9.5.9)\qquad (\mathrm{id}_{R^\lambda},\, \mathbf{m})\,\mathbf{m}(x) \;=\; (\mathbf{m},\, \mathrm{id}_{R^\rho})\,\mathbf{m}(x) \qquad \text{in } R^{\lambda} \amalg R^{\mu} \amalg R^{\rho}.$$

Let us note the explicit descriptions of the left-hand factors of each side of the above equation. $(\mathrm{id}_{R^\lambda},\ \mathbf{m})$: $R^{\lambda} \amalg R^{\rho} \to R^{\lambda} \amalg R^{\mu} \amalg R^{\rho}$ leaves each element of the form $y^{\lambda}\in|R^{\lambda} \amalg R^{\rho}|$ unchanged, while it takes an element $z^{\rho}\in|R^{\lambda} \amalg R^{\rho}|$ to the element $\mathbf{m}(z)$, but with all the superscripts ‘‘$^{\lambda}$’’ changed to ‘‘$^{\mu}$’’ (because of the way we label our 3-fold coproduct). Likewise, $(\mathbf{m},\ \mathrm{id}_{R^\rho})$ leaves each z^{ρ} unchanged, and takes each y^{λ} to the element $\mathbf{m}(y)$, with all superscripts ‘‘$^{\rho}$’’ changed to ‘‘$^{\mu}$’’.

Now let $x\in|R| - \{e\}$, suppose that $\mathbf{m}(x)$ belongs to the set $|R|^{\sigma}$ (σ a string of λ’s and ρ’s), and let the common value of the two sides of (9.5.9) belong to the set $|R|^{\tau}$ (τ a string of λ’s, μ’s and ρ’s). Note that each ‘‘λ’’ in σ yields a single λ in τ on evaluating the left-hand side of (9.5.9), but looking at the right-hand side of (9.5.9), it gives *at least* one λ in τ, because of (9.5.6). Since the two sides of (9.5.9) are equal, all of these ‘‘at least one’’s must be exactly one. For this to happen, the elements y_i in the expansion (9.5.5) must all have degree ≤ 3. By a symmetric argument (comparing occurrences of ρ in σ and in τ) we get the same conclusion for the elements z_i. Note also that if τ begins with μ, then the right-hand side of (9.5.9) tells us σ must begin with a λ, while the left-hand side says it must begin with a ρ, a contradiction. Hence τ can only begin with a λ or a ρ. In the former case, σ must begin with a λ which expands to $\lambda\mu$ on the right-hand side of (9.5.9) (so as not to yield more than one λ); in the latter case it must begin with a ρ which expands to $\rho\mu$ on the left-hand side. In either case, we conclude that the first factor in the expansion of $\mathbf{m}(x)$ must have degree 2. The same arguments apply to the last factor. In summary:

(9.5.10) All elements y_i and z_i in (9.5.5) have degree ≤ 3; hence by (9.5.5), every element of R is a product of elements of degree ≤ 3. Moreover, the elements giving the *first* and *last* factors of $\mathbf{m}(x)$ have degree 2.

But the observation about first and last factors, applied to the final equation in each line of (9.5.8), gives

(9.5.11) Every element of R of degree 3 is a product of two elements of degree 2.

(9.5.10) and (9.5.11) together allow one to express every element of R as a product of elements of degree 2, showing that R is generated by these elements. We can prove still more:

Lemma 9.5.12. *Let* $(R,\, \mathbf{m},\, \mathbf{e})$ *be a co-***Monoid** *object in* **Monoid**. *Then every element* $x\in|R|$ *has an expression as a product*

$$x_0\, \ldots\, x_{h-1} \quad (h \geq 0),$$

where all x_i *are of degree* 2, *and this expression is* unique *subject only to the condition that there be no two successive factors* $x_i,\ x_{i+1}$ *such that one of* $\mathbf{m}(x_i),\ \mathbf{m}(x_{i+1})$ *belongs to* $|R|^{(\lambda,\,\rho)}$, *the other belongs to* $|R|^{(\rho,\,\lambda)}$, *and* $x_i x_{i+1} = e$.

Proof. Since R is generated by elements of degree 2, and since any expression involving two successive factors whose product is e can be simplified to a shorter expression, we can clearly express every element in the indicated form subject to the conditions noted. To show that this form is unique, it suffices to prove that given an element and such an expression for it,

$$x \;=\; x_0 \,...\, x_{h-1} \qquad (\deg(x_i) = 2,\ \ i = 0, \ldots, h{-}1),$$

we can recover the factors x_i from x. I claim in fact that if for this x we write the common value of the two sides of (9.5.9) as a reduced product of elements of R^λ, R^μ and R^ρ, i.e., as in (9.5.4), then the sequence of factors belonging to R^μ will be precisely $x_0^\mu, \ldots, x_{h-1}^\mu$, recovering the x_i, as required.

Indeed, let us note that for any x such that $\mathbf{m}(x) \in |R|^{(\lambda,\,\rho)}$, the common value of the two sides of (9.5.9), computed using (9.5.7), is $x^\lambda x^\mu x^\rho$, while when $\mathbf{m}(x) \in |R|^{(\rho,\,\lambda)}$ it is $x^\rho x^\mu x^\lambda$. Hence when we evaluate the common value of the two sides of (9.5.9) for $x = x_0 \,...\, x_{h-1}$, the factors with superscript μ comprise, *initially*, the sequence claimed. They will continue to do so after we reduce this product to the form (9.5.4) unless, in the process of reduction, the factors with superscript ρ and/or λ separating some pair of successive μ-factors cancel, allowing these μ-factors to combine into a single element of $|R|^\mu$. Now if $\mathbf{m}(x_i)$ and $\mathbf{m}(x_{i+1})$ both belong to $|R|^{(\lambda,\,\rho)}$ or both belong to $|R|^{(\rho,\,\lambda)}$, then between x_i^μ and x_{i+1}^μ we will have exactly one λ-factor and one ρ-factor, and these cannot cancel. In the case where one belongs to $|R|^{(\lambda,\,\rho)}$ and the other to $|R|^{(\rho,\,\lambda)}$, we get adjacent factors $x_i^\rho x_{i+1}^\rho$ or $x_i^\lambda x_{i+1}^\lambda$ in the same set R^ρ or R^λ. These will in general combine into one factor, but they will cancel only if $x_i x_{i+1} = e$ in R. But this is the case excluded by our hypothesis. □

Note that in the above argument, we could have asserted that every element can be reduced to a unique product of the indicated form in which no two successive factors *whatever* have product e. However, we have proved uniqueness subject to a *weaker* condition than this, so we have a *stronger* uniqueness result. Indeed, this result implies (as the weaker uniqueness statement would not):

Corollary 9.5.13. *If* $(R,\, \mathbf{m},\, \mathbf{e})$ *is a co-***Monoid** *object in* **Monoid**, *then the monoid* R *has a presentation* $<X \mid Y>$, *where* X *is the set of elements of* R *having degree* 2 *with respect to the comultiplication* $\mathbf{m}$, *and* Y *is the set of all relations of the form* $x_0 x_1 = e$ *holding in* R *such that one of* $\mathbf{m}(x_0)$, $\mathbf{m}(x_1)$ *lies in* $|R|^{(\lambda,\,\rho)}$, *and the other in* $|R|^{(\rho,\,\lambda)}$.

Proof. We know that X generates R, and by definition the relations comprising Y are satisfied by these generators. It remains to verify that if two words w_0 and w_1 in the elements of X are equal in R, then this equality follows from the relations in Y.

Now if w_i $(i = 0 \text{ or } 1)$ contains a substring which is the left-hand side of some relation in Y, then by applying that relation, we can reduce w_i to a shorter word. Hence a finite number of applications of such relations will transform w_0 and w_1 to words w'_0 and w'_1 that contain no such substrings. The values of these words in R are still equal; hence the uniqueness statement of Lemma 9.5.12 tells us they are the same word. Thus, by applying relations in Y, we have obtained the equality of w_0 and w_1 in R, as required. □

The next step in studying our comonoid should clearly be to examine the properties of the relation $x_0 x_1 = e$ on elements of degree 2 in R. So let us make

Definition 9.5.14. *If* $(R,\,\mathbf{m},\,\mathbf{e})$ *is a co-***Monoid** *object in* **Monoid**, *then* $P(R,\,\mathbf{m},\,\mathbf{e})$ *will denote the 4-tuple* $(u,\,X^+,\,X^-,\,E)$, *where*

$$\begin{aligned} & u \;=\; e, \text{ the neutral element of } R,\\ & X^+ = \{x\in|R| \mid \mathbf{m}(x) = x^\lambda x^\rho\} \;=\; \{x\in|R| \mid \mathbf{m}(x)\in|R|^{(\lambda,\,\rho)}\}\cup\{u\},\\ & X^- = \{x\in|R| \mid \mathbf{m}(x) = x^\rho x^\lambda\} \;=\; \{x\in|R| \mid \mathbf{m}(x)\in|R|^{(\rho,\,\lambda)}\}\cup\{u\},\end{aligned}$$

and $\quad E \;=\; \{(x_0,x_1)\in|R|^2 \mid \deg(x_0),\,\deg(x_1) \le 2,\;\; x_0x_1 = e\} \;\subseteq\; (X^+\times X^-)\cup(X^-\times X^+).$

Thus, X^+ and X^- are sets intersecting in the singleton $\{u\}$, and E is a binary relation on the union of these sets, which relates certain elements of X^+ to certain elements of X^-, and vice versa. We note a key property of this relation: If both $(x_0,\,x_1)$ and $(x_1,\,x_2)$ belong to it, then since x_1 has x_0 as a left inverse and x_2 as a right inverse in R, x_0 must equal x_2.

Let us formalize the type of combinatorial object we have obtained.

Definition 9.5.15. *An E-*system *will mean a 4-tuple* $(u,\,X^+,\,X^-,\,E)$, *where* u *is an element,* X^+ *and* X^- *are sets such that*

$$X^+\cap X^- \;=\; \{u\},$$

and

$$E \;\subseteq\; (X^+\times X^-)\cup(X^-\times X^+)$$

is a relation such that

(9.5.16) $$u\,E\,u,$$

(9.5.17) $$x_0\,E\,x_1,\;\; x_1\,E\,x_2 \;\;\Rightarrow\;\; x_0 = x_2.$$

A morphism *of E-systems* $(u,\,X^+,\,X^-,\,E)\to(u',\,X'^{\,+},\,X'^{\,-},\,E')$ *will mean a map* $X^+\cup X^-\to X'^{\,+}\cup X'^{\,-}$ *carrying* u *to* u', X^+ *into* $X'^{\,+}$, X^- *into* $X'^{\,-}$, *and the relation* E *into the relation* E'.

Thus, the objects $P(R,\,\mathbf{m},\,\mathbf{e})$ constructed in Definition 9.5.14 are E-systems.

Does the concept of E-system in fact capture enough structure to model the co-**Monoid** objects of **Monoid**?

Suppose $(u,\,X^+,\,X^-,\,E)$ is an E-system, and let us try to show that it arises from a comonoid object via the construction of Definition 9.5.14. To begin the ‘‘reconstruction’’ of this comonoid object, we should clearly form the monoid with presentation

(9.5.18) $$R \;=\; <X^+\cup X^- - \{u\} \;\mid\; x_0x_1 = e \text{ whenever } x_0\,E\,x_1>.$$

On this monoid we have a unique zeroary co-operation $\mathbf{e}$, namely the trivial map $R\to\{e\}$. We now try to define a comultiplication homomorphism from this monoid into the coproduct of two copies of itself, setting

(9.5.19) $$\mathbf{m}(x) \;=\; \begin{cases} x^\lambda x^\rho & \text{if } x\in X^+ - \{u\},\\ x^\rho x^\lambda & \text{if } x\in X^- - \{u\}.\end{cases}$$

The next two exercises will show that this construction in fact inverts that of Definition 9.5.14, a result which we will then summarize as a theorem. You should therefore read these exercises through, and think about what is involved, even if you do not work out all the details.

Exercise 9.5:1. (i) Show that for any E-system $X = (u, X^+, X^-, E)$, if we define R by (9.5.18), then (9.5.19) gives a well-defined homomorphism $\mathbf{m}\colon R \to R^{\lambda} \amalg R^{\rho}$.

(ii) Show that this $\mathbf{m}$ and the trivial morphism $\mathbf{e}$ make R a comonoid object of **Monoid**. Let us denote this object $Q(X)$.

The next observation will make some subsequent results easier to state:

(iii) Verify that the presentation (9.5.18) is equivalent to the modified presentation with u included among the generators and $u = e$ added to the relations; and that (9.5.19) then holds with the ''$-\{u\}$''s deleted.

(iv) Show that the construction P of Definition 9.5.14, and the above construction Q, may be made functors in obvious ways, and that Q is then left adjoint to P.

(v) Deduce from Corollary 9.5.13 that the counit of this adjunction, i.e., the canonical morphism from the functor QP to the identity functor of the category of co-**Monoid** objects of **Monoid**, is an isomorphism. In particular, every comonoid object of **Monoid** arises under Q from an E-system.

There remains the question of whether every E-system arises from a comonoid. This is equivalent to asking whether distinct E-systems yield distinct comonoids under Q, which is in turn equivalent to asking whether the *unit* of the above adjunction, i.e., the canonical morphism from the identity functor of the category of E-systems to PQ, is also an isomorphism.

(To banish any suspicion that this conclusion might follow automatically from (v) above, consider the analogous situation where P is the forgetful functor **Group** → **Monoid**, and Q its left adjoint, taking every monoid to its universal enveloping group. Then the counit $QP \to \mathrm{Id}_{\mathbf{Group}}$ is an isomorphism, but the unit $\mathrm{Id}_{\mathbf{Monoid}} \to PQ$ is not: monoids containing noninvertible elements do not appear as values of P, and each such monoid falls together under Q with a monoid that *is* a value of P.)

To answer this question, we need a normal form result:

Exercise 9.5:2. (i) Show that given any E-system $X = (u, X^+, X^-, E)$, the monoid R with presentation (9.5.18) has for normal form the set of words in the indicated generators (including the empty word) that contain no subsequences $x_0 x_1$ with $x_0 E x_1$. (Suggestion: van der Waerden's trick.)

(ii) Deduce that the unit of the adjunction between P and Q is an isomorphism.

The above results are summarized in the first sentence of the next theorem. The second sentence translates the comonoid structure (9.5.18)-(9.5.19) into a description of the functor represented, and the final sentence follows by Corollary 9.3.5.

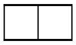
Theorem 9.5.20. *Every representable functor V from* **Monoid** *to* **Monoid** *is determined by an E-system. The functor corresponding to the E-system (u, X^+, X^-, E) can be described as a subfunctor* (*in the sense of Lemma 6.9.3 and Definition 8.4.8*) *of a direct product of copies of the* identity *functor and of the* opposite-monoid *functor; namely, as the construction taking each monoid A to the submonoid of $A^{(X^+-\{u\})} \times (A^{\mathrm{op}})^{(X^- -\{u\})}$ consisting of those elements s such that for all $(x,y) \in E - \{(u,u)\}$, the coordinate s_x is a left inverse to the coordinate s_y.*

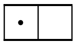
*Writing E-***System** *for the category of E-systems, the above construction yields a contravariant* equivalence E-$\mathbf{System}^{\mathrm{op}} \to \mathbf{Rep}(\mathbf{Monoid}, \mathbf{Monoid})$. □

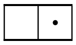
For the purpose of describing the morphism of representable functors induced by a given morphism of E-systems, it is actually most convenient to treat the functor V: **Monoid** → **Monoid** corresponding to the E-system (u, X^+, X^-, E) as taking a monoid A to a submonoid of

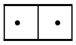
$$A^{(X^+-\{u\})} \times \{e\} \times (A^{\mathrm{op}})^{(X^- -\{u\})};$$

i.e., to introduce an extra slot, indexed by the element u of the E-system, such that the coordinate of $V(A)$ in that slot is required to be the neutral element e of A. (Cf. Exercise 9.5:1(iii).) We can then say that if $\mathbf{f}: E \to E'$ is a morphism of E-systems, and $f: V' \to V$ the corresponding morphism of representable functors, then for a monoid A and an element $\xi \in |V'(A)|$, the image $f(A)(\xi)$ has for xth coordinate the $\mathbf{f}(x)$th coordinate of ξ, whether $\mathbf{f}(x)$ happens to be u, or to be a member of $X^+ \cup X^- - \{u\}$.

Let us look at some simple examples of E-systems and the corresponding representable functors. We shall display an E-system by showing the elements of $X^+-\{u\}$ and $X^- -\{u\}$ respectively as points in two boxes, , and indicating a condition $x_0 \, E \, x_1$ by an arrow from the point x_0 to the point x_1. (The element u will not be shown; it may be thought of as embedded in the dividing line between the boxes.)

 By (9.5.18)-(9.5.19), the comonoid R corresponding to this E-system is the free monoid on one generator x, with the comultiplication under which $\mathbf{m}(x) = x^\lambda x^\rho$. We see that the functor this represents is (up to isomorphism) the *identity* functor **Monoid** → **Monoid**. This description of the functor represented can also be seen from the second sentence of the above theorem.

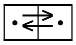
 You should verify that this E-system similarly gives the *opposite monoid* functor.

 (the relation $E - \{(u,u)\}$ still being empty). This gives the direct product of the above two functors, i.e., the functor associating to every monoid A the monoid

$$\{(\alpha, \beta) \mid \alpha, \beta \in |A|\},$$

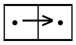
with multiplication

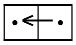
$$(\alpha_0, \beta_0)\,(\alpha_1, \beta_1) \;=\; (\alpha_0\alpha_1,\, \beta_1\beta_0). \tag{9.5.21}$$

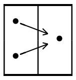
 This corresponds to the subfunctor of the preceding example determined by adding to the description of its underlying set the conditions

$$\alpha\beta \;=\; e \;=\; \beta\alpha.$$

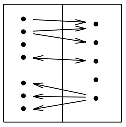
Since under these conditions α uniquely determines β, the second coordinate provides no new information, and we can describe this functor, up to isomorphism, as associating to A its *group of invertible elements* α, regarded as a monoid.

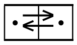
 As above, except that only the condition $\alpha\beta = e$, and not $\beta\alpha = e$ is imposed. Right inverses are *not* generally unique, so we must describe this functor as associating to A the monoid of elements $\alpha \in |A|$ given with a *specified* right inverse β. The multiplication is again as in (9.5.21).

 This associates to A the monoid of elements α given with a specified *left inverse* β, again multiplied as in (9.5.21). Set-theoretically, this construction is isomorphic to the preceding, via $(\alpha, \beta) \longleftrightarrow (\beta, \alpha)$, but the monoid structures are opposite to one another. (I have indicated this in the paraphrases by naming, after the words ‘‘monoid of’’, the elements which are multiplied as in A, while those with the opposite multiplication are referred to as specified inverses of these elements.)

 ‘‘The monoid of *pairs* of elements of A with a specified *common* right inverse’’.

And so on. We note that for a general diagram such as



the associated functor is the direct product of the functors associated with the graph-theoretic ‘‘connected components’’ of the diagram. Each of these components, *except* those of the form  must have, by (9.5.17), the property that arrows, if any, all go in the same direction, i.e., from left to right or from right to left. Subject to this restriction, the arrows are independent.

Let us pause to note the curious fact that, although for every *nonzero* cardinal r, the construction that associates to a monoid A the monoid of its right invertible elements given with a specified r-tuple of right inverses is a representable functor, this is false for $r = 0$:

Exercise 9.5:3. Let H: **Monoid** → **Monoid** be the functor associating to a monoid A its submonoid of right invertible elements (a subfunctor of the identity functor).

(i) Show that H is not representable.

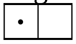
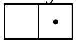
(ii) Show, however, that the composite functor HH is representable, and concisely describe this functor.

(iii) Show that H can be written as a direct limit of representable functors. (Hint: can you write the empty set as an inverse limit of nonempty sets?)

It is natural to ask how to compose two representable functors expressed in terms of E-systems.

Exercise 9.5:4. In this exercise, ‘‘functor’’ will mean ‘‘representable functor **Monoid** → **Monoid**’’.

(i) Define precisely what is meant by the connected components of an E-system, and prove the assertion made above that the functor associated with an E-system is the direct product of the functors associated with its connected components. Using this result, reduce the problem of describing the E-system of the composite of two functors to the case where the E-systems of the given functors are connected.

(ii) Characterize in terms of E-systems the results of composing an arbitrary functor on the right and on the left with the functors having the diagrams  and . (Thus, four questions are asked, though two of them are trivial to answer.)

This leaves us with the problem of describing the composite of two functors whose associated diagrams are both connected, and each have more than one element. The answer is quite simple, but the argument requires two preliminary observations:

(iii) Show that if s, t are two left invertible elements of a monoid A, or two right invertible elements, then the condition $st = e$ implies that they are both invertible.

(iv) Let V be a functor whose diagram is connected. Show that if some $\xi \in |V(A)|$ has an invertible element of A in at least one coordinate, then it has invertible elements in all coordinates, and these are determined by that one coordinate. Show that the set of elements ξ with these properties forms a submonoid of $V(A)$, isomorphic to the group of invertible elements of A. (In writing ‘‘at least one coordinate’’ above, I am understanding our description of V to be that of Theorem 9.5.20, which does not include a coordinate indexed by u.)

(v) Deduce from (iii) and (iv) a description for the composite of any two functors whose diagrams are both connected and each have more than one element (not counting u as an element of our diagrams).

Exercise 9.5:5. Suppose $f: V \to V'$ is a morphism of representable functors **Monoid** $\to$ **Monoid**, and W is another such functor. Assuming the results of the preceding exercise, show how to describe the map of E-systems corresponding to $f \circ W: VW \to V'W$, respectively $W \circ f: WV \to WV'$, in terms of the map of E-systems corresponding to f.

Exercise 9.5:6. We saw in the discussion following Corollary 9.4.1 that the object representing a composite of representable functors among varieties could be constructed from presentations $< X \mid Y >_{\mathbf{U}}$ and $< X' \mid Y' >_{\mathbf{V}}$ of representing objects for those functors, using a set of generators indexed by $X \times X'$ and a set of relations indexed by $X \times Y' \sqcup X' \times Y$. See whether you can get the results of the preceding two exercises by applying this idea to presentations of the representing objects for functors **Monoid** $\to$ **Monoid** induced by given E-systems. (If you did Exercise 9.4:1, you will be able to apply the results of that exercise here; if not, you can still work out the corresponding results for this particular case.)

9.6. Functors to and from some related categories. The characterization of representable functors **Monoid** $\to$ **Monoid** that we have obtained can be used to characterize various classes of representable functors involving the category **Group** as well.

We begin with some general observations. Let U: **Group** $\to$ **Monoid** denote the ''forgetful'' functor, let F: **Monoid** $\to$ **Group** denote the left adjoint of U, the ''universal enveloping group'' functor, and let G: **Monoid** $\to$ **Group** denote the right adjoint of U, the ''group of invertible elements'' functor. It is clear that the counit of the first adjunction and the unit of the second are isomorphisms

$$\varepsilon_{U,F}: \; FU \cong \mathrm{Id}_{\mathbf{Group}} \qquad \text{and} \qquad \eta_{G,U}: \; \mathrm{Id}_{\mathbf{Group}} \cong GU.$$

This implies that the composites of our two adjoint pairs in the reverse order, UF and UG, are retractions of **Monoid** onto U(**Group**), and that the latter is a full subcategory of **Monoid** isomorphic to **Group**. The other unit and counit of our adjunctions relate each monoid to its image in this subcategory under the corresponding retraction; let us write these

$$\eta = \eta_{U,F}: \; \mathrm{Id}_{\mathbf{Monoid}} \to UF \qquad \text{and} \qquad \varepsilon = \varepsilon_{G,U}: \; UG \to \mathrm{Id}_{\mathbf{Monoid}}$$

(breaking the convention that η and ε generally denote the unit and counit of the same adjunction). The next steps are given in the following two exercises:

Exercise 9.6:1. (i) Show that the monoids S of the form $U(A)$ (A a group) are precisely those for which the universal map $\eta(S): S \to UF(S)$ is an isomorphism, and are also those for which the universal map $\varepsilon(S): UG(S) \to S$ is an isomorphism.

(ii) Show that UF is left adjoint to UG.

(iii) Show that for any variety **V**, the representable functors **Group** $\to$ **V** can be identified with the representable functors V: **Monoid** $\to$ **V** which are invariant under composition on the right with UG (i.e., those V such that the induced map $V\varepsilon: VUG \to V$ is an isomorphism).

(iv) Show similarly that the representable functors **V** $\to$ **Group** can be identified with the representable functors **V** $\to$ **Monoid** which are invariant under composition on the left with UG (i.e., such that the induced map $\varepsilon V: UGV \to V$ is an isomorphism).

Though we shall not need it, you may also

(v) Show that the functors **Group** $\to$ **V**, respectively **V** $\to$ **Group** which have right adjoints (i.e., the left adjoints of representable functors) can be identified with the functors **Monoid** $\to$ **V**, respectively **V** $\to$ **Monoid** which have right adjoints and are invariant under composition on the right, respectively on the left with UF.

Exercise 9.6:2. Using the preceding exercise,

(i) Show that every representable functor **Group** → **Monoid** is a power (i.e., product of copies) of the forgetful functor U. (First proved by D. Kan [**69**].)

(ii) Show that every representable functor **Monoid** → **Group** is a power of the group-of-invertible-elements functor G.

(iii) Show that every representable functor **Group** → **Group** is a power of the identity functor.

Thus, in each of these three cases, all representable functors arise as powers of one ‘‘basic’’ functor, U, G or $\mathrm{Id}_{\mathbf{Group}}$ respectively. Calling this functor B in each case, so that the general representable functor between the categories in question has the form B^X, let us observe that for any set map $X \to Y$ we get a map $B^Y \to B^X$. Are these the only morphisms among these functors?

Not quite. For instance, in the case of functors **Group** → **Group**, if we take $X = Y = 1$, so that we are considering endomorphisms of the identity functor of **Group**, there is not only the identity morphism, associating to every group its identity map, and arising from the unique set map $1 \to 1$, but also the trivial morphism, associating to every group the endomorphism under which all elements go to e. To correctly describe the morphisms among our functors, let $\mathbf{Set}^{\mathrm{pt}}$ denote the category of *pointed sets*, whose objects are sets given with a single distinguished element, and whose morphisms are set maps sending distinguished element to distinguished element. (This may be identified with the variety Ω-**Alg** with Ω consisting of a single zeroary operation.) The next exercise shows that this is the right category for parametrizing these functors.

Exercise 9.6:3. (i) Let $L\colon E\text{-}\mathbf{System} \to \mathbf{Set}^{\mathrm{pt}}$ denote the functor taking every E-system $X = (u, X^+, X^-, E)$ to the pointed set (X^+, u). Show that when restricted to the full subcategory of E-systems whose ‘‘box pictures’’ have all connected components of the form $\boxed{\cdot \rightleftarrows \cdot}$, the functor L gives an equivalence of categories.

(ii) Deduce that in each of the cases of the preceding exercise, the indicated category of representable functors is equivalent to $(\mathbf{Set}^{\mathrm{pt}})^{\mathrm{op}}$. Precisely, letting B denote the ‘‘basic’’ functor in each case, show that morphisms $B^X \to B^Y$ correspond to the morphisms of pointed sets $(Y \cup \{u\}, u) \to (X \cup \{u\}, u)$ where u denotes an element not in X or Y.

Let us turn back to something mentioned at the beginning of the preceding section. In the description of a comonoid object of **Monoid**, the co-neutral-element was uniquely determined, and hence provided no information; nevertheless, the coidentities it was required to satisfy played an important role in our arguments. The next exercise shows that these coidentities were really needed for our results.

Exercise 9.6:4. Consider the following two representable functors from **Monoid** to the variety of semigroups with a distinguished element (zeroary operation) e subject to no additional identities.

(a) The functor V taking $A \in \mathrm{Ob}(\mathbf{Monoid})$ to the semigroup with underlying set $|A|$, multiplication given by $x * y = x$ for all x and y, and distinguished element given by the neutral element e of A.

(b) The functor W specifying the same underlying set and distinguished element, but with multiplication given by $x * y = e$.

Verify that in both cases the operation $*$ is indeed associative (so that the functors have domain in the variety claimed), and also that in both cases the distinguished element e is an *idempotent* with respect to $*$ (i.e., satisfies $e * e = e$). Show that in case (a), this element also satisfies the *right* neutral law, but not the left neutral law, while in case (b), neither neutral law is satisfied.

Note that in case (b) of the above exercise, the distinguished element satisfies the identities $e*x = e = x*e$. An element with this property is called a *zero* element of a semigroup, because these identities hold for 0 in the multiplicative semigroup of a ring. An element of a semigroup satisfying only the first of these identities is called a *left zero* element. We see that in case (a) *every* element is a left zero. The unique multiplication with the latter property on any set is called the *left zero multiplication*.

Little is known about general representable functors **Monoid** → **Semigroup**. Dropping the zeroary co-operations $\mathbf{e}$, the above exercise gives examples that are interesting in that construction (a) used nothing about the given monoid A but its underlying set, while (b) used only its structure of set with distinguished element e. The next exercise displays some constructions that do use the monoid operation, but in peculiar – almost random – ways.

Exercise 9.6:5. (i) Show that one can define a representable functor **Monoid** → **Semigroup** by associating to every monoid A the set of pairs (ξ, η) such that ξ is an invertible element of A and η is an arbitrary element of A, with the operation $(\xi, \eta)(\xi', \eta') = (e,\ \xi^{-1}\xi'^{-1}\xi\xi')$.
(ii) Show that if we impose on the ordered pairs in the description of the above functor the additional condition that $\xi^n = e$ for a fixed positive integer n, and/or the condition $\xi\eta = \eta$, the resulting subsets are still closed under the above operation, and hence define further representable functors.

Exercise 9.6:6. (Open question [**2**, Problem 21.7, p.94]) Find a description of (or other strong results about) all representable functors $\mathbf{W}$ → **Semigroup**, where $\mathbf{W}$ is any of the varieties **Monoid**, **Group** or **Semigroup**.

The following questions may be easy or hard to answer; I have not thought about them:

Exercise 9.6:7. Let V: **Monoid** → **Monoid** be a representable functor whose E-system has a single connected component, and is not one of $\boxed{\,\cdot \mid \,}$, $\boxed{\, \mid \cdot\,}$, $\boxed{\,\cdot \rightleftarrows \cdot\,}$. What can one say about the class of monoids of the form $V(A)$ $(A\in\mathrm{Ob}(\mathbf{Monoid}))$? How much does this class depend on the choice of V? How does it compare the with class of monoids that are embeddable in groups? With the class of monoids $H(A)$, where H is the functor of Exercise 9.5:3?

One may likewise ask these questions for the classes of monoids arising as values of the *left adjoints* of such functors.

9.7. Representable functors among categories of abelian groups and modules.

Let us now analyze representable functors from abelian groups to monoids. Let

$$V\colon\ \mathbf{Ab} \ \to\ \mathbf{Monoid}$$

be such a functor, with representing coalgebra $(R,\ \mathbf{m},\ \mathbf{e})$. Since coproducts of abelian groups are direct sums, we may write the coproduct of two copies of R as $R^{\lambda} \oplus R^{\rho}$; thus, every element of this group has the form $y^{\lambda} + z^{\rho}$ for unique $y, z\in|R|$. In particular, for each $x\in|R|$ we can write

$$\mathbf{m}(x) \ = \ y^{\lambda} + z^{\rho}.$$

As in the case of functors on **Monoid**, the co-neutral-element must be the trivial map. Applying the coneutral laws to the above equation, we immediately get $x = y = z$, i.e.,

$$\mathbf{m}(x) \ = \ x^{\lambda} + x^{\rho}.$$

Given any two elements $a, b\in|V(A)| = \mathbf{Ab}(R, A)$, this says that their ‘‘product’’ in $V(A)$ is the homomorphism taking $x\in|R|$ to $a(x)+b(x)$. In other words, the induced ‘‘multiplication’’ of homomorphisms is just the familiar addition of homomorphisms of abelian groups. It is clear that, conversely, for every abelian group R this operation on homomorphisms with domain R *does* make h_R a **Monoid**-valued functor. So for each $R\in\mathrm{Ob}(\mathbf{Ab})$, there is a unique representable functor $\mathbf{Ab}\to\mathbf{Monoid}$ whose representing coalgebra has underlying object R.

In view of the form V takes, it is natural to call the binary co-operation on R a ‘‘coaddition’’ rather than a ‘‘comultiplication’’. Of course, it is well known that addition on the sets $\mathbf{Ab}(R,A)$ is actually an operation of *group*, and, indeed, of *abelian* group, with the unique inverse operation described in the obvious way. Thus, our determination of all representable functors $\mathbf{Ab}\to\mathbf{Monoid}$ also determines all representable functors $\mathbf{Ab}\to\mathbf{Group}$ and $\mathbf{Ab}\to\mathbf{Ab}$. That is,

Lemma 9.7.1. *For every object R of* **Ab**, *there is a unique co-***Monoid** *object, a unique co-***Group** *object, and a unique co-***Ab** *object with underlying object R. Each of these has coaddition given by the diagonal map*

$$\mathbf{a}(x) \;=\; x^{\lambda}+x^{\rho}, \tag{9.7.2}$$

and co-neutral-element given by $\mathbf{e}(x)=0$. *In the co-***Group** *and co-***Ab** *structures, the co-inverse operation is given by*

$$\mathbf{i}(x) \;=\; -x.\ \square$$

Since this result was so easy to prove, let’s make some more work for ourselves, and try to generalize it!

Recall that an abelian group is equivalent to a left $\mathbf{Z}$-module, and that for any ring K, a left K-module M can be described as an abelian group with a family of abelian group endomorphisms, called ‘‘scalar multiplications’’, indexed by the elements of K, such that sums of these endomorphisms, composites of these endomorphisms, and the identity endomorphism are the endomorphisms indexed by sums of elements of K, products of elements of K, and the multiplicative neutral element $1\in|K|$. (Unless the contrary is stated, our rings are always members of the variety $\mathbf{Ring}^1$ of associative rings with multiplicative neutral element 1.) We will write K-**Mod** for the variety of left K-modules.

It is easy to see that the argument giving Lemma 9.7.1 generalizes to the case of representable functors from K-**Mod** to the varieties **Monoid**, **Group** and **Ab**.

What about functors from K-**Mod** to K-**Mod**, or better, to L-**Mod** for another ring L?

To study this question, let us write out explicitly the identities for the scalar multiplication operations of K-**Mod** which we stated above in words. The identities saying that each such multiplication is an abelian group endomorphism say that for all $c\in|K|$ and $x,x'\in|M|$,

$$c(x+x') \;=\; cx+cx' \tag{9.7.3}$$

(We are again, for simplicity, taking advantage of the fact that group homomorphisms can be characterized as set-maps respecting the binary group operation alone.) The identities characterizing sums and composites of scalar multiplications, and scalar multiplication by $1\in|K|$, say that for $c,c'\in|K|$, $x\in|M|$,

$$(c+c')x \;=\; cx+c'x \tag{9.7.4}$$

(9.7.5) $$(cc')x \;=\; c(c'x)$$

(9.7.6) $$1x \;=\; x.$$

Now suppose L is another ring, and $(R,\ \mathbf{a},\ \mathbf{i},\ \mathbf{e},\ (\mathbf{s}_d)_{d\in|L|})$ a co-L-module object in K-**Mod**, where R is the underlying K-module, $\mathbf{a}$, $\mathbf{i}$ and $\mathbf{e}$ give the co-abelian-group structure of R, and for each $d\in|L|$, $\mathbf{s}_d$ is the co-operation corresponding to scalar multiplication by d. The co-abelian-group structure will, as we have noted, have the form described in Lemma 9.7.1. The $\mathbf{s}_d$ will be unary co-operations, i.e., K-module homomorphisms $R\to R$, which can thus be looked at as unary *operations* on the set $|R|$. We now need some basic observations:

Exercise 9.7:1. Let R be any K-module, and $\mathbf{a}$, $\mathbf{i}$, $\mathbf{e}$ the coaddition, coinverse and cozero morphisms defining the unique co-**Ab** structure on R in K-**Mod**.

(i) Show that every K-module endomorphism $\mathbf{s}: R\to R$ satisfies the coidentity corresponding to the identity (9.7.3); i.e., show that the unary operation induced by such an $\mathbf{s}$ on each $h_R(A)$ is an abelian group endomorphism.

(ii) Show that such an operation $\mathbf{s}$ induces the identity operation on each $h_R(A)$ (cf. (9.7.6)) if and only if it is the identity endomorphism of R.

(iii) Show that if $\mathbf{s}_d$, $\mathbf{s}_{d'}$ and $\mathbf{s}_{d''}$ are three endomorphisms of R, then the operations on the abelian groups $h_R(A)$ induced by $\mathbf{s}_d$ and $\mathbf{s}_{d'}$ sum to the operation induced by $\mathbf{s}_{d''}$ if and only if $\mathbf{s}_d+\mathbf{s}_{d'}=\mathbf{s}_{d''}$.

(iv) Show likewise that the operation induced by $\mathbf{s}_{d''}$ is the composite in a given order of the operations induced by $\mathbf{s}_d$ and $\mathbf{s}_{d'}$ (cf. (9.7.5)) if and only if $\mathbf{s}_{d''}$ is the composite of $\mathbf{s}_d$ and $\mathbf{s}_{d'}$ in the *opposite* order.

From the above results we deduce that

(9.7.7) If K and L are rings, and R a left K-module, then a co-left-L-module structure on R is equivalent to a system of R-module endomorphisms $(\mathbf{s}_d)_{d\in|L|}$ which for all $d,\ d'\in|L|$ satisfy

(9.7.8) $$\mathbf{s}_1 \;=\; \mathrm{id}_R$$

(9.7.9) $$\mathbf{s}_{d+d'} \;=\; \mathbf{s}_d+\mathbf{s}_{d'}$$

(9.7.10) $$\mathbf{s}_{dd'} \;=\; \mathbf{s}_{d'}\,\mathbf{s}_d.$$

This is a nice result, but we can make it more elegant with a change of notation. The reversal of the order of composition in (9.7.10) can be cured if we write the operators $\mathbf{s}_d$ on the *right* of their arguments, instead of on the left, and compose them accordingly. Moreover, once the operation of elements of L (by co-scalar-multiplications) is written on a different side from the operation of elements of K (by scalar multiplication), there is no real danger of confusion if we drop the symbols $\mathbf{s}$, i.e., replace the above notation $\mathbf{s}_d(x)$ by xd $(x\in|R|,\ d\in|L|)$. We now find that the scalar multiplications by elements of K and the co-scalar-multiplications by elements of L satisfy a very symmetrical set of conditions, namely, that for all $c,\ c'\in|K|$, $x,\ x'\in|R|$, $d,\ d'\in|L|$,

$$(9.7.11)\qquad 1x = x \qquad\qquad x1 = x$$

$$(9.7.12)\qquad c(x+x') = cx+cx' \qquad\qquad (x+x')d = xd+x'd$$

$$(9.7.13)\qquad (c+c')x = cx+c'x \qquad\qquad x(d+d') = xd+xd'$$

$$(9.7.14)\qquad (cc')x = c(c'x) \qquad\qquad x(dd') = (xd)d'$$

$$(9.7.15)\qquad c(xd) = (cx)d$$

Here (9.7.15), and the right hand equation of (9.7.12), say that the co-scalar-multiplications are endomorphisms of the K-module R. The conditions in the left-hand column, together with the identities for the abelian group structure of R, constitute the identities of a left K-module, while the remaining three conditions on the right say that the co-scalar-multiplication endomorphisms behave as required to give a co-left-L-module structure. (Only three such conditions are needed, as against the four on the left, because of Exercise 9.7:1(i).)

We have, in fact, rediscovered a standard concept of ring theory:

Definition 9.7.16. *An abelian group on which a ring K operates by maps written on the left and a ring L operates by maps written on the right so that* (9.7.11)-(9.7.15) *are satisfied is called a* (K, L)-bimodule.

*For given K and L, the variety of (K, L)-bimodules will be denoted K-**Mod**-L.*

Note that given two arbitrary varieties of algebras $\mathbf{V}$ and $\mathbf{W}$, the category of $\mathbf{V}$-coalgebra objects of $\mathbf{W}$ cannot in general be regarded as a variety of algebras, because the co-operations $\mathbf{s}: R \to \coprod_{\mathrm{ari}(s)} R$ do not have the form of maps $|R|^{\beta} \to |R|$, *unless* $\mathrm{ari}(s) = 1$. In the present case, it happened that the two *non-unary* co-operations of our objects, the coaddition and the cozero, were uniquely determined, so that the structure could be defined wholly by unary co-operations, and so, atypically, the category of these coalgebras could be identified with a variety of algebras.

Ring-theorists often write a (K, L)-bimodule R as ${}_K R_L$. Here the subscripts are not part of the ''name'' of the object, but reminders that K operates on the left, and L on the right. (Actually, ring-theorists more often use other letters, such as B, for ''bimodule'', or M, for ''module'', reserving R for rings. But in this chapter we are using R wherever possible for ''representing object''.) That such a bimodule structure makes R a co-L-module in K-**Mod** is equivalent to the result familiar to ring-theorists, that the set of left K-module homomorphisms from a (K, L)-bimodule to a left K-module,

$$(9.7.17)\qquad K\text{-}\mathbf{Mod}({}_K R_L,\ {}_K A)$$

has a natural structure of left L-module. Let us describe how this L-module structure arises without using the language of coalgebras. If we regard the actions of the elements of L on R as K-module endomorphisms, then the functoriality of K-**Mod**$(-,-)$ in its first variable turns these into endomorphisms of the abelian group K-**Mod**$({}_K R,\ {}_K A)$, and since this functoriality is contravariant, the order of composition of these endomorphisms is reversed; so from the right L-module structure on R, we get a left L-module structure on that hom-set. Explicitly, given any $f \in K$-**Mod**(R, A) and $d \in |L|$, the action of d on f in this induced left L-module structure is given by

$$(9.7.18)\qquad (df)(x) = f(xd).$$

This takes a more elegant form if we adopt

(9.7.19) (*Frequent convention in ring theory.*) If possible, write homomorphisms of *left* modules on the *right* of their arguments, and homomorphisms of *right* modules on the *left* of their arguments, and use the notation for composition of such homomorphisms appropriate to the side on which they are written.

This says we should write elements $f \in K\text{-}\mathbf{Mod}(R,\,A)$ on the right of elements $x \in |A|$. When we do so, (9.7.18) takes the form

$$x(df) \;=\; (xd)f. \tag{9.7.20}$$

In summary:

Lemma 9.7.21. *If K and L are unital associative rings, then a $(K,\,L)$-bimodule ${}_K R_L$ is equivalent to a co-L-**Mod** object of K-**Mod**. The left L-module structure on the functor K-**Mod**$(R,\,-)$ is given by the standard abelian group structure on hom-sets, together with the scalar multiplications* (9.7.18)*, or in right-operator notation,* (9.7.20). □

9.8. More on modules: left adjoints of representable functors. Let us now find the left adjoint to the functor induced as above by a $(K,\,L)$-bimodule R. This must take a left L-module B to a left K-module A with a universal left L-module homomorphism

$$h\colon\; B \;\to\; K\text{-}\mathbf{Mod}(R,\,A). \tag{9.8.1}$$

To find this object A, let us apply our standard heuristic approach: We consider an arbitrary left K-module A with an L-module homomorphism (9.8.1), and see what elements of A, and what relations among these elements, this map gives us.

For each $y \in |B|$, (9.8.1) gives a homomorphism $h(y)\colon R \to A$; and such a homomorphism gives us, for each $x \in |R|$, an element of A. With (9.7.19) in mind, let us write this as

$$x{*}y \;=\; x\,h(y) \qquad (x \in |R|,\; y \in |B|).$$

I claim that the conditions that these elements must satisfy are that for all $x, x' \in |R|$, $y, y' \in |B|$, $c \in |K|$, $d \in |L|$,

$$(x+x'){*}y \;=\; x{*}y + x'{*}y \qquad\qquad x{*}(y+y') \;=\; x{*}y + x{*}y' \tag{9.8.2}$$

$$(cx){*}y \;=\; c(x{*}y) \qquad\qquad \text{——} \tag{9.8.3}$$

$$x{*}(dy) \;=\; (xd){*}y. \tag{9.8.4}$$

Indeed, the two equations on the left are the conditions for the maps $h(y)$ to be left K-module homomorphisms, while the equations on the right and at the bottom are the conditions for the map (9.8.1) to be a homomorphism of left L-modules with respect to the given L-module structure on B and the operations (9.7.20) on K-**Mod**$(R,\,A)$. We note the gap on the right-hand side of (9.8.3); since nothing acts on the *right* on the L-module B, there is nothing to put there. (But do not lose heart; this asymmetry will presently repair itself.) So the universal A with a homomorphism (9.8.1) will be presented by generators $x{*}y$ $(x \in |R|,\, y \in |B|)$ and relations (9.8.2)-(9.8.4).

Again we have discovered a standard concept. The K-module presented by this system of generators and relations is denoted

$$R\otimes_L B,$$

and called the *tensor product over* L of the $(K,\,L)$-bimodule R and the left L-module B. The generators of this module corresponding to the $x*y$ of the above discussion are written $x\otimes y$ $(x\in|R|,\ y\in|B|)$.

We reiterate that $R\otimes_L B$ is only a left K-module. Intuitively, when we form the tensor product $({}_K R_L)\otimes_L({}_L B)$, the operation of tensoring over L ‘‘eats up’’ the two L-module structures, leaving the K-module structure. This is dual to the situation of (9.7.17), where the construction of taking the hom-set over K ‘‘eats up’’ the two K-module structures, leaving only the L-module structure.

We have shown:

Lemma 9.8.5. *If* ${}_K R_L$ *is a bimodule, then the left adjoint to the functor*

$$K\text{-}\mathbf{Mod}(R,-):\ K\text{-}\mathbf{Mod}\ \to\ L\text{-}\mathbf{Mod}$$

is the functor

$$R\otimes_L -:\ L\text{-}\mathbf{Mod}\ \to\ K\text{-}\mathbf{Mod}.$$

Thus, given the bimodule R, *a left* K*-module* A, *and a left* L*-module* B, *we have a functorial isomorphism of abelian groups*

$$L\text{-}\mathbf{Mod}(B,\ K\text{-}\mathbf{Mod}(R,A))\ \cong\ K\text{-}\mathbf{Mod}(R\otimes_L B,\ A).\ \ \square$$

An interesting consequence of Lemmas 9.7.21 and 9.8.5 is that every *representable* functor between module categories, and likewise the left adjoint of every such functor, respects **Ab**-structures, i.e., sends sums of morphisms to sums of morphisms. This is not true of general functors between module categories, as the reader can see from the functor $A\mapsto A\otimes A$ of Exercise 9.4:4(i).

In defining the tensor product over L, I said that one presents it as a left K-module using the relations (9.8.2)-(9.8.4). But another standard definition is to present it as an *abelian group* using only the relations corresponding to (9.8.2) and (9.8.4), and then to use (9.8.3) to define a left K-module structure on this group. Not every abelian group with elements $x*y$ $(x\in|R|,\ y\in|B|)$ satisfying (9.8.2) and (9.8.4) has a left K-module structure satisfying (9.8.3); but the *universal* abelian group with these properties does, because the universal construction is functorial in R as a right L-module, and the left K-module structure of R constitutes a system of right-L-module endomorphisms; these induce endomorphisms of the constructed abelian group which make it a K-module.

This approach shows that the underlying abelian group structure of $R\otimes_L B$ depends only on the right L-module structure of R and the left L-module structure of B; this is again analogous to the situation for the hom-set $K\text{-}\mathbf{Mod}({}_K R_L,\ {}_K A)$, which starts out as an abelian group constructed using only the left K-module structures of R and A, and then acquires a left L-module structure from the right L-module structure of R, by functoriality.

We should now learn how to compose the representable functors we have described. Suppose we have three rings, H, K, L, and adjoint pairs determined by an $(H,\,K)$-bimodule R and a $(K,\,L)$-bimodule S:

(9.8.6)
$$\begin{tikzcd}
H\text{-}\mathbf{Mod} \arrow[r, shift left, "H\text{-}\mathbf{Mod}({}_H R_K,\,-)"] & K\text{-}\mathbf{Mod} \arrow[l, shift left, "{}_H R_K\ \otimes_K\ -"] \arrow[r, shift left, "K\text{-}\mathbf{Mod}({}_K S_L,\,-)"] & L\text{-}\mathbf{Mod}. \arrow[l, shift left, "{}_K S_L\ \otimes_L\ -"]
\end{tikzcd}$$

By observations we made in §9.4, the underlying left K-module of the coalgebra determining the composite adjoint pair can be gotten by applying the left adjoint functor $R \otimes_K -$ to the underlying object of the coalgebra determining the other adjoint pair; hence it is the left H-module $R \otimes_K S$. It remains to find the coalgebra structure, i.e., the right L-module structure, on this object; this arises from the right L-module structure on S, by the same ‘‘functoriality’’ effect noted above for the left module structure of $R \otimes_L B$. So the composite of the adjoint pairs shown above is determined by an (H, L)-bimodule ${}_H(R \otimes_K S)_L$.

At this point, we have discussed enough kinds of structure on tensor products so that we are ready to put them all into a definition, after which we will state formally the above characterization of representing objects for composite functors.

Definition 9.8.7. *If K is a ring, R a right K-module and S a left K-module, then*

$$R \otimes_K S$$

will denote the abelian group *presented by generators $x \otimes y$ $(x \in |R|,\ y \in |S|)$ and the relations* (*for all $x, x' \in |R|$, $d \in |K|$, $y, y' \in |S|$*)

(9.8.8) $$(x + x') \otimes y \ = \ x \otimes y + x' \otimes y, \quad x \otimes (y + y') \ = \ x \otimes y + x \otimes y'$$

(9.8.9) $$(xd) \otimes y \ = \ x \otimes (dy).$$

If R is in fact an (H, K)-bimodule, respectively if S is a (K, L)-bimodule, respectively if both are true (*by which we mean, if the right K-module structure of R is given as part of an (H, K)-bimodule structure, and/or if the left K-module structure of S is given as part of a (K, L)-bimodule structure, for some rings H, L*), *then the abelian group $R \otimes_K S$ becomes a left H-module, respectively a right L-module, respectively an (H, L)-bimodule, with scalar multiplications characterized by* (*one or both of*) *the following identities for $c \in |H|$, $e \in |L|$:*

(9.8.10) $$c(x \otimes y) \ = \ (cx) \otimes y \qquad\qquad (x \otimes y)\, e \ = \ x \otimes (y\, e)$$

The four cases of the above definition (tensoring a right module R_K or a bimodule ${}_H R_K$ with a left module ${}_K S$ or a bimodule ${}_K S_L$) reduce to a single case if for every K we identify **Mod**-K with **Z**-**Mod**-K and K-**Mod** with K-**Mod**-**Z**, and likewise identify **Ab** with **Z**-**Mod**-**Z**.

Note that (9.8.10) has restored the symmetry that was missing in (9.8.3)!

Let us now set down the result sketched before this definition.

Lemma 9.8.11. *In the situation shown in* (9.8.6), *the composite of the functors among left module categories represented by bimodules ${}_H R_K$ and ${}_K S_L$ is represented by the* (H, L)*-bimodule $R \otimes_K S$.* □

Terminological note: Given bimodules ${}_H R_K$ and ${}_K S_L$, we may call a map $*$ from $|R| \times |S|$ into an (H, L)-bimodule ${}_H T_L$ satisfying the equations corresponding to (9.8.8)-(9.8.10) a ‘‘bilinear map $R \times S \to T$’’, generalizing the term we have already used in the case of abelian groups (§3.9), so that we may describe $R \otimes_K S$ as an (H, L)-bimodule with a universal bilinear map of these bimodules into it. However, many authors feel that the term ‘‘bilinear’’ should logically only mean ‘‘left H-linear and right L-linear’’, i.e., the conditions of (9.8.8) and (9.8.10), and they use the adjective ‘‘balanced’’ to express (9.8.9). So they would call $R \otimes_K S$ an (H, L)-bimodule with a universal *K-balanced bilinear map* of $R \times S$ into it.

The case of modules and bimodules developed above may be regarded as a ‘‘model case’’ in terms of which to think of the general theory of representable functors among varieties of algebras, and their representing coalgebras. Indeed, Freyd entitled the paper [**8**] in which he introduced the theory of such functors and coalgebras ‘‘*On algebra-valued functors in general, and tensor products in particular*’’, and he called the coalgebra that represents a composite of representable functors between arbitrary varieties of algebras the ‘‘tensor product’’ of the coalgebras representing the given functors. I recommend that paper to the interested student, though with one word of advice: Ignore the roundabout way the author treats zeroary operations, and simply consider them, as we have done, to be morphisms from the empty product in the category to the object in question.

Further remarks: In the paragraph following (9.7.10), we chose a notation which ‘‘separated’’ the actions of elements of K and elements of L, writing them on opposite sides of elements of R. It is also worth seeing what happens if we do not separate them, but continue to write them both to the left of their arguments. The actions of elements of L will then compose in the opposite way to the *multiplication* of those elements in the ring L. This can be thought of as making R a left module over L^{op}, the opposite of the ring L (defined to have the same underlying set and additive group structure as L, but with the order of multiplication reversed). Thus we have on R both a left K-module structure and a left L^{op}-module structure, related by the conditions that the additive group operations of the two module structures are the same, and that the scalar multiplications of the L^{op}-module structure are endomorphisms of the K-module structure. The latter condition says that the images of the elements of K and of elements of L^{op} in the endomorphism ring of the common abelian group R *commute* with one another. Now we saw in §3.13 that given two rings P and Q, if we form the tensor product of their underlying abelian groups, this can be given a ring structure such that the maps $p \mapsto p \otimes 1$ and $q \mapsto 1 \otimes q$ are homomorphisms of P and Q into $P \otimes Q$, whose images commute elementwise, and which is *universal* among rings given with such a pair of homomorphisms from P and Q. Thus, in our present situation, the mutually commuting left K-module and left L^{op}-module structures on R are equivalent to a single structure of left $K \otimes L^{\mathrm{op}}$-module. That is

(9.8.12) $$K\text{-}\mathbf{Mod}\text{-}L \;\cong\; (K \otimes L^{\mathrm{op}})\text{-}\mathbf{Mod}.$$

Hence one can study bimodules with the help of the theory of tensor products of rings, and vice versa.

This also shows us that if we want to study representable functors between categories of *bimodules*, we do not need to undertake a new investigation, but can reduce this situation to the one we have already studied by using rings $K_0 \otimes K_1^{\mathrm{op}}$, etc., in place of K, etc..

Exercise 9.8:1. (i) If you did Exercise 3.13:4(ii), translate the results you got there to a partial or complete description of all $(\mathbf{Q}(2^{1/3}),\ \mathbf{Q}(2^{1/3}))$-bimodules.

(ii) If you did Exercise 3.13:4(i), translate the results you got there to a partial or complete description of all $\mathbf{R}$-centralizing $(\mathbf{C}, \mathbf{C})$-bimodules B, where ‘‘$\mathbf{R}$-centralizing’’ means satisfying the identity $rx = xr$ for all $r \in \mathbf{R}$, $x \in B$.

The student familiar with the theory of modules over *commutative* rings may have been surprised at my saying that when we form a hom-set $K\text{-}\mathbf{Mod}({}_K R,\ {}_K A)$ or a tensor product $R_L \otimes_L {}_L B$, the K-module structure, respectively the L-module structure, is ''eaten up'' in the process; for in the commutative case, these sets inherit natural K- and L-module structures. You can discover the general statement, of which these apparently contradictory observations are cases, by doing the next exercise. (The answer comes out in parts (iii) and (iv).)

Exercise 9.8:2. Let K be a ring (not assumed commutative) and M a left K-module.

(i) Determine the *structure*, in the sense of §8.10, of the functor $K\text{-}\mathbf{Mod}({}_K M,\ -)\colon K\text{-}\mathbf{Mod} \to \mathbf{Set}$.

(ii) Determine similarly the structure of $K\text{-}\mathbf{Mod}(-,\ {}_K M)\colon K\text{-}\mathbf{Mod}^{\mathrm{op}} \to \mathbf{Set}$.

(iii) Determine the structure of $K\text{-}\mathbf{Mod}(-,-)\colon K\text{-}\mathbf{Mod}^{\mathrm{op}} \times K\text{-}\mathbf{Mod} \to \mathbf{Set}$.

(iv) Answer the corresponding three questions for tensor products in place of hom-sets (with or without the help of Corollary 7.11.6).

Let us note another basic ring-theoretic tool that we can understand with the help of the results of this section. Suppose $f\colon L \to K$ is a ring homomorphism. Then we can make any left K-module A into a left L-module by keeping the same abelian group structure, and defining the new scalar multiplication by $d\cdot x = f(d)x$ $(d \in |L|)$. This functor preserves underlying sets, hence it is representable. It is called ''restriction of scalars along f'', and its left adjoint is called ''extension of scalars along f''. (When f is the inclusion of a subring L in a ring K these are obvious terms to use. The usage in the case of arbitrary homomorphisms f is a generalization from that case.) You should find the first part of the next exercise straightforward, and the second not too hard.

Exercise 9.8:3. Let $f\colon L \to K$ be a ring homomorphism.

(i) Describe the bimodule representing the restriction-of-scalars functor, and get a description of the extension-of-scalars construction $K\text{-}\mathbf{Mod} \to L\text{-}\mathbf{Mod}$ as a tensor product operation.

(ii) If K and L are commutative, we may also consider the ''restriction of base-ring'' functor from K-algebras to L-algebras, defined to preserve underlying ring-structures, and act as restriction of scalars on module structures. (You may here take ''algebras'' over K and L either to mean commutative algebras, or not-necessarily commutative (but as always, unless the contrary is stated, associative) algebras, depending on what you are familiar with.) We know this functor is representable. (Why?) Describe its representing coalgebra. Show that the left adjoint of this functor acts on the underlying modules of algebras by extension of scalars. How is the ring structure on the resulting modules defined?

At the beginning of the preceding section, when we determined the form of the general *comonoid* object of $\mathbf{Ab}$, recall that our argument used only the fact that we had a binary co-operation satisfying the coneutral laws with respect to the unique zeroary co-operation – the coassociative law was never needed! Thus, if we let $\mathbf{Binar}^e$ denote the variety of sets with a binary operation and a neutral element e for that operation, then co-$\mathbf{Binar}^e$ objects of $\mathbf{Ab}$ are automatically co-**Monoid** objects, and even co-**AbMonoid** objects, and, as we noted, these have unique coinverse operations making them co-**Group** and co-**Ab** objects.

On the other hand, if we drop the co-neutral-element, the associativity and commutativity conditions do make a difference:

Exercise 9.8:4. Characterize all representable functors from $\mathbf{Ab}$ to each of the following varieties:

(i) **Binar**, the variety of sets given with a binary operation.

(ii) **Semigroup** (a subvariety of **Binar**).

(iii) **AbSemigroup** (a subvariety of **Semigroup**).

In the last two cases, you should discover that every such functor decomposes as a direct sum of a small number of functors whose structures are easily described.

9.9. Some general results on representable functors, mostly negative. As we mentioned in Exercise 9.6:6, the form of the general representable functor **Monoid** → **Semigroup** is not known. What about representable functors going the other way, from **Semigroup** to **Monoid**?

It is easy to show that in this case there are no nontrivial examples. The idea is that in working in **Semigroup**, one has no distinguished elements available, so there is no way to pin down a zeroary ''neutral element'' operation.

Before having you prove this, let me indicate the exception implied in the word ''nontrivial''. If **C** is any category with an initial object I, then the functor h_I takes every object of **C** to a one-element set, which of course has a unique structure of **V**-algebra for every variety **V**; hence for every variety **V**, the object I admits co-operations making it a **V**-coalgebra. Let us call the functor represented by this coalgebra, which takes every object of **C** to the one-element algebra (the terminal object) of **V**, the ''trivial functor'' $\mathbf{C} \to \mathbf{V}$. We can now state

Exercise 9.9:1. Show that if **W** is a variety without zeroary operations, and **V** a variety with at least one zeroary operation, then there is no nontrivial representable functor $\mathbf{W} \to \mathbf{V}$.

More generally, can you give a condition on a general category **C** with finite coproducts that assures that there are no nontrivial representable functors from **C** to a variety with at least one zeroary operation?

Here is another observation about specific varieties from which we can extract a similar general principle. We began this chapter with an example of a representable functor from rings to groups; but if one looks for a representable functor from groups to rings, it is hard to imagine how one might be constructed (aside from the trivial functor), because a nontrivial ring must have distinct 0 and 1, and we have only one distinguished group element, e, to use in the coordinates of a distinguished element of a ring we are constructing. You might like to think about how you would turn this idea into a proof, and then how to abstract what is involved in category-theoretic terms, before reading the next definition and exercise.

Definition 9.9.1. *If* **C** *is a category with a terminal object* T, *let us* (*as in Exercise 6.8:3*) *define a* pointed object *of* **C** *to mean a pair* (A, p) *where* A *is an object of* **C** *and* p *a morphism* $T \to A$. (*Thus, since* T *is the product of the empty family of copies of* A, *this is an object of* **C** *given with a single zeroary operation.*) *A morphism* $(A, p) \to (A', p')$ *of such objects will mean a morphism* $A \to A'$ *making a commuting triangle with* p *and* p'. *The category of pointed objects of* **C**, *with these morphisms, will be denoted* $\mathbf{C}^{\mathrm{pt}}$.

Dually, if **C** *is a category with an initial object* I, *an* augmented object *of* **C** *will mean a pair* (A, a) *where* A *is an object of* **C** *and* a *is a morphism* $A \to I$ (*an ''augmentation map''*), *equivalently, a zeroary co-operation on* A. *Again using the obvious commuting triangles as morphisms, we denote the category of augmented objects of* **C** *by* $\mathbf{C}^{\mathrm{aug}}$.

Thus, in comma category notation, $\mathbf{C}^{\mathrm{pt}} = (T \downarrow \mathbf{C})$, *and* $\mathbf{C}^{\mathrm{aug}} = (\mathbf{C} \downarrow I)$.

A category **C** *will be called ''pointed'' if it has a* zero object (*an object that is both initial and terminal; Definition 6.8.1*).

Exercise 6.8:3 shows that if **C** is a category with a terminal object, then $\mathbf{C}^{\mathrm{pt}}$ is a pointed category. By duality, if **C** is a category with an initial object, then $\mathbf{C}^{\mathrm{aug}}$ is likewise pointed.

The next exercise begins with a few more observations of the same sort, then gets down to business.

Exercise 9.9:2. Prove the following:

(i) Let $\mathbf{C}$ be a category with a terminal (respectively initial) object. Then the forgetful functor $\mathbf{C}^{\mathrm{pt}} \to \mathbf{C}$ (respectively $\mathbf{C}^{\mathrm{aug}} \to \mathbf{C}$) is an equivalence if and only if $\mathbf{C}$ is a pointed category.

(ii) If $\mathbf{V}$ is a variety of algebras, then $\mathbf{V}^{\mathrm{pt}}$ is equivalent to a variety of algebras.

(iii) A variety of algebras $\mathbf{V}$ is a pointed category if and only if $\mathbf{V}$ has at least one zeroary operation, and all derived zeroary operations of $\mathbf{V}$ are equal.

Now suppose $\mathbf{V}$ is a variety, and $\mathbf{C}$ a category having small coproducts.

(iv) Show that

$$\mathbf{Rep}(\mathbf{C}, \mathbf{V}^{\mathrm{pt}}) \approx \mathbf{Rep}(\mathbf{C}^{\mathrm{aug}}, \mathbf{V}^{\mathrm{pt}}) \approx \mathbf{Rep}(\mathbf{C}^{\mathrm{aug}}, \mathbf{V}).$$

(If you don’t see how to begin, you might think first about the case $\mathbf{V} = \mathbf{Set}$.)

(v) Show that **Group** is pointed, and that $(\mathbf{Ring}^1)^{\mathrm{pt}}$ consists only of the trivial ring. Deduce that there are no nontrivial functors from **Group** or any of its subvarieties to $\mathbf{Ring}^1$ or any of its subvarieties (e.g., $\mathbf{CommRing}^1$).

Incidentally, I believe the term ‘‘augmented’’ comes from ring theory, where an ‘‘augmentation’’ on a k-algebra R means a k-algebra homomorphism $\varepsilon: R \to k$. This ring-theoretic concept in turn probably originated in algebraic topology, where the cohomology of a pointed space acquires, by contravariance of the cohomology ring functor, such an augmentation.

Here is yet another sort of nonexistence result:

Exercise 9.9:3. Let R be an object of a variety $\mathbf{V}$, and let $\tau: R^\lambda \amalg R^\rho \to R^\lambda \amalg R^\rho$ denote the automorphism that interchanges x^λ and x^ρ for all $x \in |R|$. Denote by $\mathrm{Sym}(R^\lambda \amalg R^\rho)$ the fixed-point algebra of τ; i.e., the algebra of ‘‘(λ, ρ)-symmetric’’ elements of $R^\lambda \amalg R^\rho$.

(i) Show that a binary co-operation $\mathbf{m}: R \to R^\lambda \amalg R^\rho$ is cocommutative (i.e., satisfies the coidentity making the induced operations on all sets $\mathbf{V}(R, A)$ commutative) if and only if it carries R into $\mathrm{Sym}(R^\lambda \amalg R^\rho)$.

(ii) Show that in the variety **Group**, one has $\mathrm{Sym}(R^\lambda \amalg R^\rho) = \{e\}$ for all R.

(iii) Deduce that there are no nontrivial representable functors **Group** → **Ab**, hence also no nontrivial representable functors $\mathbf{Group} \to \mathbf{Ring}^1$; and that there are no nontrivial representable functors **Group** → **Semilattice**, hence also no nontrivial representable functors **Group** → **Lattice**.

Seeing that there are no nontrivial representable functors from groups to lattices, we may ask, what about functors in the reverse direction? The category **Lattice** has no zeroary operations, so there can be no functors from it or any of its subvarieties to **Group** by Exercise 9.9:1; but we can get out of this hole by considering lattices with one or more distinguished elements. I do not know the answer to the first part of the next exercise, though I do know the answer to the second.

Exercise 9.9:4. (i) Is there a variety $\mathbf{L}$ of lattices for which there exists a nontrivial representable functor $\mathbf{L}^{\mathrm{pt}} \to \mathbf{Group}$?

(ii) For $\mathbf{C}$ a category with a terminal object T, let $\mathbf{C}^{\text{2-pt}}$ denote the category of 3-tuples (A, p_0, p_1) where A is an object of $\mathbf{C}$, and p_0, p_1 are morphisms $T \rightrightarrows C$. Is there any variety of lattices $\mathbf{L}$ for which there exists a nontrivial representable functor $\mathbf{L}^{\text{2-pt}} \to \mathbf{Group}$?

Of course, not every plausible heuristic argument restricting the properties of representable functors is valid. For instance, every primitive operation of lattices, and hence also every derived operation of lattices, is isotone with respect to the natural ordering of the underlying set, while Boolean rings have the operation of complementation, which is not. Nevertheless, we have the construction of the following exercise.

Exercise 9.9:5. Let $\mathbf{DistLat}^{0,1}$ denote the variety of distributive lattices (Exercise 5.1:14) with least element 0 and greatest element 1. An element x of such a lattice L is called *complemented* if there exists $y \in |L|$ such that $x \wedge y = 0$ and $x \vee y = 1$.

Show that for $L \in \mathrm{Ob}(\mathbf{DistLat}^{0,1})$, the set of complemented elements of L can be made a Boolean ring, whose natural partial ordering is the restriction of the natural partial ordering of L, and that this construction yields a representable functor $C: \mathbf{DistLat}^{0,1} \to \mathbf{Bool}$. Give a description of this functor in terms of ‘‘tuples of elements satisfying certain relations’’, and describe the Boolean operations on such tuples.

Here is a triviality question of a different sort.

Exercise 9.9:6. If $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are varieties such that there exist nontrivial representable functors $\mathbf{W} \to \mathbf{V}$ and $\mathbf{V} \to \mathbf{U}$, must there exist a nontrivial representable functor $\mathbf{W} \to \mathbf{U}$?

Let us turn to positive results. We recall from Exercise 7.3:5 that every equivalence of categories is also an adjunction. We deduce

Lemma 9.9.2. *Suppose* γ *is a regular cardinal, and* $\mathbf{C} \underset{j}{\overset{i}{\rightleftarrows}} \mathbf{V}$ *is an equivalence between a category* $\mathbf{C}$ *with* $<\gamma$*-fold coproducts and a variety* $\mathbf{V}$ *of algebras all of whose operations have arities* $<\gamma$. *Then* $i: \mathbf{C} \to \mathbf{V}$ *is representable, and has a representing coalgebra with underlying object* $j(F_{\mathbf{V}}(1))$. □

The above fact is used in [**42**] to study the *self-equivalences* of the variety of rings, and more generally, of the variety of algebras over a commutative ring k. (The self-equivalences of any category $\mathbf{C}$, modulo isomorphism of functors, form a group, called the *automorphism class group* of $\mathbf{C}$. When $\mathbf{C} = \mathbf{Ring}^1$, this group is shown in [**42**] to be isomorphic to $\mathbf{Z}_2$; the nonidentity element arises from the self-equivalence $K \mapsto K^{\mathrm{op}}$. For k a commutative ring, the variety of k-algebras has a more complicated automorphism class group if k has nontrivial idempotents or nontrivial automorphisms.)

Exercise 9.9:7. We saw in Exercise 6.9:18 that for R a ring, the varieties R-**Mod** and $M_n(R)$-**Mod** were equivalent. By the above lemma, the equivalence must be representable. Determine the bimodules that yield this equivalence.

This suggests the question: Given rings K and L and an object R of K-**Mod**-L, under what conditions is the functor K-**Mod** → L-**Mod** represented by R an equivalence? In a future edition of these notes, I hope to add a section introducing *Morita theory*, which answers this question, and to give the generalization of that theory that answers the corresponding question for arbitrary varieties of algebras.

A challenging related problem is

Exercise 9.9:8. Characterize those functors between module categories, $F: K$-**Mod** → L-**Mod**, which have both a left and a right adjoint.

Another useful result is given in

Exercise 9.9:9. Let $\mathbf{C}$ be a category with small colimits, and $\mathbf{V}$ a variety of algebras. Show that the category $\mathbf{Rep}(\mathbf{C}, \mathbf{V})$ is closed under taking small limits within the functor category $\mathbf{V}^{\mathbf{C}}$.

As an example, let n be a positive integer, and consider the functors $\mathrm{GL}(n)$, $\mathrm{GL}(1) \in \mathbf{Rep}(\mathbf{CommRing}^1, \mathbf{Group})$. (Note that $\mathrm{GL}(1)$ is just the ‘‘group of units’’ functor.) We can define morphisms

$$e,\ \det\colon\ \mathrm{GL}(n) \rightrightarrows \mathrm{GL}(1),$$

where the first takes every invertible matrix to 1, and the second takes every invertible matrix to its determinant. By the preceding exercise, the limit (difference kernel) of this diagram of functors and morphisms is representable. This difference kernel is in fact the functor $\mathrm{SL}(n)$, with which we began this chapter.

Exercise 9.9:10. In §9.5 we described the general object of $\mathbf{Rep}(\mathbf{Monoid}, \mathbf{Monoid})$. Find a finite family S of such objects with the property that every object of this category is the limit of a system of objects in S and morphisms among these.

9.10. A few ideas and techniques. In §§9.6-9.8, we considered some cases of the problem, ‘‘Given varieties $\mathbf{V}$ and $\mathbf{W}$, find all representable functors $\mathbf{W} \to \mathbf{V}$’’. We can also turn this around and ask, ‘‘Given an object R of a variety $\mathbf{W}$, what kinds of algebras can we make out of the values of the functor h_R?’’ This question asks for the *structure* on the set-valued functor h_R, in the sense of §8.10, i.e., for the operations admitted by that functor and the identities that they satisfy.

I gave some examples of this question in Exercise 8.10:3; we can now see what you probably discovered (without having terminology in which to state it precisely) if you did that exercise: that to find the operations on such a functor and the identities they satisfy, one needs to look for the co-operations admitted by its representing object, and the coidentities satisfied by these.

Let us work out an example here.

Suppose we are interested in the algebraic structure one can put, in a functorial way, on the set of *elements of exponent* 2 in a general group G. This means we want to study the structure on the functor taking G to the set of such elements, i.e., the set-valued functor represented by the group $\mathbf{Z}_2$; so our task is equivalent to describing the clone of co-operations admitted by $\mathbf{Z}_2$ in $\mathbf{Group}$.

An n-ary co-operation on $\mathbf{Z}_2$ means a group homomorphism $\mathbf{Z}_2 \to \mathbf{Z}_2 \amalg \ldots \amalg \mathbf{Z}_2$, and hence corresponds to an element of exponent 2 in the latter group. Though in $\mathbf{Z}_2$ one usually uses additive notation, these coproduct groups are noncommutative, so let us write $\mathbf{Z}_2$ multiplicatively, calling the identity element e and the nonidentity element t. Then the coproduct of n copies of $\mathbf{Z}_2$ will be generated by elements $t_0, \ldots, t_{n-1}$ of exponent 2, and (by the observations in §3.6 on the structure of coproduct groups), the general element of this coproduct can be written uniquely

$$(9.10.1)\qquad t_{\alpha_0} t_{\alpha_1} \ldots t_{\alpha_{h-1}}, \ \text{ where } h \geq 0, \ \text{ all } \alpha_i \in n, \ \text{ and } \alpha_i \neq \alpha_{i+1} \ \text{ for } 0 \leq i < h-1.$$

Let us begin by seeing what structure on $h_{\mathbf{Z}_2}$ is apparent to the naked eye, and translating it into the above terms. Since the identity element of every group is of exponent 2, $h_{\mathbf{Z}_2}$ admits

$$(9.10.2)\qquad \begin{array}{c}\text{the zeroary operation } e, \text{ determined by} \\ \text{the unique homomorphism } \mathbf{Z}_2 \to \{e\}.\end{array}$$

Also, any conjugate of an element of exponent 2 has exponent 2, hence $h_{\mathbf{Z}_2}$ admits

(9.10.3) the binary operation $(x,y) \mapsto x^y = y^{-1}xy = yxy$, determined by the homomorphism $\mathbf{Z}_2 \to \mathbf{Z}_2 \amalg \mathbf{Z}_2$ taking t to $t_1\, t_0\, t_1$.

To see whether these generate all functorial operations on $h_{\mathbf{Z}_2}$, let us consider a general element (9.10.1) of the n-fold coproduct of copies of $\mathbf{Z}_2$. If (9.10.1) has exponent 2, all factors must cancel when we square it, which we see means that we must have $\alpha_0 = \alpha_{h-1}$, $\alpha_1 = \alpha_{h-2}$, etc.. If h is even and positive, this gives, in particular, $\alpha_{(h/2)-1} = \alpha_{h/2}$, which contradicts the final condition of (9.10.1). Hence the only element (9.10.1) of even length having exponent 2 is e. This element induces the constant n-ary operation e, and we see that for each n, this is a derived operation of the zeroary operation (9.10.2).

On the other hand, for $h = 2k+1$, we see that (9.10.1) will have exponent 2 if and only if it has the form

(9.10.4) $$t_{\alpha_0} \cdots\, t_{\alpha_{k-1}}\, t_{\alpha_k}\, t_{\alpha_{k-1}} \cdots\, t_{\alpha_0} \quad \text{with } k \geq 0, \text{ and } \alpha_i \neq \alpha_{i+1} \text{ for } 0 \leq i < k.$$

The operation that this induces can clearly be expressed in terms of the operation (9.10.3); it is

(9.10.5) $$(x_0, \ldots, x_n) \mapsto (\ldots((x_{\alpha_k})^{x_{\alpha_{k-1}}})^{x_{\alpha_{k-2}}} \ldots)^{x_{\alpha_0}}.$$

So the operations (9.10.2) and (9.10.3) do indeed generate the clone of operations on $h_{\mathbf{Z}_2}$.

How can we find a generating set for the identities these operations satisfy? This is shown in

Exercise 9.10:1. Note that the set of all terms in the two operations (9.10.2) and (9.10.3) includes terms not of the form e or (9.10.5); e.g., e^{x_0}, $x_0^{(x_1^{x_2})}$, $(x_0^{x_1})^{x_1}$. (The last is not of the form (9.10.5) because it fails to satisfy the condition on the α's inherited from (9.10.4)).

(i) For each of these *terms*, show how the resulting *derived operation* of $h_{\mathbf{Z}_2}$ can be expressed either as e or in the form (9.10.5), and extract from each such observation an identity satisfied by (9.10.2) and (9.10.3). Do the same with other such terms, until you can show that you have enough identities to reduce every term in (9.10.2) and (9.10.3) either to e or to the form (9.10.5).

(ii) Deduce that all identities of the operations (9.10.2) and (9.10.3) of $h_{\mathbf{Z}_2}$ are consequences of the identities in your list.

Exercise 9.10:2. Let $\mathbf{V}$ denote the variety defined by a zeroary operation e and a binary operation $(-)^{-}$, subject to the identities of our two operations on $h_{\mathbf{Z}_2}$, and let V: $\mathbf{Group} \to \mathbf{V}$ be the functor represented by $\mathbf{Z}_2$ with the co-operations defined above. Is every object of $\mathbf{V}$ embeddable in an object of the form $V(G)$ for G a group? Translate your answer (or if you don't get an answer, translate the question) into a property or question concerning the functor V and its left adjoint.

Exercise 9.10:3. Analyze similarly the structure of $h_{\mathbf{Z}_n}$ for a general positive integer n.

Let me present, next, an interesting problem which, though not obviously related to the concepts of this chapter, turns out, like the question examined above, to be approachable by studying the structure on a functor.

If y is an element of a group G, recall that the map $x \mapsto y^{-1}xy$ is an *automorphism* of G, and that an automorphism that has this form (for some $y \in |G|$) is called an *inner* automorphism. Now suppose one is handed an automorphism $\alpha \in \mathbf{Group}(G,\, G)$. Is it possible to say whether α is inner, using only its properties within the category $\mathbf{Group}$, i.e., conditions statable in terms of objects and morphisms, without reference to the ''internal'' nature of the objects?

Well, observe that if α is an inner automorphism of G, induced as above by an element $y\in|G|$, then given any homomorphism h from G to another group H, there exists an automorphism of H which forms a commuting square with α and h; namely, the inner automorphism of H induced by $h(y)$. In fact, this construction associates to every such pair (H, h) an automorphism $\alpha_{(H,\,h)}$ in a ''coherent manner'', in the sense that given two such pairs (H_0, h_0) and (H_1, h_1), and a morphism $f: H_0 \to H_1$ such that $h_1 = f h_0$, the automorphisms $\alpha_{(H_0,\,h_0)}$ of H_0 and $\alpha_{(H_1,\,h_1)}$ of H_1 form a commuting square with f.

I claim, conversely, that any automorphism α of a group G which can be ''extended coherently'', in the above sense, to all groups H with maps of G into them, is inner. The next exercise formalizes this ''coherence'' property in a general category-theoretic setting, then asks you to prove this characterization of inner automorphisms of groups.

Exercise 9.10:4. Given an object C of a category $\mathbf{C}$, let $(C\downarrow\mathbf{C})$ denote the category whose objects are pairs (D, d), $(D\in\mathrm{Ob}(\mathbf{C}),\ d\in\mathbf{C}(C, D))$ and where morphisms $(D_0, d_0) \to (D_1, d_1)$ are morphisms $D_0 \to D_1$ making commuting triangles with d_0 and d_1. (Cf. Exercise 6.8:24.) Let $U_C: (C\downarrow\mathbf{C}) \to \mathbf{C}$ denote the forgetful functor sending (D, d) to D.

Call an endomorphism α of the object C ''functorializable'' if there exists an endomorphism a of the forgetful functor U_C which, when applied to the initial object of $(C\downarrow\mathbf{C})$, namely (C, id_C), yields α.

Show that for $\mathbf{C} = \mathbf{Group}$, an automorphism of an object G is functorializable if and only if it is an inner automorphism. In fact, determine the monoid of endomorphisms of $U_G: (G\downarrow\mathbf{Group}) \to \mathbf{Group}$ and its image in the monoid of endomorphisms of G.

(Some related questions you can also look at: How does the above group compare with the group of automorphisms of the *identity* functor of $(G\downarrow\mathbf{Group})$? Can you characterize functorializable endomorphisms of objects of other interesting varieties?)

On to another topic. The next exercise is unexpectedly hard (unless there is a trick I haven't found), but is interesting.

Exercise 9.10:5. Let $\mathbf{V}$ and $\mathbf{W}$ be varieties of algebras (finitary if you wish). Show that the category $\mathbf{Rep}(\mathbf{V}, \mathbf{W})$ has an *initial object*.

The next exercise develops some results and examples regarding these initial representable functors.

Exercise 9.10:6. Suppose we classify varieties into three sorts: (a) those with no zeroary operations, (b) those with a unique derived zeroary operation, and (c) those with more than one derived zeroary operation. Applying this classification to the varieties $\mathbf{V}$ and $\mathbf{W}$ in the preceding exercise, we get nine cases.

(i) Show that in *most of* these cases, the initial object of $\mathbf{Rep}(\mathbf{V}, \mathbf{W})$ must be trivial, in the weak sense that it takes every object A *either* to the one-element algebra *or* to the empty algebra.

(ii) Determine the initial object of $\mathbf{Rep}(\mathbf{Set}, \mathbf{Semigroup})$.

(iii) Determine the initial object of $\mathbf{Rep}(\mathbf{Set}, \mathbf{Binar})$, where $\mathbf{Binar}$ is the variety of sets with a single (unrestricted) binary operation.

(iv) Interpret the result of Exercise 8.3:9 as describing the initial object of $\mathbf{Rep}(\mathbf{Binar}, \mathbf{Binar})$.

(v) The three preceding examples all belong to the same one of the nine cases referred to in (i). Give an example belonging to a different case, in which $\mathbf{Rep}(\mathbf{V}, \mathbf{W})$ also has nontrivial initial object.

When I first learned about the concept of ''coidentities'' in coalgebra objects of a category $\mathbf{C}$, I was a little disappointed that the possible coidentities merely corresponded to the identities of set-based algebras of the same type – I thought it would have been be more interesting if this ‘‘exotic’’ version of the concept of algebra led to an ‘‘exotic’’ concept of identity as well. But perhaps there is still hope for something exotic, if the question is posed differently. Recall that in §8.6 we characterized varieties of algebras as those classes of algebras that were closed under three operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$.

Exercise 9.10:7. Define analogs of the operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ for classes of objects of $\mathbf{Rep}(\mathbf{C},\,\Omega\textbf{-Alg})$. Presumably, for every variety $\mathbf{V}$ of Ω-algebras, $\mathbf{Rep}(\mathbf{C},\,\mathbf{V})$ will be closed in $\mathbf{Rep}(\mathbf{C},\,\Omega\textbf{-Alg})$ under your operators; but will these be the only closed classes?

If not, try to characterize the classes closed under your operators (possibly assuming some restrictions on $\mathbf{C}$ and Ω).

9.11. Contravariant representable functors.

In §9.2 we defined the concept of an *algebra*-object of a category, but we immediately passed from this to that of a *coalgebra* object in §9.3, and showed that a covariant functor has an adjoint if and only if it is represented by such an object. Let us now look at the version of this result for algebra objects, and the contravariant functors these represent. We recall from §7.12 that a contravariant adjunction involves a pair of *mutually right adjoint* or of *mutually left adjoint* functors. Putting ‘‘$\mathbf{C}^{\mathrm{op}}$’’ in place of ‘‘$\mathbf{C}$’’ in Theorem 9.3.6, we get

Theorem 9.11.1. *Let* $\mathbf{C}$ *be a category with small limits,* $\mathbf{V}$ *a variety of algebras, and* $V\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ *a contravariant functor. Then the following conditions are equivalent*:

(i) V *has a right adjoint* $W\colon \mathbf{V}^{\mathrm{op}} \to \mathbf{C}$ (*so that* V *and* W *form a pair of mutually right adjoint contravariant functors*).

(ii) $V\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ *is representable, i.e., is isomorphic to* $\mathbf{C}(-,\,R)$ *for some* $\mathbf{V}$*-algebra object* R *of* $\mathbf{C}$ (*Definition 9.2.8*).

(iii) *The composite of* V *with the underlying-set functor* $U_{\mathbf{V}}\colon \mathbf{V} \to \mathbf{Set}$ *is representable, i.e., is isomorphic to* $h^{|R|} = \mathbf{C}(-,\,|R|)$ *for some object* $|R|$ *of* $\mathbf{C}$. □

Now suppose that in the above situation we take for $\mathbf{C}$ another variety of algebras, $\mathbf{W}$; what will a $\mathbf{V}$-object of $\mathbf{W}$ look like? Its $\mathbf{V}$-operations will be $\mathbf{W}$-algebra homomorphisms $t_R\colon |R|^{\mathrm{ari}(t)} \to |R|$; that is, set maps $\|R\|^{\mathrm{ari}(t)} \to \|R\|$ which respect the $\mathbf{W}$-operations of $|R|$. Let us write down the condition for an n-ary operation t on a set to ‘‘respect’’ an m-ary operation s:

$$\begin{aligned} &s(t(x_{0,0},\dots,x_{0,n-1}),\ \dots,\ t(x_{m-1,0},\dots,x_{m-1,n-1}))\\ &\quad = \ t(s(x_{0,0},\dots,x_{m-1,0}),\ \dots,\ s(x_{0,n-1},\dots,x_{m-1,n-1})).\end{aligned}$$

The above equation assumes the arities m and n are natural numbers. For operations of arbitrary arity, the condition may be written

$$s((t(x_{ij})_{j\in\mathrm{ari}(t)})_{i\in\mathrm{ari}(s)}) \ = \ t((s(x_{ij})_{i\in\mathrm{ari}(s)})_{j\in\mathrm{ari}(t)}). \tag{9.11.2}$$

Note that this condition is symmetric in s and t, and that when s and t are both *unary*, it says that $s(t(x)) = t(s(x))$, i.e., that as elements of the monoid of set maps $\|R\| \to \|R\|$, s and t commute. Generalizing this term, one calls operations s and t of arbitrary arities which satisfy (9.11.2) *commuting* operations. This condition is equivalent to commutativity of the diagram

$$\begin{tikzcd}
\|R\|^{\mathrm{ari}(s)\times\mathrm{ari}(t)} \arrow[r, "t^{\mathrm{ari}(s)}"] \arrow[d, "s^{\mathrm{ari}(t)}"] & \|R\|^{\mathrm{ari}(s)} \arrow[d, "s"] \\
\|R\|^{\mathrm{ari}(t)} \arrow[r, "t"] & \|R\|.
\end{tikzcd}$$

To get some feel for this concept, you might do

Exercise 9.11:1. (i) Show that two zeroary operations commute if and only if they are equal. More generally, when will an n-ary operation s commute with a zeroary operation t?

(ii) Verify that every zeroary or unary operation on a set commutes with itself.

(iii) Show that not every binary operation s on a set X commutes with itself. In fact, consider the following four conditions on a binary operation s: (a) s commutes with itself, (b) s satisfies the commutative identity $s(x,y) = s(y,x)$, (c) s satisfies the associative identity $s(s(x,y),z) = s(x,s(y,z))$, and (d) there exists a neutral element $e\in X$ for s, i.e., an element satisfying the identities $s(x,e) = x = s(e,x)$. Determine which of the 16 possible combinations of truth values for these conditions can be realized.

Summarize your results as one or more implications which hold among these conditions, and such that any combination of truth-values consistent with those implications can be realized.

We see that if $\mathbf{V}$ is a variety of Ω-algebras and $\mathbf{W}$ a variety of Ω'-algebras, then a $\mathbf{V}$-algebra object of $\mathbf{W}$ is equivalent to a set-based algebra $R = (|R|,\ (s_R)_{s\in|\Omega'|\sqcup|\Omega|}),$ where the operations indexed by $|\Omega'|$, respectively, $|\Omega|$, are of the arities specified in Ω', respectively, Ω, and satisfy the identities of $\mathbf{W}$, respectively $\mathbf{V}$, and where, moreover, for every $s\in|\Omega'|$ and $t\in|\Omega|$, the commutativity identity (9.11.2) is satisfied. Since all these conditions are identities, the category of such objects forms a variety!

Given such an object R, and an ordinary object A of $\mathbf{W}$, we see that the operations of the $\mathbf{V}$-algebra $\mathbf{W}(A,\ R)$ are given by ‘‘pointwise’’ application of the $\mathbf{V}$-operations of R to $\mathbf{W}$-homomorphisms $A \to R$. (In general, if A and B are objects of a variety $\mathbf{W}$ and one combines a family of algebra homomorphisms $f_\alpha \colon A \to B$ $(\alpha\in\beta)$ by pointwise application of a β-ary operation t on the set $|B|$, the result is not a homomorphism of $\mathbf{W}$-algebras. What makes this true here is the fact that t is an operation on R *as an object of the category* $\mathbf{W}$, i.e., that it commutes with all the $\mathbf{W}$-operations.)

Since the functor $\mathbf{W}(-,\ R)\colon \mathbf{W}^{\mathrm{op}} \to \mathbf{V}$ belongs to a *mutually* right adjoint pair, its adjoint will also satisfy condition (i) of Theorem 9.11.1, and hence the other two equivalent conditions; that is, this adjoint will *also* be a representable contravariant functor, but going the other way, $\mathbf{V}^{\mathrm{op}} \to \mathbf{W}$. As the next exercise shows, the representing object for this functor is gotten by very slightly modifying the representing object R for the original functor.

Exercise 9.11:2. Let $V\colon \mathbf{W}^{\mathrm{op}} \to \mathbf{V}$ be a representable contravariant functor, whose representing $\mathbf{V}$-algebra object R is, in the above formulation $(|R|,\ (s_R)_{s\in|\Omega'|\sqcup|\Omega|})$. Show that the right adjoint to V is the functor $\mathbf{V}(-,\ R')$, where R' has the same underlying set as R, and the same operations, but with the roles of the $\mathbf{W}$-operations and the $\mathbf{V}$-operations as ‘‘primary’’ and ‘‘secondary’’ interchanged, so that it becomes a $\mathbf{W}$-algebra object of $\mathbf{V}$.

A basic contrast between *covariant* and *contravariant* representable functors on a variety $\mathbf{W}$ is that the former, as we saw in §9.3, define their operations *using* derived operations of $\mathbf{W}$, while the objects representing the latter have operations that must *commute* with those of $\mathbf{W}$. A consequence is that, generally speaking, the ‘‘richer’’ the structure of $\mathbf{W}$, the richer is the class of covariant representable functors on $\mathbf{W}$, and the scarcer are the contravariant representable functors. Hence, for a case in which it should be particularly easy to get contravariant representable functors, let us look at the variety with the *least* family of operations, namely **Set**.

A **V**-algebra object of **Set** is in fact just an ordinary **V**-algebra. Let us take the smallest nontrivial object in **Set**, and find the richest algebra structure we can put on it, and the functor this represents.

Exercise 9.11:3. (i) Show that the clone of all finitary operations on the object $2 = \{0,1\}$ of **Set** can be described as the clone of derived operations of the *ring* $\mathbf{Z}_2$, and that this is isomorphic to the clone of operations of the variety $\mathbf{Bool}^1$.

(ii) Describe the contravariant adjunction between **Set** and $\mathbf{Bool}^1$ determined by this $\mathbf{Bool}^1$-structure on 2.

As an interesting sideline,

(iii) Regarding $\mathbf{Bool}^1$ as the variety generated by the 2-element Boolean ring, obtain a cardinality-bound for the free Boolean ring on n generators by considerations analogous to those applied to the free group on 3 generators in $\mathbf{Var}(S_3)$ in the discussion leading up to Exercise 2.3:2. If you did that exercise and Exercise 3.14:1, compare these two cases with respect to how close the resulting bounds are to the actual cardinalities of these free algebras.

More generally,

Exercise 9.11:4. For n any integer >1, let $\mathbf{X}^{[n]}$ denote the clone of all finitary operations on the set $n = \{0, \ldots, n-1\}$.

(i) Show that if p is a prime, $\mathbf{X}^{[p]}$ can be described as the clone of derived operations of the ring $\mathbf{Z}_p$. Show moreover that the variety $\mathbf{X}^{[p]}$**-Alg**, regarded as a subvariety of $\mathbf{CommRing}^1$, is equivalent to $\mathbf{Bool}^1$ by the ‘‘Boolean ring of idempotent elements’’ functor (Exercise 3.14:3). Describe the functor going the other way.

(ii) Show that if n is not a prime, then $\mathbf{X}^{[n]}$**-Alg** does not coincide with the clone of derived operations of the ring $\mathbf{Z}_n$.

(iii) For n not a prime, is it still true that $\mathbf{X}^{[n]}$**-Alg** is equivalent to $\mathbf{Bool}^1$?

Let us look next at a contravariant representable algebra-valued functor on a category **C** other than a variety of algebras, which nonetheless has properties very similar to the those of the functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Bool}^1$ considered above.

Exercise 9.11:5. In the category **POSet** of partially ordered sets and isotone maps, let 2 denote the object with underlying set $\{0,1\}$, ordered so that $0 < 1$.

(i) Show that the finitary structure on this object of **POSet**, i.e., the clone of all operations $2^n \to 2$ that are morphisms of **POSet**, is a structure of *distributive lattice* (Exercise 5.1:14) with *least* element 0 and *greatest* element 1. Describe the resulting functor $\mathbf{POSet}^{\mathrm{op}} \to \mathbf{DistLat}^{0,1}$.

(You will need to know the form that products take in **POSet**; for this see Definition 4.1.4. Note also that it will be natural to speak of 2 as having a ‘‘structure of partially ordered set’’; but you should beware confusion with Lawvere’s technical sense of ‘‘structure’’, i.e., the operations which an object admits, as in (i) above.)

(ii) Verify that **POSet** has small limits, so that Theorem 9.11.1 is applicable to this functor.

(iii) Show that the adjoint to this functor, a functor $(\mathbf{DistLat}^{0,1})^{\mathrm{op}} \to \mathbf{POSet}$, can be characterized as taking every object of $\mathbf{DistLat}^{0,1}$ to the set of its morphisms into the object 2, with the partial ordering on 2 being used to get a partial ordering on the set of morphisms. (Cf. Exercise 6.6:5.)

(iv) Suppose instead that we consider $2 = \{0,1\}$ as an object of $\mathbf{POSet}^{0,1}$, the category whose objects are partially ordered sets with least and greatest elements, and whose morphisms are the isotone maps that respect those elements. Show that the structure on 2 in this category

leads to a contravariant right adjunction with the variety **DistLat**.
What if you start with $\mathbf{POSet}^0$ or $\mathbf{POSet}^1$?

Can a contravariant representable functor give a contravariant *equivalence* between varieties, i.e., an equivalence between one variety and the opposite of another? This is addressed in the next exercise.

Exercise 9.11:6. Let us call a variety ''nontrivial'' if it does not satisfy the identity $x = y$.

(i) Find a condition on categories which is invariant under equivalence of categories, and is satisfied by all nontrivial varieties, but is not satisfied by the *opposite* of any nontrivial variety. (Essentially, any condition on categories that does not refer to how many isomorphic copies an object has will be invariant under equivalence. What is hard is finding one that distinguishes between varieties and their opposites. I know some ways to do this, but they are not obvious. Perhaps you can find a more natural one. If you wish, take ''variety'' to mean ''finitary variety''.)

(ii) Deduce that there can exist no contravariant equivalences (representable or not) between nontrivial varieties.

However, some of the contravariant representable functors considered above come surprisingly close to being equivalences; namely, when restricted to the *finitely generated* objects of one category, they yield finitely generated objects of the other, and give equivalences between these subcategories of finitely generated objects. In the case of duality of vector spaces, this is a category-theoretic translation of some well-known facts of linear algebra. In the cases of Boolean rings (Exercise 9.11:3) and of distributive lattices (Exercise 9.11:5), the results in question are likewise translations of classical fundamental results about these two kinds of object ([**4**, §III.3]; cf. also Exercise 6.9:17 above). For the variety **Ab** the functor $\mathbf{Ab}(-, \mathbf{Q}/\mathbf{Z})$ is a self-duality on the category of finite (though not on the category of finitely generated) abelian groups (see [**21**, §4.6], noting the comment after *ibid.* Theorem 6.2).

It turns out, moreover, that the above dualities on finite objects can be extended to equivalences between *all* objects of one variety and certain *topologized* objects of the other. The reader interested in learning about a large class of such results might look at the interesting paper [**33**] (though the results there are not stated in category-theoretic language). The result on $\mathbf{Ab}(-, \mathbf{Q}/\mathbf{Z})$ does not fall within the scope of that article, but it, too, has a generalization to topological abelian groups, the theory of *Pontryagin duality* of locally compact abelian groups [**93**]. The topological approach to duality of not necessarily finite-dimensional vector spaces is implicit in Exercises 5.5:5 and 7.5:16. A new book on dualities, which I have not yet had a chance to look at, is [**51**].

Exercise 9.11:7. (i) Show from Exercise 3.14:5 that our functors connecting $\mathbf{Bool}^1$ and **Set** do indeed induce a contravariant equivalence between the subcategories of finite objects.

(ii) Deduce that if **V** is any variety of finitary algebras, and A a finite object of **V**, then there exists a **V**-*coalgebra* object R of $\mathbf{Bool}^1$ such that $\mathbf{Bool}^1(R, 2) \cong A$.

If you or the class succeeded in characterizing derived operations of the ''majority vote function'' M_3 on $\{0,1\}$ in Exercise 1.7:1, you can now try:

Exercise 9.11:8. (i) Can you find some structure (in the nontechnical sense, i.e., not necessarily given by operations!) on $\{0,1\}$, such that the clone of operations generated by the majority vote function M_3 is precisely the clone of finitary operations respecting that structure?

(ii) Can you prove a duality result, to the effect that the set $\{0,1\}$, with this structure on the one hand, and with the operation M_3 on the other, induces an adjunction, which, when restricted to finite objects, gives a contravariant equivalence between finite algebras in the variety generated by $(\{0,1\}, M_3)$, and finite objects of an appropriate category?

I have not thought hard about the following question:

Exercise 9.11:9. Suppose $\mathbf{V}$ and $\mathbf{W}$ are varieties, and we have a contravariant equivalence between their subcategories of finite (finitely generated? finitely presented?) objects. Will this necessarily be the restriction of a pair of mutually right adjoint representable functors between all of $\mathbf{V}$ and all of $\mathbf{W}$?

What can we say about *composition* of contravariant representable functors? We know that for adjoint pairs of *covariant* functors

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "U"] & \mathbf{D} \arrow[l, shift left, "F"] \arrow[r, shift left, "V"] & \mathbf{E}, \arrow[l, shift left, "G"] \end{tikzcd}$$

the composites $\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "VU"] & \mathbf{E} \arrow[l, shift left, "FG"] \end{tikzcd}$ are also adjoint; so let us look at the results we get on replacing some subset of the three categories $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ in this result by their opposites. This will give 8 statements, saying that composites of certain combinations of covariant adjoint pairs, contravariant right adjoint pairs, and contravariant left adjoint pairs are again adjoint pairs of one sort or another.

These statements will break into pairs of statements which have the same translations after some relabeling, because Theorem 7.3.5 itself is invariant under replacing all three categories by their opposites and interchanging the roles of $\mathbf{C}$ and $\mathbf{E}$. Of the resulting four statements, one is, of course, the original Theorem 7.3.5. Two of the others involve contravariant *left* adjunctions, of which, as I have mentioned, there are no interesting cases among varieties of algebras [**39**]. I state the one remaining case as the next corollary. In that corollary, for a functor between arbitrary categories, $A: \mathbf{C} \to \mathbf{D}$, the ‘‘same’’ functor regarded as going from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}^{\mathrm{op}}$ is written A^{op} (though for most purposes, it is safe to write this A).

Corollary 9.11.3 (to Theorem 7.3.5). *Suppose*

$$\begin{tikzcd} \mathbf{C}^{\mathrm{op}} \arrow[r, "V"] & \mathbf{D} \\ \mathbf{C} & \mathbf{D}^{\mathrm{op}} \arrow[l, "V'"'] \end{tikzcd}$$

is a pair of mutually right adjoint contravariant functors, and

$$\begin{tikzcd} \mathbf{D} \arrow[r, shift left, "U"] & \mathbf{E} \arrow[l, shift left, "F"] \end{tikzcd}$$

a pair of covariant adjoint functors (U *the right adjoint and* F *the left adjoint*). *Then the composite functors* UV *and* $V'F^{\mathrm{op}}$ (*in less discriminating notation,* $V'F$)*:*

$$\begin{tikzcd} \mathbf{C}^{\mathrm{op}} \arrow[r, "V"] & \mathbf{D} \arrow[r, "U"] & \mathbf{E} \\ \mathbf{C} & \mathbf{D}^{\mathrm{op}} \arrow[l, "V'"'] & \mathbf{E}^{\mathrm{op}} \arrow[l, "F^{\mathrm{op}}"'] \end{tikzcd}$$

are also mutually right adjoint contravariant functors.

In particular, the class of contravariant functors admitting right adjoints is closed under postcomposition with right adjoint covariant functors, and under precomposition with left adjoint covariant functors. □

Exercise 9.11:10. (i) Derive the above result from Theorem 7.3.5, and also derive the two other statements mentioned which involve contravariant *left* adjunctions.

(ii) Give a (nontrivial) example of Corollary 9.11.3, verifying directly the adjointness.

Exercise 9.11:11. Suppose in the context of the above corollary that $\mathbf{C}$ and $\mathbf{E}$ are both varieties of algebras. Thus the pair of mutually right adjoint functors UV and $V'F$ are induced by some object with commuting $\mathbf{C}$- and $\mathbf{E}$-algebra structures. Describe this object and its $\mathbf{C}$- and $\mathbf{E}$-algebra structures in terms of the representing objects R and S for the given functors $V\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ and $U\colon \mathbf{D} \to \mathbf{E}$.

Corollary 9.11.3 does *not* say anything about a composite of two contravariant representable functors. This will be a covariant functor, but as the first part of the next exercise shows, it need not have an adjoint on either side.

Exercise 9.11:12. (i) Let K be a field, and $V\colon (K\text{-}\mathbf{Mod})^{\mathrm{op}} \to K\text{-}\mathbf{Mod}$ the contravariant representable functor taking each K-vector space to its dual. Show that the composite of V with itself, VV, or more accurately, VV^{op}, a covariant functor $K\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$, has no left or right adjoint.

(ii) Show by examples that the class of representable contravariant functors between varieties is closed neither under precomposition with right adjoint covariant functors nor under postcomposition with left adjoint covariant functors.

The ‘‘double dual’’ functor of part (i) above *does* belong to a class of functors which have interesting properties, namely, composites of functors (covariant or contravariant) with their own adjoints. I hope to develop some of these properties in the next chapter, when I have time to write it.

We have mentioned the principle that the richer the structure of a variety of algebras, the more covariant representable functors it admits, and the fewer contravariant representable functors, and we then looked at contravariant representable functors on the variety with the least algebraic structure. In the opposite direction, rings have a particularly rich structure, and the next exercise shows that they are quite poor when it comes to contravariant representable functors.

Exercise 9.11:13. Let R be a nonzero ring (commutative if you wish).

(i) Show that if R has no zero divisors, then any finitary operation $R^n \to R$ can be expressed as a composite $a\,p_{i,n}$ where a is an endomorphism of R, and $p_{i,n}$ is the ith projection map on R^n. Deduce that any clone of finitary operations on R as an object of $\mathbf{Ring}^1$ or of $\mathbf{CommRing}^1$ is generated by unary operations.

(ii) Can you generalize these observations to a wider class of rings than those without zero divisors?

(iii) Choose a simple example of a ring R with zero divisors for which the conclusion of (i) fails, and see whether you can describe the clone of operations on that ring.

9.12. More on commuting operations.

We have seen that for varieties $\mathbf{V}$ and $\mathbf{W}$, the $\mathbf{V}$-algebra objects of $\mathbf{W}$ correspond to sets given with two families of operations which commute with one another in the sense of (9.11.2). Let us look further at this concept of commuting operations.

Lemma 9.12.1. *If s is an operation on a set A, then the set of operations on A which commute with s forms a clone.*

Idea of Proof. Given a family of operations on A for which the map $s\colon A^{\mathrm{ari}(s)} \to A$ is a homomorphism, it will clearly be a homomorphism for all derived operations of that family. □

Exercise 9.12:1. Give a detailed proof of the above lemma. Remember that proving a set of operations to be a clone includes proving that it contains the projections maps.

Definition 9.12.2. *If s is an operation on a set A, then the clone of operations on A which commute with s will be called the* centralizer *of s. If S is a set of operations on A, the intersection of the centralizers of these operations will be called the centralizer of S.*

If C is a clone of operations on A and S a set of operations on A (which may or may not be contained in C), then the intersection of C with the centralizer of S will be called the centralizer of S in C. *The centralizer of C in C will be called the* center *of C. A clone which is its own center will be called* commutative.

Let us fix a notation for a construction we defined in the preceding section.

Definition 9.12.3. *If Ω and Ω' are types, then $\Omega \sqcup \Omega'$ will denote the type whose set of operation-symbols is $|\Omega| \sqcup |\Omega'|$, and where the arity function on this set is induced by the arity functions of Ω and Ω'.*

If $\mathbf{V}$ and $\mathbf{W}$ are varieties of algebras of types Ω and Ω' respectively, then the variety of algebras of type $\Omega' \sqcup \Omega$ such that the operations of Ω satisfy the identities of $\mathbf{V}$, the operations of Ω' satisfy identities of $\mathbf{W}$, and all $\mathbf{V}$-operations commute with all $\mathbf{W}$-operations, will be denoted $\mathbf{V} \bigcirc \mathbf{W}$.

Note that in the above definition, $\mathbf{V}$ and $\mathbf{W}$ are specified as *varieties*, i.e., in terms of given primitive operations. However, if we are not interested in distinguishing ‘‘primitive’’ from ‘‘derived’’ operations, e.g., if we are interested in varieties as categories of representations of given clonal categories, the above construction ‘‘$\bigcirc$’’ also induces a construction on these, since by Lemma 9.12.1, the property that two sets of primitive operations commute is equivalent to the property that their sets of derived operations commute. Likewise, if we consider varieties merely to be a certain class of *concrete categories*, ‘‘$\bigcirc$’’ yields a construction on these, since the ‘‘Structure’’ functor of §8.10 allows us to recover their clones of operations from these concrete categories, and so apply the preceding observation. Finally, if we are interested in varieties only up to equivalence as categories, without reference to concretization (e.g., if we are not interested in distinguishing the varieties K-**Mod** and $M_n(K)$-**Mod**), then $\mathbf{V} \bigcirc \mathbf{W}$ is also determined up to equivalence on these, namely, as the category of contravariant right adjunctions between $\mathbf{V}$ and $\mathbf{W}$.

(Freyd introduces essentially the concept we have called $\mathbf{V} \bigcirc \mathbf{W}$ in [**8**, pp.93-95], but rather than naming the resulting variety, he names its clonal theory $T_1 \otimes T_2$, where T_1 and T_2 are the clonal theories of the given varieties. We have made the opposite choice so as to minimize the dependence of this chapter on the view of a variety as the category of representations of a clonal theory.)

In the case of covariant representable functors, we found that certain *differences* between two varieties $\mathbf{V}$ and $\mathbf{W}$ regarding the number of derived zeroary operations led to restrictions on representable functors between these varieties. For contravariant functors, on the other hand, it is when both varieties *have* such operations that one gets a restriction:

Lemma 9.12.4 ([**8**, p.94]). *Suppose $\mathbf{V}$ and $\mathbf{W}$ are varieties of algebras, each having at least one zeroary operation. Then in $\mathbf{V} \bigcirc \mathbf{W}$, all derived zeroary operations of $\mathbf{V}$ and all derived zeroary operations of $\mathbf{W}$ fall together, and the result is the unique derived zeroary operation of $\mathbf{V} \bigcirc \mathbf{W}$, which thus defines a one-element subalgebra of every $\mathbf{V} \bigcirc \mathbf{W}$-object.*

Proof. The fact that each derived zeroary operation coming from $\mathbf{V}$ commutes with each derived zeroary operation coming from $\mathbf{W}$ means that each of the former is equal to each of the latter (Exercise 9.11:1(i)). Hence, as both these families are nonempty, all of these derived zeroary operations are equal. Since zeroary operations from $\mathbf{V}$ commute with arbitrary operations from $\mathbf{W}$ and vice versa, the resulting zeroary operation of $\mathbf{V}\bigcirc\mathbf{W}$ is central. It is easy to verify that this means that it defines a one-element subalgebra of every algebra, equivalently, is the unique derived zeroary operation of $\mathbf{V}\bigcirc\mathbf{W}$. □

Exercise 9.12:2. Deduce from the above lemma that if $\mathbf{V}$ is a variety having at least one zeroary operation, then the variety $\mathbf{V}\bigcirc\mathbf{Ring}^1$ is trivial; equivalently, that there is no nontrivial contravariant representable functor $\mathbf{V}^{\mathrm{op}} \to \mathbf{Ring}^1$ or $(\mathbf{Ring}^1)^{\mathrm{op}} \to \mathbf{V}$. (So, for instance, there is no nontrivial contravariant representable functor $(\mathbf{Ring}^1)^{\mathrm{op}} \to \mathbf{Ring}^1$.)

It is not only zeroary operations that are forced to fall together when one applies ‘‘$\bigcirc$’’. The next result shows the same for associative binary operations with neutral element.

Lemma 9.12.5 ([**8**, p.94])**.** *Suppose* $\mathbf{V}$ *and* $\mathbf{W}$ *are varieties of algebras, each having at least one derived binary operation with a neutral zeroary operation. Then in* $\mathbf{V}\bigcirc\mathbf{W}$, *the operations induced by all such binary operations of* $\mathbf{V}$ *and all such binary operations of* $\mathbf{W}$ *fall together, and give the unique binary operation with neutral element in this clone. The resulting operation with neutral element is a structure of abelian monoid, and is central in the clone of operations of* $\mathbf{V}\bigcirc\mathbf{W}$.

Proof. We shall show that if in any variety a binary operation $*$ with a neutral element and a binary operation $\circ$ with a neutral element commute, and their neutral elements likewise commute, then $* = \circ$, and their common value satisfies the commutative and associative identities. The remaining assertions will follow as in the proof of the preceding lemma.

The neutral elements of $*$ and $\circ$, being commuting zeroary operations, are equal; let us write e for their common value. We now write down several cases of the commutativity of $*$ with $\circ$. The equation $(x*e)\circ(e*y) = (x\circ e)*(e\circ y)$ reduces to $x\circ y = x*y$, proving equality of the two operations. On the other hand, $(e*x)\circ(y*e) = (e\circ y)*(x\circ e)$ reduces to $x\circ y = y*x$, so the common value of $*$ and $\circ$ is abelian. Finally, $(x*y)\circ(e*z) = (x\circ e)*(y\circ z)$ yields associativity. □

The above result fails without the assumption that *both* binary operations have a neutral element. E.g., the variety $\mathbf{Set}$ has the binary ‘‘derived operation’’ $p_{2,0}$ (projection of an ordered pair on its first component); but it is easy to show that for every variety $\mathbf{V}$, one has $\mathbf{V}\bigcirc\mathbf{Set} \cong \mathbf{V}$; so a binary operation of $\mathbf{V}$ with neutral element is not forced in $\mathbf{V}\bigcirc\mathbf{Set}$ to become associative, or commutative, or to fall together with $p_{2,0}$.

Recall that we denote the variety of algebras with a single binary operation with neutral element by $\mathbf{Binar}^e$.

Corollary 9.12.6. *If each of* $\mathbf{V}$ *and* $\mathbf{W}$ *is one of* $\mathbf{Binar}^e$, $\mathbf{Monoid}$, $\mathbf{AbBinar}^e$ *or* $\mathbf{AbMonoid}$, *then* $\mathbf{V}\bigcirc\mathbf{W} \cong \mathbf{AbMonoid}$.

Proof. Applying the preceding lemma, we see that the given zeroary and binary operations of $\mathbf{V}$ and $\mathbf{W}$ fall together in $\mathbf{V}\bigcirc\mathbf{W}$ to give a single zeroary and a single binary operation that generate the clone of operations of $\mathbf{V}\bigcirc\mathbf{W}$ and satisfy the identities of $\mathbf{AbMonoid}$. To show that $\mathbf{V}\bigcirc\mathbf{W}$

satisfies no other identities, it suffices to note that the multiplication and neutral element of **AbMonoid** satisfy all the identities of **V** and of **W** (clear in each case), and commute with themselves and one another (a quick calculation). □

The above corollary shows that the representing object for any *contravariant representable functor* between any two of the varieties listed is, up to notational adjustment, an abelian monoid.

The next result concerns the case where our abelian monoid structures are in fact abelian group structures. Given a binary derived operation $*$ of a variety having a neutral element e, a *left*, respectively *right* inverse operation for $*$ will mean a unary operation ι satisfying the identity $\iota(x)*x = e$, respectively $x*\iota(x) = e$.

Theorem 9.12.7 (cf. [**8**, p.95])**.** *Suppose* **V** *and* **W** *are varieties of algebras, each having at least one binary operation with a neutral element, and such that at least one such operation of* **V** or *of* **W** *has a right or left inverse operation* ι. *Then in* $\mathbf{V}\bigcirc\mathbf{W}$, ι *becomes a* 2*-sided inverse to the unique* **AbMonoid** *operation of this variety, making this an* **Ab** *structure, again central in the clone of operations.*

Moreover, any clone of operations admitting a homomorphism of the clone of operations of **Ab** *into its center is, up to isomorphism, the clone of operations of a variety* K-**Mod**, *where* K *is the set of unary operations of the clone, made a ring in a natural way.* □

Exercise 9.12:3. Prove the above theorem, with the help of previous results.

Where the above result characterizes clones with a central image of **Ab**, Freyd [**8**, p.95] gives the analogous characterization of clones with a central image of **AbMonoid**, with ‘‘half-ring’’ in place of ring. (The term ‘‘half-ring’’ is not standard. He presumably means an abelian monoid given with a bilinear multiplication having a neutral element 1; the more common term would be ‘‘semiring with 0 and 1’’. A module over such a semiring K means an abelian monoid R with a 0- and 1-respecting homomorphism of K into its semiring of endomorphisms.)

Exercise 9.12:4. (i) Show that $\mathbf{Group}\bigcirc\mathbf{Group} \cong \mathbf{Ab}$. Translate this result into a description of all representable functors $\mathbf{Group}^{\mathrm{op}} \to \mathbf{Group}$.

(ii) Your proof of (i) should also show that $\mathbf{Ab}\bigcirc\mathbf{Ab} \cong \mathbf{Ab}$. Thus, every abelian group yields a contravariant right adjunction between **Ab** and **Ab**. Describe the functors involved, and express the universal property of the adjunction as a certain bijection of hom-sets.

Exercise 9.12:5. (i) If K, L are rings, describe $(K\text{-}\mathbf{Mod})\bigcirc(L\text{-}\mathbf{Mod})$, and determine the general form of a representable contravariant functor $K\text{-}\mathbf{Mod} \to L\text{-}\mathbf{Mod}$.

(ii) Bring the above result into conformity with (9.7.19) by turning it into a characterization of representable contravariant functors $K\text{-}\mathbf{Mod} \to \mathbf{Mod}\text{-}L$. Write the associated contravariant right adjunctions as functorial isomorphisms of hom-sets.

(iii) If K is any ring, the natural (K,K)-bimodule structure of $|K|$ induces, via the result of (ii), a functor $(K\text{-}\mathbf{Mod})^{\mathrm{op}} \to \mathbf{Mod}\text{-}K$. Describe this functor, and show that in the case where K is a field, it is ordinary ‘‘duality of vector spaces’’.

(iv) Given any pair of contravariant mutually right adjoint functors among categories, $U\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$, $V\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{C}$, one has universal maps $\mathrm{Id}_{\mathbf{C}} \to VU$, $\mathrm{Id}_{\mathbf{D}} \to UV$. Determine these in case (iii) above.

Here is an interesting way of getting sets with two mutually commuting algebra structures.

Lemma 9.12.8. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be varieties of algebras in which all operations have arities less than some regular cardinal* γ, *let* $\mathbf{C}$ *be any category having* $<\gamma$*-fold products and* $<\gamma$*-fold coproducts, and let* R *and* S *be a* $\mathbf{V}$-coalgebra *object and a* $\mathbf{W}$-algebra *object of* $\mathbf{C}$ *respectively. Then* $\mathbf{C}(|R|, |S|)$ *has a natural structure of* $\mathbf{V}\bigcirc\mathbf{W}$*-algebra* (*which we may denote* $\mathbf{C}(R, S)$). □

Exercise 9.12:6. (i) Prove the above lemma.

(ii) If you are familiar with basic algebraic topology, deduce from the lemma and Theorem 9.12.7 that the fundamental group of any topological group is abelian.

(You will first need to verify that a topological group induces a group object of $\mathbf{HtpTop}^{(\mathrm{pt})}$ (see Exercise 9.3:1). The key fact to use is that the forgetful functor $\mathbf{Top}^{\mathrm{pt}} \to \mathbf{HtpTop}^{(\mathrm{pt})}$ respects products.)

In fact, the method of part (ii) above shows that all $\mathbf{Binar}^{e}$-objects of $\mathbf{HtpTop}^{(\mathrm{pt})}$ (called ‘‘H-spaces’’ by topologists) have abelian fundamental group. For a brute force proof see [**64**, Proposition II.11.4, p.81].

Exercise 9.12:7. Describe $\mathbf{Heap}\bigcirc\mathbf{Heap}$. (Hint: If A is a nonempty object of $\mathbf{Heap}\bigcirc\mathbf{Heap}$, show that any choice of a zeroary operation allows one to regard A as an object of $\mathbf{Group}\bigcirc\mathbf{Group}$.)

Generalize your result if possible; i.e., show that conditions weaker than the heap identities are enough to force two commuting ternary operations on a set to coincide, and to satisfy the identities you established for $\mathbf{Heap}\bigcirc\mathbf{Heap}$.

Exercise 9.12:8. Recall that **Semilattice** denotes the variety of sets with a single idempotent commutative associative binary operation.

(i) Show that in $\mathbf{Semilattice}\bigcirc\mathbf{Semilattice}$, the two binary operations fall together.

(ii) Deduce that $\mathbf{Semilattice}\bigcirc\mathbf{Lattice}$ and $\mathbf{Lattice}\bigcirc\mathbf{Lattice}$ are trivial.

(iii) Show that $\mathbf{Semilattice}\bigcirc\mathbf{AbMonoid} \cong \mathbf{Semilattice}^{0}$, the variety of semilattices with neutral element (which we are writing as a least element, arbitrarily interpreting the semilattice operation as ‘‘join’’).

(iv) Again, can you get a similar result using a smaller set of identities than the full identities of **Semilattice**?

In this section we have seen several parallel results; let us put in abstract form what they involve.

Exercise 9.12:9. Let **CommClone** denote the full subcategory of **Clone** consisting of all *commutative* clonal categories (Definition 9.12.2). Show that a variety $\mathbf{V}$ is idempotent under $\bigcirc$, in the sense that the two natural maps $\mathbf{V} \to \mathbf{V}\bigcirc\mathbf{V}$ are isomorphisms, if and only if its clone of operations is commutative, and is an epimorph of the initial object in **CommClone**.

It would be interesting to investigate and perhaps try to determine all varieties with the above property. The epimorphs of the clone of operations of **Ab** in **CommClone** are the clones of operations of the varieties K-**Mod** for all epimorphs K of $\mathbf{Z}$ in $\mathbf{CommRing}^{1}$. For a nice classification of these rings K (of which there are uncountably many) see [**53**]. More generally, if K is a semiring with 0 and 1 (cf. paragraph following Exercise 9.12:3) which is an epimorph of the semiring $\mathbf{N}$ of natural numbers in the category of such semirings, then the clonal theory of the variety of K-modules is an epimorph of the clonal theory of **AbMonoid**. Clonal theories of this sort include those arising as described above from epimorphs of $\mathbf{Z}$ (essentially because $\mathbf{Z}$ is an epimorph of $\mathbf{N}$ in the semiring category).

In most of the results in this section that yielded varieties $\mathbf{V}$ with the equivalent properties of the above exercise, we also found larger classes of varieties, generally not commutative, whose

$\bigcirc$-products with themselves and each other gave $\mathbf{V}$. I don't know what is going on here, but a minor question this suggests (to which I also don't know the answer) is

Exercise 9.12:10. If $\mathbf{V}$ is a variety such that the clonal theory of $\mathbf{V}\bigcirc\mathbf{V}\bigcirc\mathbf{V}$ is commutative, must the clonal theory of $\mathbf{V}\bigcirc\mathbf{V}$ also be commutative?

On an easier note, recall from Exercise 6.9:5 that the *monoid of endomorphisms* of the identity functor of any category was commutative. This generalizes to

Lemma 9.12.9. *If* $\mathbf{C}$ *is a category with finite products, then the* clone of operations *of the identity functor of* $\mathbf{C}$ *is commutative.* □

Exercise 9.12:11. Prove the above lemma.

9.13. Some further reading. Covariant representable functors among varieties of algebras are studied extensively in [**2**]. Indeed, §§9.1-9.4 above were adapted from the introductory sections of [**2**], and §§9.5-9.6 from a couple of later sections. Most of [**2**] deals with representable functors on varieties of associative and commutative rings; for the former case, the representable functors to many other varieties are precisely determined. Thus, [**2**] may be considered a natural sequel to this chapter. Many open questions are noted there. (The notation, language, and viewpoint of [**2**] are close to those of these notes. One difference is that where we here use the word ‘‘monoid’’, in that work we say ‘‘semigroup with neutral element’’, and write the variety of those objects $\mathbf{Semigp}^{e}$.)

I sketched some of the material I hope to include in Chapter 10 many years ago in [**1**]. I can give you a reprint of that if you are interested. Some further ideas that may be included in that chapter are found in [**2**, §§63-64].

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Word and phrase index

I have tried to include in this index not only the location where each term is defined, but also all significant occurrences of the concepts in question; but it has not been easy to decide which occurrences are significant. I would welcome your observations on the types of cases you would find it useful to have in the index, and on any entries that are erroneous, unnecessary, or missing.

Pages where terms are defined or where conventions relating to them are set are shown with boldface page numbers. (Sometimes a formal definition occurs after the first page of discussion of a topic, and sometimes more than one version of a concept is defined, leading to occasional entries such as 100-**101**-115, **130**-140.) At some future time, I may try to provide other information in similar ways: e.g., perhaps small type for brief tangential references, a raised dot after a page-number to signal the approximate height on the page at which a term occurs, etc.. I would be interested in your thoughts as to what information you would find useful.

I have generally grouped specific concepts under more general ones. Thus, ‘‘dihedral group’’ appears under ‘‘group’’, not ‘‘dihedral’’. But when a subtopic would have a large number of entries relating to it, it is often made a separate heading. E.g., ‘‘abelian groups’’ has its own heading; although items relating to ‘‘commutative rings’’ are shown under ‘‘rings’’, since we don’t discuss as many facets of that subtopic.

When two broad concepts intersect, it is hard to give a rule as to where the intersection is indexed. Concepts studied in this course that cut across various types of algebras outrank the specific types of algebras; so, ‘‘free group’’ appears under ‘‘free’’. Occasionally, I reference the same topic under more than one heading.

The great majority of the entries in this index concern algebraic topics. Most of the remainder concern foundations and logic, topology, or meta-topics such as ‘‘heuristics’’ and ‘‘open questions’’. For convenience, the handful of entries relating to still other subjects are grouped under the heading ‘‘miscellaneous areas’’.

Terms used by other authors for which different words are used in this work are, if referenced, put in single quotes; e.g., ‘free product’, for what we call ‘‘coproduct’’.

In secondary headings, the words of the main heading are abbreviated ‘‘–’’. In cross-references, the form ‘‘*see* main-heading: subheading’’ is used. Hence ‘‘*see* –: subheading’’ refers you to another subheading under the same main heading. Cross-references are often abbreviated using ‘‘ ... ’’, while ‘‘etc.’’ means ‘‘and similar topics’’. When a cross-reference is given simply as ‘‘*see* main-heading: ...’’, the ‘‘...’’, points to a subheading with the same first word(s) as the present heading (ignoring initial ‘‘–’’s, if any).

Symbol Index

As in the word index, boldface page-numbers indicate pages where definitions are given. Symbols of standard and uncontroversial usage are generally not included here.

If a symbol is defined in one place and used again without explanation more than a page or so away, I show the page(s) where it is defined, and often some of the pages where it is used or where the entity it symbolizes is discussed. But I do not attempt, as in the word index, to show all significant occurrences of each subject. For this, the word index, with its headings and subheadings, is more useful.

Order of entries: Under each letter of the alphabet, the lower-case letter is followed by the upper-case letter, then Greek and miscellaneous related symbols, in a somewhat arbitrary order. (For a particularly complicated example, the order I have set up under p is p, P, π, Π, $\prod$, $\coprod$, φ, Φ, ψ, Ψ, though not all of these symbols actually occur.) Symbols that are not even approximately alphabetical are alphabetized by assigning them spellings; e.g., $\wedge$ and $\vee$ are alphabetized as meet and join; the symbol $=$, and related symbols such as $\cong$, are alphabetized, in an arbitrary order, under equal; and $\mapsto$ and $\downarrow$ are similarly alphabetized under arrow. Fortunately, you do not have to know all the details, though some symbols will require more search than others.

Font-differences, and ‘‘punctuation’’ such as brackets, do not affect ordering unless everything else is equal. Operator-symbols are often shown in combination with letters with which they are commonly used, e.g., $<X \mid Y>$ is alphabetized under XY.

Of the many categories that are given names in Chapter 6 and subsequently, we do not record cases where the meaning is obvious, like **Group**, nor cases discussed only in passing, like **GermAnal** (germs of analytic functions, which can be found under ‘‘functions’’ in the word index), but only category-names used in more than one place for which some aspect of the definition (e.g., the associativity assumption in $\mathbf{Ring}^1$) or the abbreviation (as with **Ab**) is not obvious.

List of Exercises

I hope the telegraphic descriptions given below will help you recall in most cases roughly what the exercises were. When they don't, you can always look back to the page in question (shown at the left). If any of the descriptions are incorrect, or if you think of a more sugegstive wording to briefly describe some exercise, let me know.

Errata to Edition/Printing 2.1 of

An Invitation to General Algebra and Universal Constructions

by George M. Bergman

Explanation of ‘‘Edition/Printing’’: I am retroactively calling the many partial versions of these notes handed out to my classes in early years Edition/Printings 0.1, 0.2, etc., the two Berkeley Lecture Notes versions of 1995, Edition/Printings 1.1 and 1.2, and the current version, published with Henry Helson in 1998, Edition/Printing 2.1. Starting with Edition/Printing 2.2, which should appear some time in 2000, the cover will show the Edition/Printing number.

In the lists below, I ignore minor corrections (corrections in font, centering of displays, etc.).

A. Non-obvious important corrections.

P. 33, displayed equation in Exercise 2.4:2: The ‘‘outermost’’ superscripts on the three factors should be $y^{\pm 1}$, $x^{\pm 1}$, $z^{\pm 1}$, in that order.

P. 108, 2nd sentence of 3rd paragraph: The union α of a set S of ordinals will be $\geq$ each member of the set, so to get an ordinal $>$ all members of S, we should take the *successor* of α, i.e., $\alpha \cup \{\alpha\}$. (The assertion of the sentence as it stands, with S a putative ‘‘set of all ordinals’’, could be justified by further argument; but the above is simplest.)

P. 185, Exercise 6.8:10: This should be moved to p.206, after Exercise 7.2:4, since ‘‘representable functor’’ hasn’t been defined yet on p.185.

P. 257, first displayed equation: This should be $i(f) = (V \circ ((\varepsilon_{U,F})(f \circ U)))\, \eta_{V,G} \circ U$.

P. 291, Exercise 8.7:2: Part (ii) should be deleted. (I don’t now see a workable argument of the sort I asked for.)

P. 292, first sentence of longest paragraph: This result does not require that we topologize R and restrict attention to continuous derivations. (Edition/Printing 2.2 will contain an exercise proving the assertion, in this generality.)

P. 293, 4 lines above display: After ‘‘identify’’ add ‘‘in a C^{∞} manner’’.

P. 300, last sentence of Lemma 8.7.9: Move this assertion (and the parenthetical explanation of ‘‘$\mathbf{X}_{\mathbf{V}}$’’ in the preceding sentence) to the end of the preceding paragraph, reworded ‘‘*In this situation* $\mathbf{X}_{\mathbf{V}}$ (*see first paragraph of Definition 8.9.6*) *is naturally isomorphic in* **Clone** *to* **X**’’. (The ‘‘$\mathbf{X}_{\mathbf{X\text{-}Alg}}$’’ of the present wording is undefined, since **X-Alg**, though equivalent to the variety **V**, is not itself a variety.)

P. 306, two lines after Exercise 8.10:1: ‘‘clonal category’’ should be ‘‘concrete category’’.

P. 308, Exercise 8.10:6: Part (ii) is incorrect in the context of this exercise, which allows varieties of infinitary algebras. (In Edition/Printing 2.2, part (ii) will be made a separate exercise, asking the reader to show that **Clone** is not equivalent to a variety of *finitary* algebras, and asking whether this remains so without the restriction ‘‘finitary’’.)

P. 327, display just after commutative diagram: Both occurrences of $\mathbf{e}$ should be $i_R\,\mathbf{e}$, where i_R denotes the unique map from the initial object I to R.

P. 360, first line of second paragraph of Theorem 9.12.7: ‘‘*clone of operations*’’ should be ‘‘*clone of finitary operations*’’.

B. Obvious, and less important corrections.

P. 12, next-to-last line of Exercise 1.2:2: X should be $|X|$.

P. 17, line after Exercise 1.6:1: $\to G$ should be $\to |G|$.

P. 28, 5th line from bottom: $\in G$ should be $\in |G|$.

P. 34, Exercise 2.4:3: The reference to Proposition 2.4.5(ii) should be to Lemma 2.4.2(ii).

P. 54, lines 8, 9 and 13 from bottom: The references to (i) and (ii) should be to (a) and (b).

P. 55, middle of page, line before (b): G should be G_1.

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P. 89, second line of Exercise 4.1:7: The reference to Exercise 4.1:4(i) should be to Exercise 4.1:4(ii).

P. 92, paragraph before §4.2: The journal *Order* no longer regularly contains a list of open questions.

P. 189, first two displays: The symbols ‘‘1_X’’ etc. should everywhere be ‘‘id_X’’ etc., in accordance with our general notation. Also, in the second diagram the 1_X should be id_Y, not id_X.

P. 194, last line of Exercise 6.9:6: F should be G.

P. 195, Lemma 6.9.3(c), second line: $\mathbf{C} \to \mathbf{D}$ should be $\mathbf{D} \to \mathbf{C}$.

P. 231, Definition 7.6.1: The two occurrences of ‘‘$(X \in \mathrm{Ob}(\mathbf{C}))$’’, in the second and third paragraphs, should both be ‘‘$(X \in \mathrm{Ob}(\mathbf{D}))$’’.

P. 259, paragraph before Exercise 7.12:1: This should refer not to **Lattice** but to $\mathbf{Lattice}^{0,1}$, the variety of lattices with least and greatest element.

P. 274, last line of exercise begun on preceding page: After ‘‘but that’’ add ‘‘if S has more than one point’’.

P. 291, the two lines preceding Definition 8.7.8: The translation of the Jacobi identity as saying that ‘‘the commutator bracket operation is a derivation with respect to itself’’ should say ‘‘the commutator bracket operation with one variable fixed is a derivation with respect to the commutator bracket operation as a function of two variables’’.

P. 292, Exercise 8.7:3, next-to-last line: ‘‘(satisfies (8.7.5))’’ should be ‘‘(satisfies $x*y + y*x = 0$)’’.

P. 317, last line of Definition 9.2.8: $\mathbf{X}_\Omega\text{-}\mathbf{Alg}$ should be $\mathbf{X}_{\Omega\text{-}\mathbf{Alg}}$.

P. 318, end of line after last display: $\mathbf{C}(|R|, -)$ should be $\mathbf{W}(|R|, -)$.

P. 320, line 5: $\mathbf{V}$ should be $\mathbf{W}$.

P. 327, bottom line of displayed diagram: 1_R should be id_R.

P. 332, paragraph following Exercise 9.5:1: It is asserted that the desired property is equivalent to the unit of the adjunction being an isomorphism; but to establish this would require a nontrivial argument. However, it is easy to see that showing the unit to be an isomorphism will *imply* the desired statement.

P. 348, Lemma 9.9.2: The hypothesis that $\mathbf{C}$ has γ-fold coproducts is not needed, since any category equivalent to a variety has small colimits. Hence mention of γ can be dropped from the lemma.

P. 351, final parenthetical paragraph of Exercise 9.10:4: The words ‘‘How does the above group compare with the group of automorphisms’’ should be ‘‘How does the above monoid compare with the monoid of endomorphisms’’.

P. 360, 2nd and 4th lines of Exercise 9.12:5: To conform with our general notational conventions, ‘‘representable contravariant functor $K\text{-}\mathbf{Mod} \to L\text{-}\mathbf{Mod}$’’ should read ‘‘representable functor $(K\text{-}\mathbf{Mod})^{\mathrm{op}} \to L\text{-}\mathbf{Mod}$’’.

Pp. 363, 365: There are two errors in the alphabetical ordering of references: **4** should come before **3**, and **51** should come before **48**.