

*\mathcal{I} -polynomial data types:
adjunctions, equations, and theories*

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Peter Padawitz, TU Dortmund, Germany

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\mathcal{I} -polynomial types

Let S be a finite set and \mathcal{I} be a set of nonempty sets (of indices), implicitly including the one-element set $1 = \{\epsilon\}$, the two-element set $2 = \{0, 1\}$ and the n -element set $[n] = \{1, \dots, n\}$ for all $n > 1$.

The set $\mathcal{T}(S, \mathcal{I})$ of \mathcal{I} -polynomial types over S is inductively defined as follows:

- $S \cup \mathcal{I} \subseteq \mathcal{T}(S, \mathcal{I})$.
- For all $I \in \mathcal{I}$ and $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$, $\coprod_{i \in I} e_i, \prod_{i \in I} e_i \in \mathcal{T}(S, \mathcal{I})$.

For alle $I \in \mathcal{I}$, $n > 1$ and $e, e_1, \dots, e_n \in \mathcal{T}(S, \mathcal{I})$ we use the following short notations:

$$\begin{aligned} e_1 \times \cdots \times e_n &=_{def} \prod_{i \in [n]} e_i, \\ e_1 + \cdots + e_n &=_{def} \coprod_{i \in [n]} e_i, \\ e^I &=_{def} \prod_{i \in I} e, \\ e^n &=_{def} e^{[n]}, \\ e^+ &=_{def} e + \coprod_{n > 1} e^n, \\ e^* &=_{def} 1 + e^+. \end{aligned}$$

Signatures

A **signature** $\Sigma = (S, \mathcal{I}, F)$ consists of sets S and \mathcal{I} as above and a finite set F of typed function symbols (“operations”) $f : e \rightarrow e'$ with $e, e' \in \mathcal{T}(S, \mathcal{I})$.

$f : e \rightarrow e' \in F$ is a **constructor** if $e' \in S$ and a **destructor** if $e \in S$.

Σ is **constructive** if F consists of constructors and for all $s \in S$, \mathcal{I} implicitly contains $\{s\}$ and $\{f \in F \mid \text{ran}(f) = s\}$.

Σ is **destructive** if F consists of destructors and for all $s \in S$, \mathcal{I} implicitly contains $\{s\}$ and $\{f \in F \mid \text{dom}(f) = s\}$.

Terms and coterms

$A \dashrightarrow B$ denotes the set of partial functions from A to B .

$L \subseteq A^*$ is **prefix closed** if for all $w \in A^*$ and $a \in A$, $wa \in L$ implies $w \in L$.

A **deterministic tree** is a partial function $f : A^* \dashrightarrow B$ with prefix closed domain.

f may be written as a kind of record:

$$t_f = f(\epsilon)\{x \rightarrow t_{\lambda w.f(xw)} \mid x \in \text{def}(t) \cap A\}.$$

f is **well-founded** if there is $n \in \mathbb{N}$ with $|w| \leq n$ for all $w \in \text{def}(t)$, intuitively: all paths emanating from the root are finite.

$\text{dtr}(A, B)$ denotes the set of all deterministic trees from A^* to B .

$\text{wdtr}(A, B)$ denotes the set of all wellfounded trees of $\text{dtr}(A, B)$.

Let $\Sigma = (S, \mathcal{I}, F)$ be a signature, V be an S -sorted set,

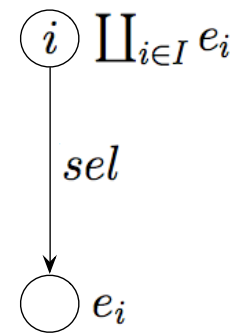
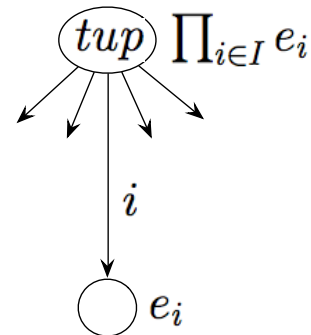
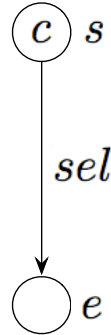
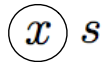
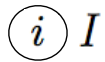
$$EL_\Sigma = \bigcup \mathcal{I} \cup \{sel\}, \quad (\text{edge labels})$$

$$NL_{\Sigma, V} = \bigcup \mathcal{I} \cup V \cup \{tup\}. \quad (\text{node labels})$$

Let Σ be *constructive*.

The set $CT_{\Sigma}(V)$ Σ -terms over V is the *greatest* $\mathcal{T}(S, \mathcal{I})$ -sorted set M of subsets of $dtr(EL_{\Sigma}, NL_{\Sigma, V})$ with the following properties: Let $I \in \mathcal{I}$ and $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$.

- $M_I = I$. (1)
- For all $s \in S$ and $t \in M_s, t \in V_s$ (2)
 or $t = c\{sel \rightarrow t'\}$ for some $c : e \rightarrow s \in F$ and $t' \in M_e$. (3)
- For all $t \in M_{\prod_{i \in I} e_i}$ and $i \in I, t = tup\{i \rightarrow t_i \mid i \in I\}$ for some $t_i \in M_{e_i}$. (4)
- For all $t \in M_{\coprod_{i \in I} e_i}, t = i\{sel \rightarrow t'\}$ for some $i \in I$ and $t' \in M_{e_i}$. (5)



(1)

(2/6)

(3/7)

(4)

(5)

$i \in I$
 $I \in \mathcal{I}$

$s \in S$
 $x \in V_s$

$c : e \rightarrow s \in C$

Terms with their respective types.

The elements of $CT_\Sigma =_{def} CT_\Sigma(\emptyset)$ are called **ground Σ -terms**.

$T_\Sigma(V) =_{def} CT_\Sigma(V) \cap wdtr(EL_\Sigma, NL_{\Sigma, V})$ is the **least** $\mathcal{T}(S, \mathcal{I})$ -sorted set M of subsets of $dtr(EL_\Sigma, NL_{\Sigma, V})$ with (1) and the following properties:

Let $I \in \mathcal{I}$ and $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$.

- For all $s \in S$, $V_s \subseteq M_s$. (6)

- For all $c : e \rightarrow s \in F$ and $t \in M_e$, $c\{sel \rightarrow t\} \in M_s$. (7)

- For all $t_i \in M_{e_i}$, $i \in I$, $tup\{i \rightarrow t_i \mid i \in I\} \in M_{\prod_{i \in I} e_i}$. (8)

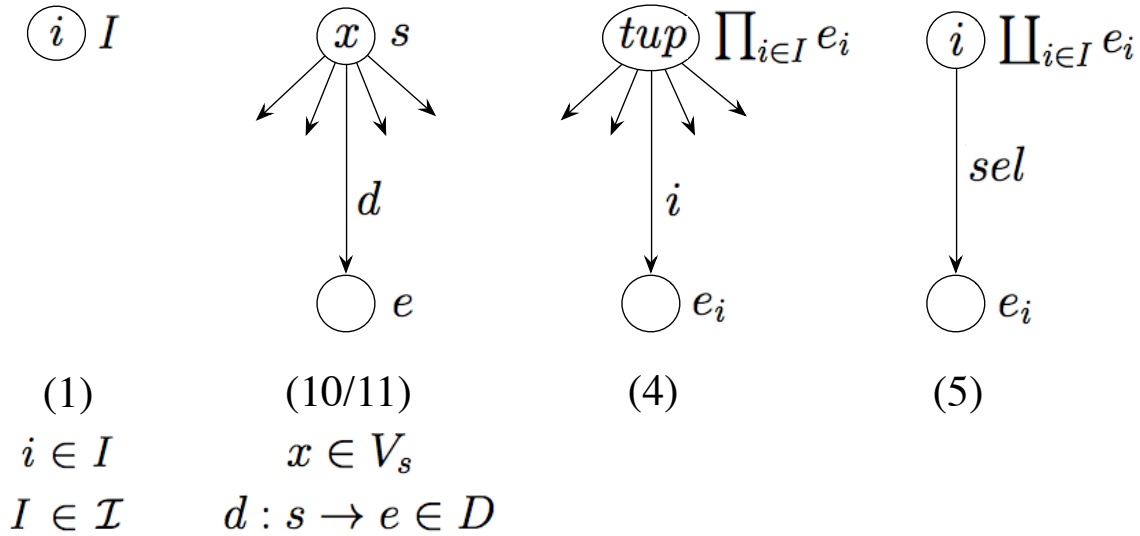
- For all $i \in I$ and $t \in M_{e_i}$, $i\{sel \rightarrow t\} \in M_{\prod_{i \in I} e_i}$. (9)

$$T_\Sigma =_{def} T_\Sigma(\emptyset).$$

Let Σ be *destructive*.

The set $DT_{\Sigma}(V)$ of Σ -**coterms** over V is the *greatest* $\mathcal{T}(S, \mathcal{I})$ -sorted set M of subsets of $dtr(EL_{\Sigma}, NL_{\Sigma, V})$ with (1), (4), (5) and the following property:

- For all $s \in S$ and $t \in M_s$ there is $x \in V_s$ and for all $d : s \rightarrow e \in F$ there is $t_d \in M_e$ with $t = x\{d \rightarrow t_d \mid d : s \rightarrow e \in F\}$. (10)



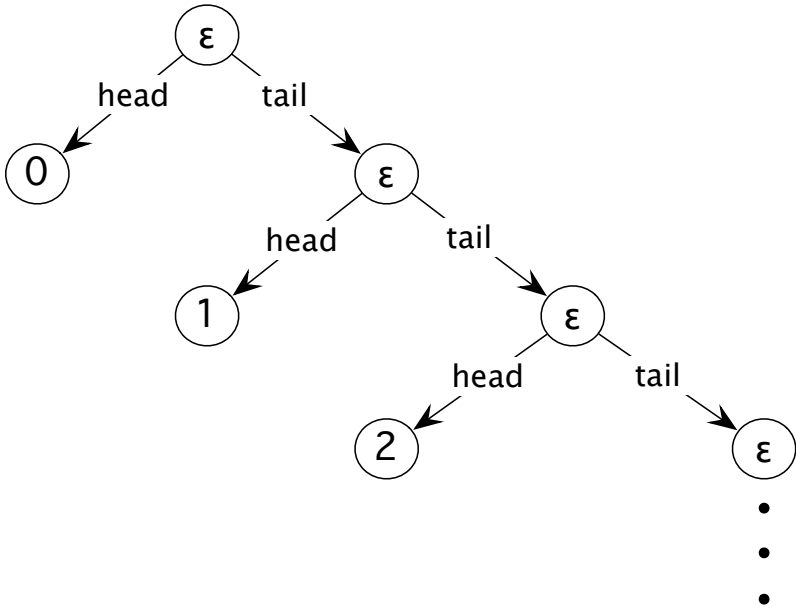
Coterms with their respective types.

The elements of $DT_{\Sigma} =_{def} DT_{\Sigma}(1)$ are called **ground Σ -coterms**.

Examples

Let $Stream(X) = (\{stream\}, \{X\}, \{head : stream \rightarrow X, tail : stream \rightarrow stream\})$.

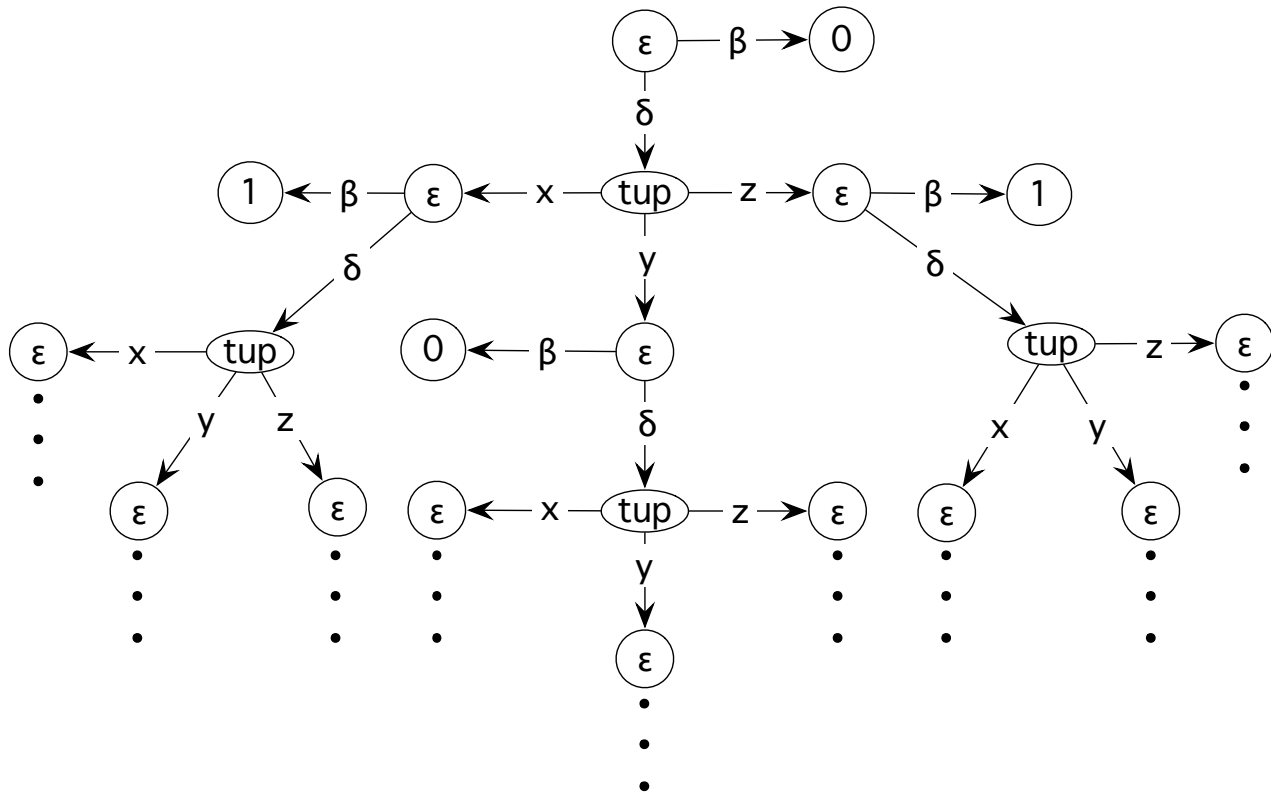
The following $Stream(\mathbb{N})$ -coterms represents the stream of natural numbers:



Let $Acc(X) = (\{stream\}, \{X, 2\}, \{\delta : state \rightarrow state^X, \beta : state \rightarrow 2\})$.

Let $X = \{x, y, z\}$.

The following $Acc(X)$ -coterm represents an acceptor of all words over X that contain x or z :



$coT_\Sigma(V) =_{def} DT_\Sigma(V) \cap wdtr(EL_\Sigma, NL_{\Sigma,V})$ is the **least** $\mathcal{T}(S, \mathcal{I})$ -sorted set M of subsets of $dtr(EL_\Sigma, NL_{\Sigma,V})$ with (1), (8), (9) and the following property:

- For all $s \in S$, $x \in V_s$, $d : s \rightarrow e \in F$ and $t_d \in M_e$, $x\{d \rightarrow t_d \mid d : s \rightarrow e \in F\} \in M_s$. (11)

$$coT_\Sigma =_{def} coT_\Sigma(1).$$

The set $T_\Sigma(V)$ of **well-founded** Σ -terms over V , however, is defined as if Σ were constructive:

$T_\Sigma(V)$ is the **least** $\mathcal{T}(S, \mathcal{I})$ -sorted set M of subsets of $dtr(EL_\Sigma, NL_{\Sigma,V})$ with (1), (6), (8), (9), but the following property instead of (7):

- For all $s \in S$, $d : s \rightarrow e \in F$ and $t_d \in M_e$, $\epsilon\{d \rightarrow t_d \mid d : s \rightarrow e \in F\} \in M_s$. (12)

Type compatible $\mathcal{T}(S, \mathcal{I})$ -sorted sets

A $\mathcal{T}(S, \mathcal{I})$ -sorted set A is **type compatible** if for all $I \in \mathcal{I}$,

- $A_I = I$,
- for all $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$
- there are tuples

$$\pi = (\pi_i : A_{\prod_{i \in I} e_i} \rightarrow A_{e_i})_{i \in I} \quad \text{and} \quad \iota = (\iota_i : A_{e_i} \rightarrow A_{\coprod_{i \in I} e_i})_{i \in I}$$

such that $(A_{\prod_{i \in I} e_i}, \pi)$ is a **product** and $(A_{\coprod_{i \in I} e_i}, \iota)$ is a **sum** or **coproduct** of $(A_{e_i})_{i \in I}$.

Let A be type compatible, $I \in \mathcal{I}$ and $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$

- (1) For all $a \in A_{\prod_{i \in I} e_i}$ there are unique $i \in I$ and $b \in A_{e_i}$ such that $\iota_i(b) = a$.
- (2) For all $a, b \in A_{\prod_{i \in I} e_i}$, $a = b$ if for all $i \in I$, $\pi_i(a) = \pi_i(b)$.

Let A, B be type compatible $\mathcal{T}(S, \mathcal{I})$ -sorted sets.

A $\mathcal{T}(S, \mathcal{I})$ -sorted function $h : A \rightarrow B$ is **type compatible** if for all $I \in \mathcal{I}$,

- $h_I = id_I$,
- for all $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$, $h_{\prod_{i \in I} e_i} = \prod_{i \in I} h_{e_i}$ and $h_{\coprod_{i \in I} e_i} = \coprod_{i \in I} h_{e_i}$.

$Set^{S, \mathcal{I}}$ denotes the subcategory of $Set^{\mathcal{T}(S, \mathcal{I})}$ with type compatible $\mathcal{T}(S, \mathcal{I})$ -sorted sets as objects and type compatible $\mathcal{T}(S, \mathcal{I})$ -sorted functions as morphisms.

Let $\Sigma = (S, \mathcal{I}, F)$ be a signature.

A Σ -**algebra** $\mathcal{A} = (A, Op)$ consists of a type compatible $\mathcal{T}(S, \mathcal{I})$ -sorted set A and an F -sorted set

$$Op = (f^{\mathcal{A}} : A_e \rightarrow A_{e'})_{f:e \rightarrow e' \in F}$$

of functions.

Let \mathcal{A}, \mathcal{B} be Σ -algebras. A type compatible $\mathcal{T}(S, \mathcal{I})$ -sorted function $h : \mathcal{A} \rightarrow \mathcal{B}$ is a Σ -**homomorphism** if for all $f : e \rightarrow e' \in F$,

$$h_{e'} \circ f^{\mathcal{A}} = f^{\mathcal{B}} \circ h_e.$$

Alg_{Σ} denotes the subcategory of $Set^{S, \mathcal{I}}$ with Σ -algebras as objects and Σ -homomorphisms as morphisms.

If Σ is **constructive**, then $CT_{\Sigma}(V)$ is a Σ -algebra:

Let $I \in \mathcal{I}$ and $\{e_i\} \subseteq \mathcal{T}(S, \mathcal{I})$.

- For all $c : e \rightarrow s \in C$, $t \in CT_{\Sigma}(V)_e$, $c^{CT_{\Sigma}(V)}(t) =_{def} c\{sel \rightarrow t\}$.
- For all $t_i \in CT_{\Sigma}(V)_{e_i}$, $i \in I$, and $k \in I$, $\pi_k(tup\{i \rightarrow t_i \mid i \in I\}) =_{def} t_k$.
- For all $i \in I$ and $t \in CT_{\Sigma}(V)_{e_i}$, $\iota_i(t) =_{def} i\{sel \rightarrow t\}$.

If Σ is **destructive**, then $DT_\Sigma(V)$ is a Σ -algebra:

Let $I \in \mathcal{I}$ and $\{e_i\} \subseteq \mathcal{T}(S, \mathcal{I})$.

- For all $d : s \rightarrow e \in D$, $x \in V_s$ and $t'_d \in DT_\Sigma(V)_e$, $d' : s \rightarrow e' \in D$,

$$d^{DT_\Sigma(V)}(x\{d \rightarrow t'_d \mid d' : s \rightarrow e' \in D\}) =_{def} t_d.$$
- For all $t_i \in DT_\Sigma(V)_{e_i}$, $i \in I$, and $k \in I$, $\pi_k(\text{tup}\{i \rightarrow t_i \mid i \in I\}) =_{def} t_k.$
- For all $i \in I$ and $t \in DT_\Sigma(V)_{e_i}$, $\iota_i(t) =_{def} i\{sel \rightarrow t\}.$

Let $e \in \mathcal{T}(S, \mathcal{I})$, $I \in \mathcal{I}$ and $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$.

$\{c_i : A_{e_i} \rightarrow A_e \mid i \in I\}$ is a **set of constructors for** e if $\langle c_i \rangle_{i \in I} : \coprod_{i \in I} A_{e_i} \rightarrow A_e$ is iso.

$\{d_i : A_e \rightarrow A_{e_i} \mid i \in I\}$ is a **set of destructors for** e if $\langle d_i \rangle_{i \in I} : A_e \rightarrow \prod_{i \in I} A_{e_i}$ is iso.

- The injections of A for a sum type form a set of constructors for this type.
- The projections of A for a product type form a set of destructors for this type.
- If Σ is **constructive** and \mathcal{A} is **initial** in Alg_Σ , then for all $s \in S$, $\{f^{\mathcal{A}} \mid f : e \rightarrow s \in F\}$ is a set of constructors for s .
- If Σ is **destructive** and \mathcal{A} is **final** in Alg_Σ , then for all $s \in S$, $\{f^{\mathcal{A}} \mid f : s \rightarrow e \in F\}$ is a set of destructors for s .

Let $\Sigma = (S, \mathcal{I}, F)$ be a **constructive** signature.

Σ induces the functor $H_\Sigma : \text{Set}^S \rightarrow \text{Set}^S$:

For all $A, B \in \text{Set}^S$, $h \in \text{Set}^S(A, B)$ and $s \in S$,

$$H_\Sigma(A)_s = \coprod_{f:e \rightarrow s \in F} A_e,$$

$$H_\Sigma(h)_s = \coprod_{f:e \rightarrow s \in F} h_e.$$

For all $s \in S$ and $f : e \rightarrow s \in F$,

$$\begin{array}{ccc}
 H_\Sigma(A)_s & \xrightarrow{\alpha_s = [f^A]_{f:e \rightarrow s \in F}} & A_s \\
 \uparrow \wr & \searrow & \uparrow \wr \\
 A_e & &
 \end{array}
 \quad (1) \quad
 \begin{array}{l}
 f^A = \alpha_s \circ \wr_f
 \end{array}$$

Let $\Sigma = (S, \mathcal{I}, F)$ be a **destructive** signature.

Σ induces the functor $H_\Sigma : \text{Set}^S \rightarrow \text{Set}^S$:

For all $A, B \in \text{Set}^S$, $h \in \text{Set}^S(A, B)$ and $s \in S$,

$$H_\Sigma(A)_s = \prod_{f:s \rightarrow e \in F} A_e,$$

$$H_\Sigma(h)_s = \prod_{f:s \rightarrow e \in F} h_e.$$

For all $s \in S$ and $f : s \rightarrow e \in F$,

$$\begin{array}{ccc}
 A_s & \xrightarrow{\alpha_s = \langle f^A \rangle_{f:s \rightarrow e \in F}} & H_\Sigma(A)_s \\
 & \searrow & \downarrow \pi_f \\
 & & A_e
 \end{array}
 \quad (2)$$

$f^A = \pi_f \circ \alpha_s$

Term folding und state unfolding

Let $\Sigma = (S, \mathcal{I}, C)$ be a **constructive** signature, $\mathcal{A} = (A, Op)$ be a Σ -algebra, V be an S -sorted set of “variables” and $g : V \rightarrow A$ be an S -sorted **valuation** of V .

The **extension** of g ,

$$g^* : T_\Sigma(V) \rightarrow A,$$

is the $\mathcal{T}(S, \mathcal{I})$ -sorted function that is inductively defined as follows:

Let $I \in \mathcal{I}$ and $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$.

- $g_I^* = id_I$. (1)

- For all $s \in S$ and $x \in V_s$, $g_s^*(x) = g_s(x)$. (2)

- For all $c : e \rightarrow s \in F$ and $t \in T_\Sigma(V)_e$, $g_s^*(c\{sel \rightarrow t\}) = c^{\mathcal{A}}(g_e^*(t))$. (3)

- For all $t_i \in T_\Sigma(V)_{e_i}$, $i \in I$, and $k \in I$, $\pi_k(g_{\prod_{i \in I} e_i}^*(\{tup \rightarrow t_i \mid i \in I\})) = g_{e_k}^*(t_k)$. (4)

- For all $k \in I$ and $t \in T_\Sigma(V)_{e_k}$, $g_{\prod_{i \in I} e_i}^*(k\{sel \rightarrow t\}) = \iota_k(g_{e_k}^*(t))$. (5)

Intuitively, g^* evaluates each wellfounded Σ -term over V in \mathcal{A} .

Theorem FREE

g^* is the only Σ -homomorphism from $T_\Sigma(V)$ to \mathcal{A} that satisfies (2):

$$\begin{array}{ccc} V & \xrightarrow{\text{inc}_V} & T_\Sigma(V) \\ & \searrow g & \swarrow g^* \\ & & \mathcal{A}_s \end{array} \quad (2)$$

The restriction of g^* to ground terms does not depend on g and is denoted by

$$\text{fold}^{\mathcal{A}}: T_\Sigma \rightarrow \mathcal{A}.$$

Since g^* is the only Σ -homomorphism from $T_\Sigma(V)$ to \mathcal{A} that satisfies (2), $\text{fold}^{\mathcal{A}}$ is the only Σ -homomorphism from T_Σ to \mathcal{A} , i.e., T_Σ is initial in Alg_Σ .

\mathcal{A} is **reachable** (or **generated**) if $\text{fold}^{\mathcal{A}}$ is epi.

\mathcal{A} is **equationally consistent** if $\text{fold}^{\mathcal{A}}$ is mono.

Let $\Sigma = (S, \mathcal{I}, D)$ be a **destructive** signature, $\mathcal{A} = (A, Op)$ be a Σ -algebra, V be an S -sorted set of “colors” and $g : A \rightarrow V$ be an S -sorted **coloring** of A .

The **coextension** of g ,

$$g^\# : A \rightarrow DT_\Sigma(V),$$

is the $\mathcal{T}(S, \mathcal{I})$ -sorted function that is inductively defined as follows:

Let $I \in \mathcal{I}$ and $\{e_i\}_{i \in I} \subseteq \mathcal{T}(S, \mathcal{I})$.

$$\bullet g_I^\# = id_I. \tag{1}$$

$$\bullet \text{For all } s \in S \text{ and } a \in A_s, g_s^\#(a) = g_s(a)\{d \rightarrow g_e^\#(d^{\mathcal{A}}(a)) \mid d : s \rightarrow e \in D\}. \tag{2}$$

$$\bullet \text{For all } a \in A_{\prod_{i \in I} e_i}, g_{\prod_{i \in I} e_i}^\#(a) = tup\{i \rightarrow g_{e_i}^\#(\pi_i(a)) \mid i \in I\}. \tag{3}$$

$$\bullet \text{For all } k \in I \text{ and } a \in A_{e_k}, g_{\prod_{i \in I} e_i}^\#(\iota_k(a)) = k\{sel \rightarrow g_{e_k}^\#(a)\}. \tag{4}$$

Intuitively, $g^\#$ unfolds each “state” $a \in A$ into the Σ -cotermin that represents the “behavior” of a w.r.t. \mathcal{A} .

In particular, the coextension $id_A^\# : A \rightarrow DT_\Sigma(A)$ “runs” (the destructors of) \mathcal{A} on its arguments.

Theorem COFREE

$g^\#$ is the only Σ -homomorphism from \mathcal{A} to $DT_\Sigma(V)$ that satisfies (5):

$$\begin{array}{ccc}
 V & \xleftarrow{\text{root} =_{\text{def}} \lambda t.t(\epsilon)} & DT_\Sigma(V) \\
 & \nearrow g & \nwarrow g^\# \\
 & A &
 \end{array}
 \quad (5)$$

The restriction of $g^\#$ to ground coterms does not depend on g and is denoted by

$$\text{unfold}^{\mathcal{A}}: \mathcal{A} \rightarrow DT_\Sigma.$$

Since $g^\#$ is the only Σ -homomorphism from \mathcal{A} to $DT_\Sigma(V)$ that satisfies (5), $\text{unfold}^{\mathcal{A}}$ is the only Σ -homomorphism from \mathcal{A} to DT_Σ , i.e., DT_Σ is final in Alg_Σ .

\mathcal{A} is **observable** (or **cogenerated**) if $\text{unfold}^{\mathcal{A}}$ is mono.

\mathcal{A} is **behaviorally complete** if $\text{unfold}^{\mathcal{A}}$ is epi.

From constructors to destructors and backwards

Lambek's Lemma

- (1) Suppose that Alg_F has an initial object $\alpha : F(A) \rightarrow A$. α is iso.
- (2) Suppose that $coAlg_F$ has a final object $\beta : A \rightarrow F(A)$. β is iso.

Lambek's Lemma allows us to transform every **constructive** or **destructive** signature Σ into a **destructive** resp. **constructive** signature $co\Sigma$ such that

$$DT_{co\Sigma} \cong CT_{\Sigma} \quad \text{resp.} \quad T_{co\Sigma} \cong coT_{\Sigma}.$$

Here are the details:

Let $\Sigma = (S, \mathcal{I}, C)$ be a **constructive** signature,

$$\begin{aligned} D &= \{s : s \rightarrow \coprod_{c:e \rightarrow s \in C} e \mid s \in S\}, \\ \mathit{co}\Sigma &= (S, \mathcal{I}, D). \end{aligned}$$

By Lambek's Lemma (1), the **initial** H_Σ -algebra

$$\alpha = \{\alpha_s : H_\Sigma(T_\Sigma)_s \xrightarrow{[c^{T_\Sigma}]_{c:e \rightarrow s \in C}} T_{\Sigma,s} \mid s \in S\}$$

is iso. Hence there is the H_Σ -coalgebra

$$\{\alpha_s^{-1} : T_{\Sigma,s} \rightarrow H_\Sigma(T_\Sigma)_s \mid s \in S\}$$

that corresponds to the $\mathit{co}\Sigma$ -algebra $\mathcal{A} = (T_\Sigma, \mathit{Op})$ with $s^{\mathcal{A}} = \alpha_s^{-1}$ for all $s \in S$.

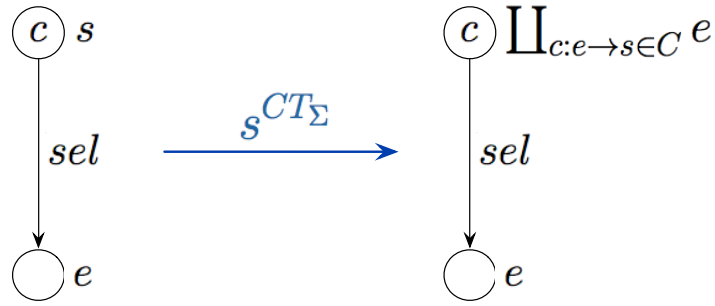
Since $\mathit{co}\Sigma$ is destructive, Theorem COFREE implies that $DT_{\mathit{co}\Sigma}$ is final in $\mathit{Alg}_{\mathit{co}\Sigma}$.

CT_Σ is also final in $\mathit{Alg}_{\mathit{co}\Sigma}$:

CT_Σ is a $\mathit{co}\Sigma$ -algebra: Let $I \in \mathcal{I}$ and $\{e_i\} \subseteq \mathcal{T}(S, \mathcal{I})$.

- For all $c : e \rightarrow s \in C$, $t \in CT_{\Sigma,e}$,

$$s^{CT_\Sigma}(c\{sel \rightarrow t\}) =_{\mathit{def}} c\{sel \rightarrow t\}.$$

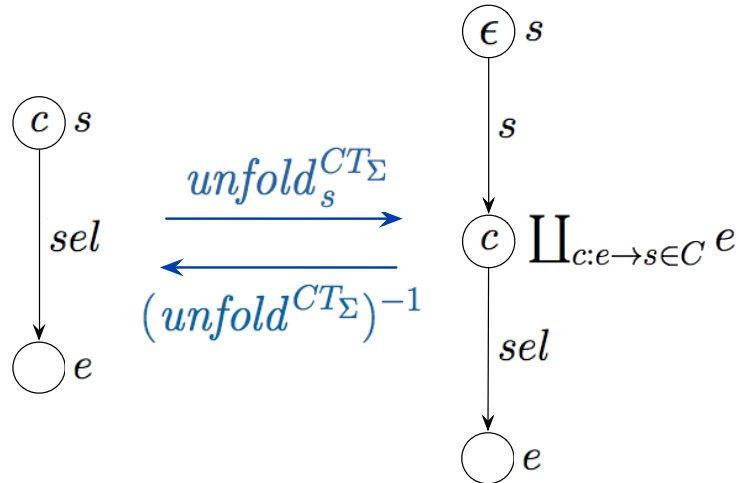


- For all $t_i \in CT_{\Sigma, e_i}$, $i \in I$, and $k \in I$, $\pi_k(\text{tup}\{i \rightarrow t_i \mid i \in I\}) =_{\text{def}} t_k$.
- For all $i \in I$ and $t \in CT_{\Sigma, e_i}$, $\iota_i(t) =_{\text{def}} i\{\text{sel} \rightarrow t\}$.

CT_Σ and $DT_{\text{co}\Sigma}$ are $\text{co}\Sigma$ -isomorphic. Equivalently,

$$\text{unfold}^{CT_\Sigma} : CT_\Sigma \rightarrow DT_{\text{co}\Sigma}$$

is bijective.



Let $\Sigma = (S, \mathcal{I}, D)$ be a **destructive** signature,

$$\begin{aligned} \mathcal{C} &= \{s : \prod_{d:s \rightarrow e \in D} e \rightarrow s \mid s \in S\}, \\ \text{co}\Sigma &= (S, \mathcal{I}, \mathcal{C}). \end{aligned}$$

By Lambek's Lemma (2), the **final** H_Σ -coalgebra

$$\alpha = \{\alpha_s : DT_{\Sigma,s} \xrightarrow{\langle d^{DT_\Sigma} \rangle_{d:s \rightarrow e \in D}} H_\Sigma(DT_\Sigma)_s \mid s \in S\}$$

is iso. Hence there is the H_Σ -algebra

$$\{\alpha_s^{-1} : H_\Sigma(DT_\Sigma)_s \rightarrow DT_{\Sigma,s} \mid s \in S\}$$

that corresponds to the $\text{co}\Sigma$ -algebra $\mathcal{A} = (DT_\Sigma, Op)$ with $s^{\mathcal{A}} = \alpha_s^{-1}$ for all $s \in S$.

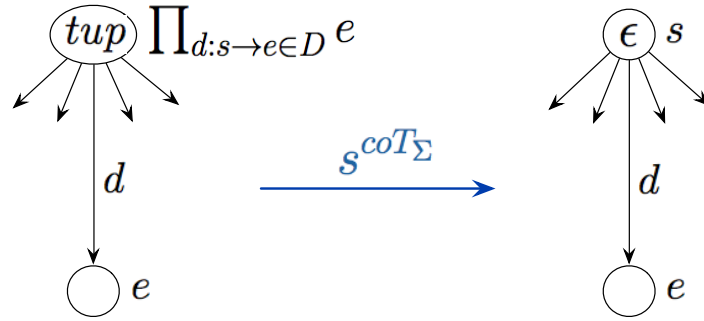
Since $\text{co}\Sigma$ is constructive, Theorem FREE implies that $T_{\text{co}\Sigma}$ is initial in $\text{Alg}_{\text{co}\Sigma}$.

$\text{co}T_\Sigma$ is also initial in $\text{Alg}_{\text{co}\Sigma}$:

$\text{co}T_\Sigma$ is a $\text{co}\Sigma$ -algebra: Let $I \in \mathcal{I}$ and $\{e_i\} \subseteq \mathcal{T}(S, \mathcal{I})$.

- For all $s \in S$, $c : e \rightarrow s \in \mathcal{C}$ and $t_d \in \text{co}T_{\Sigma,e}$, $d : s \rightarrow e \in D$,

$$s^{\text{co}T_\Sigma}(\text{tup}\{d \rightarrow t_d \mid d : s \rightarrow e \in D\}) =_{\text{def}} \epsilon\{d \rightarrow t_d \mid d : s \rightarrow e \in D\}.$$

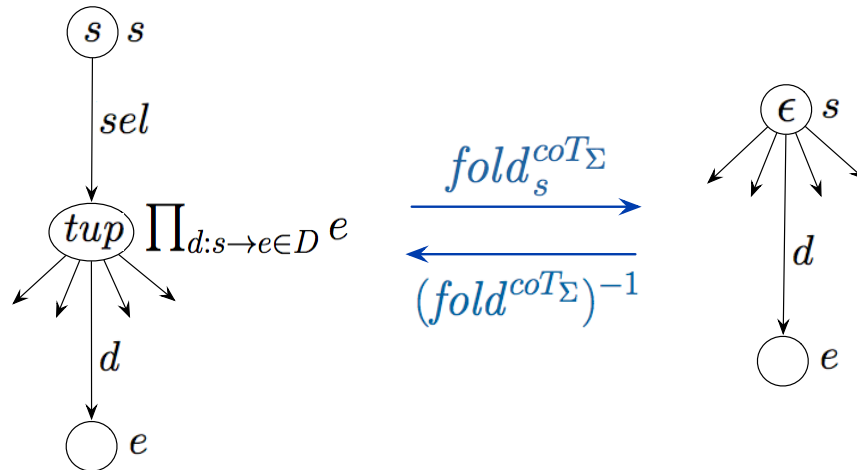


- For all $t_i \in coT_{\Sigma, e_i}$, $i \in I$, and $k \in I$, $\pi_k(tup\{i \rightarrow t_i \mid i \in I\}) =_{def} t_k$.
- For all $i \in I$ and $t \in coT_{\Sigma, e_i}$, $\iota_i(t) =_{def} i\{sel \rightarrow t\}$.

$T_{co\Sigma}$ and coT_Σ are $co\Sigma$ -isomorphic. Equivalently,

$$fold^{coT_\Sigma} : T_{co\Sigma} \rightarrow coT_\Sigma$$

is bijective.



Iterative Σ -equations

Let $\Sigma = (S, \mathcal{I}, F)$ be a constructive or destructive signature and V be a finite S -sorted set. An S -sorted function

$$E : V \rightarrow T_\Sigma(V)$$

with $\text{img}(E) \cap V = \emptyset$ is called a **system of iterative Σ -equations**.

E is usually written as $\{x = E(x) \mid x \in V\}$.

Let Σ be **constructive**, $\mathcal{A} = (A, Op)$ be a Σ -algebra and A^V be the set of S -sorted functions from V to A .

$g \in A^V$ **solves E in \mathcal{A}** if $g^* \circ E = g$.

E turns $T_\Sigma(V)$ into a $co\Sigma$ -algebra: Let $s \in S$, $I \in \mathcal{I}$ and $\{e_i\} \subseteq \mathcal{T}(S, \mathcal{I})$.

- For all $x \in V_s$, $s^{T_\Sigma(V)}(x) =_{def} s^{T_\Sigma(V)}(E(x))$.
- For all $c : e \rightarrow s \in F$, $t \in T_\Sigma(V)_e$, $s^{T_\Sigma(V)}(c\{sel \rightarrow t\}) =_{def} c\{sel \rightarrow t\}$.
- For all $t_i \in T_\Sigma(V)_{e_i}$, $i \in I$, and $k \in I$, $\pi_k(\text{tup}\{i \rightarrow t_i \mid i \in I\}) =_{def} t_k$.
- For all $i \in I$ and $t \in T_\Sigma(V)_{e_i}$, $\iota_i(t) =_{def} i\{sel \rightarrow t\}$.

Theorem SOL

$$V \xrightarrow{\text{incy}} T_{\Sigma}(V) \xrightarrow{\text{unfold}^{T_{\Sigma}(V)}} DT_{co\Sigma} \xrightarrow{(\text{unfold}^{CT_{\Sigma}})^{-1}} CT_{\Sigma}$$

solves E in CT_{Σ} uniquely.

Proof. See Theorem SOL (coalgebraic version) in [Fixpoints, Categories, and \(Co\)Algebraic Modeling](#). □

Example

Let $List(X) = (\{list\}, \{X\}, \{nil : 1 \rightarrow list, cons : X \times list \rightarrow list\})$.

Let $V = \{blink, blink'\}$. The following system of $List(\mathbb{Z})$ -equations over V has a unique solution in $CT_{List(\mathbb{Z})}$ and thus defines two elements of $CT_{List(\mathbb{Z})}$:

$$\begin{aligned} blink &= cons\{sel \rightarrow tup\{1 \rightarrow 0, 2 \rightarrow blink'\}\}, \\ blink' &= cons\{sel \rightarrow tup\{1 \rightarrow 1, 2 \rightarrow blink\}\}. \end{aligned} \tag{1}$$

Infinite Σ -terms that are representable as unique solutions of iterative equations are called **rational**. A Σ -term is rational iff it has only finitely many subterms.

Let Σ be [destructive](#) and h be the bijection between $T_{\Sigma}(V)$ and $T_{co\Sigma}(V)$ that is the identity on V and agrees with $(fold^{coT_{\Sigma}})^{-1}$ on $T_{\Sigma} = coT_{\Sigma}$.

Corollary $h \circ E$ has a unique solution in DT_Σ .

Proof. DT_Σ is a $co\Sigma$ -algebra: For all $s \in S$,

$$s^{DT_\Sigma}(\epsilon\{d \rightarrow t_d \mid d : s \rightarrow e \in F\}) =_{def} s\{sel \rightarrow tup\{d \rightarrow t_d \mid d : s \rightarrow e \in F\}\}.$$

By Theorem SOL, $h \circ E$ has a unique solution in $CT_{co\Sigma}$. Since $CT_{co\Sigma}$ is final in $Alg_{coco\Sigma}$, $CT_{co\Sigma}$ is $coco\Sigma$ -isomorphic to $A =_{def} DT_{coco\Sigma}$. A is a Σ -algebra: For all $s \in S$ and $d : s \rightarrow e$ and $t_d \in A_e$, $d : s \rightarrow e \in F$,

$$d^A(\epsilon\{s \rightarrow s\{sel \rightarrow tup\{d \rightarrow t_d \mid d : s \rightarrow e \in F\}\}\}) =_{def} t_d.$$

$unfold^A : A \rightarrow DT_\Sigma$ is bijective: The inverse maps $\epsilon\{d \rightarrow t_d \mid d : s \rightarrow e \in F\} \in DT_\Sigma$ to

$$\epsilon\{s \rightarrow s\{sel \rightarrow tup\{d \rightarrow t_d \mid d : s \rightarrow e \in F\}\}\}.$$

Hence $CT_{co\Sigma} \cong A \cong DT_\Sigma$ and thus the solutions of $h \circ E$ in $CT_{co\Sigma}$ and DT_Σ , respectively, coincide up to isomorphism. \square

Example

Let $V = \{esum, osum\}$. Given the following system E of $Acc(\mathbb{Z})$ -equations over V , $h \circ E$ has a unique solution in $DT_{Acc(\mathbb{Z})}$ and thus defines two elements of $DT_{Acc(\mathbb{Z})}$:

$$\begin{aligned} esum &= \epsilon\{\delta \rightarrow tup(\{x \rightarrow esum \mid x \in even\} \cup \{x \rightarrow osum \mid x \in odd\}), \beta \rightarrow 1\}, \\ osum &= \epsilon\{\delta \rightarrow tup\{x \rightarrow osum \mid x \in even\} \cup \{x \rightarrow esum \mid x \in odd\}, \beta \rightarrow 0\}. \end{aligned} \quad (2)$$

Typed theories

Let $\Sigma = (S, \mathcal{I}, F)$ be a signature.

The set der_Σ of **derived Σ -operations** is inductively defined as follows:

Let $I \in \mathcal{I}$ and $\{e_i\} \subseteq \mathcal{T}(S, \mathcal{I})$.

- $F \subseteq der_\Sigma$.
- For all $e \in \mathcal{T}(S, \mathcal{I})$ and $i \in I$, $\bar{i} : e \rightarrow I \in der_\Sigma$.
- For all $f : e \rightarrow e'$, $g : e' \rightarrow e'' \in der_\Sigma$, $g \circ f : e \rightarrow e'' \in der_\Sigma$.
- $\pi_i : \prod_{i \in I} e_i \rightarrow e_i$, $\iota_i : e_i \rightarrow \coprod_{i \in I} e_i \in der_\Sigma$ (also written as id if I is a singleton).
- For all $f_i : e \rightarrow e_i \in der_\Sigma$, $i \in I$, $\langle f_i \rangle : e \rightarrow \prod_{i \in I} e_i \in der_\Sigma$.
- For all $f_i : e_i \rightarrow e \in der_\Sigma$, $i \in I$, $[f_i] : \prod_{i \in I} e_i \rightarrow e \in der_\Sigma$.
- **λ -abstraction:**
For all $c_i : e_i \rightarrow e$, $f_i : e_i \rightarrow e' \in der_\Sigma$, $i \in I$, $\lambda\{c_i.f_i\}_{i \in I} : e \rightarrow e' \in der_\Sigma$.
- **κ -abstraction:**
For all $d_i : e \rightarrow e_i$, $f_i : e' \rightarrow e_i \in der_\Sigma$, $i \in I$, $\kappa\{d_i.f_i\}_{i \in I} : e' \rightarrow e \in der_\Sigma$.

$Th(\Sigma) = (S, \mathcal{I}, F \cup der_\Sigma)$ is called the (algebraic) Σ -**theory**.

Let $\mathcal{A} = (A, Op)$ be a Σ -algebra.

The $Th(\Sigma)$ -algebra $\mathcal{B} = Th(\mathcal{A})$ with $\mathcal{B}|_{\Sigma} = \mathcal{A}$ and the following interpretation of der_{Σ} is called the **theory of \mathcal{A}** .

Let $I \in \mathcal{I}$ and $\{e_i\} \subseteq \mathcal{T}(S, \mathcal{I})$.

- For all $e \in \mathcal{T}(S, \mathcal{I})$, $id^{\mathcal{B}} = id_{\mathcal{A}}$.
- For all $e \in \mathcal{T}(S, \mathcal{I})$, $i \in I$ and $a \in A_e$, $\bar{i}^{\mathcal{B}} = \lambda x. i$.
- Compositions, projections, injections, product and coproduct extensions are defined as usually.
- For all $c_i : e_i \rightarrow e$, $f_i : e_i \rightarrow e' \in der_{\Sigma}$, $i \in I$, such that $\{c_i^{\mathcal{B}} \mid i \in I\}$ is a set of constructors for e , for all $k \in I$,

$$(\lambda\{c_i.f_i\}_{i \in I})^{\mathcal{B}} \circ c_k^{\mathcal{B}} = f_k^{\mathcal{B}}.$$

- For all $d_i : e \rightarrow e_i$, $f_i : e' \rightarrow e_i \in der_{\Sigma}$, $i \in I$, such that $\{d_i^{\mathcal{B}} \mid i \in I\}$ is a set of destructors for e , for all $k \in I$,

$$d_k^{\mathcal{B}} \circ (\kappa\{d_i.f_i\}_{i \in I})^{\mathcal{B}} = f_k^{\mathcal{B}}.$$

The following lemma implies that λ - and κ -abstractions are well-defined:

(1) Let $\{f_i : A_{e_i} \rightarrow A_e \mid i \in I\}$ be a set of constructors for e .

For all $a \in A_e$ there are unique $i \in I$ and $b \in A_{e_i}$ such that $f_i^A(b) = a$.

(2) Let $\{f_i : A_e \rightarrow A_{e_i} \mid i \in I\}$ be a set of destructors for e .

For all $a, b \in A_e$, $a = b$ if $f_i(a) = f_i(b)$ for all $i \in I$.

For ease of notation, $Th(\mathcal{A})$ may be regarded as the category with $\mathcal{T}(S, \mathcal{I})$ as the set of objects and the operations of $Th(\mathcal{A})$ as morphisms: Every $Th(\mathcal{A})$ -morphism $f : e \rightarrow e'$ denotes the interpretation of some derived Σ -operation in \mathcal{A} .

Example

Let $p : e \rightarrow 2$ and $f, g : e \rightarrow e'$ be $Th(\mathcal{A})$ -morphisms. The conditional

$$\textit{if } p \textit{ then } f \textit{ else } g : e \rightarrow e'$$

can be derived as follows:

$$\textit{if } p \textit{ then } f \textit{ else } g = e \xrightarrow{\langle id, p \rangle} e \times 2 \xrightarrow{\lambda\{\langle id, \bar{1} \rangle.f, \langle id, \bar{0} \rangle.g\}} e'.$$

Recursive Σ -equations

$$\mathit{factorial} : \mathbb{N} \rightarrow \mathbb{N}$$

$$\mathit{factorial} = \lambda\{\bar{0}.\bar{1}, (+1).(*) \circ \langle \mathit{id}, \mathit{factorial} \circ (-1) \rangle\}$$

$$\mathit{factorial} : \mathbb{N}^2 \rightarrow \mathbb{N}^2$$

$$\mathit{factorial} = [\mathit{id}, \mathit{factorial} \circ (x \leftarrow x - 1) \circ (y \leftarrow x * y)] \circ (x = 0) \quad \text{or}$$

$$\mathit{factorial} = \text{if } x \equiv 0 \text{ then } \mathit{id} \text{ else } \mathit{factorial} \circ (x \leftarrow x - 1) \circ (y \leftarrow x * y)$$

where $(x = 0)(m, n) = \text{if } m = 0 \text{ then } \iota_1(m, n) \text{ else } \iota_2(m, n)$

$$(x \equiv 0)(m, n) = \text{if } m = 0 \text{ then } 1 \text{ else } 0$$

$$(x \leftarrow x - 1)(m, n) = (m - 1, n)$$

$$(y \leftarrow x * y)(m, n) = (m, m * n)$$

$$\mathit{zip} : X^{\mathbb{N}} \times X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$$

$$\mathit{zip} = \kappa\{\mathit{head}.\mathit{head} \circ \pi_1, \mathit{tail}.\mathit{tail} \circ \mathit{zip} \circ \langle \pi_2, \mathit{tail} \circ \pi_1 \rangle\}$$

Where do such equations have unique solutions?