Terminal coalgebras for endofunctors on sets

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Abstract

This paper shows that the main results of Aczel and Mendler on the existence of terminal coalgebras for an endofunctor on the category of sets do not, for the main part of the results require looking at functors on the category of (possibly proper) classes. We will see here that the main results are valid for sets up to some regular cardinal. Should that cardinal be inaccessible, then Aczel and Mendler's results are derived. In addition we discuss the canonical map from the initial algebra for an endofunctor on sets to the terminal coalgebra and show that in many cases it embeds the former as a dense subset of the latter in a certain natural topology. By way of example, we calculate the terminal coalgebra for various simple endofunctors.

Introduction

Let T be an endofunctor on a category $\mathscr{C}$. A T-coalgebra in $\mathscr{C}$ is a pair $(C, \psi: C \longrightarrow TC)$. A morphism $f:(C,\psi) \longrightarrow (C',\psi')$ is a morphism $f: C \longrightarrow C'$ such that

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "f"'] & TC \arrow[d, "Tf"] \\ C' \arrow[r, "\psi'"'] & TC' \end{tikzcd}$$

commutes. This defines a category called the category of T-coalgebras and denoted $\mathscr{C}_T$.

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It should be noted that this terminology and notation do not conflict with those of coalgebra for a cotriple $\mathbf{T} = (T, \epsilon, \delta)$ in which the category $\mathscr{C}_{\mathbf{T}}$ of $\mathbf{T}$- coalgebras is a full subcategory of $\mathscr{C}_T$.

Let **Set** and **SET** denote the categories of small sets and of classes (or large sets), respectively. Let $T\colon \mathbf{SET} \longrightarrow \mathbf{SET}$ be a "Sets-based functor" (this notion defined by Aczel and Mendler will be defined below; it is the same as a SET-accessible functor in the sense of Makkai and Paré [1990]. The existence of a terminal coalgebra for such functors is of interest in theoretical computer science in connection with bisimulations. A brief discussion and further reference may be found in [Aczel & Mendler, 1989]. A theorem guaranteeing a terminal coalgebra in general will also guarantee a right adjoint to the forgetful functor $\mathbf{Set}_T \longrightarrow \mathbf{Set}$ (Theorem 1.2).

A Set-based functor on SET is a functor $T\colon \mathbf{SET} \longrightarrow \mathbf{SET}$ such that for any class X, TX is a colimit of the TA where A is a subSet of X. If we interpret SET as the category of sets of cardinality up to and including some inaccessible cardinal κ and Set as the category of all those of cardinality less than κ, then a Set-based functor is the same thing as a κ-accessible functor whose value on every set of cardinality less than κ has cardinality at most κ. Thus Aczel and Mendler's result can thus be interpreted as saying that if κ is an inaccessible cardinal and if T is a κ-accessible functor whose value on a set of cardinality less than κ is at most κ, then there is a terminal T-coalgebra of size at most κ. (Note that "accessible" in the sense of Makkai & Paré has nothing to do with accessible and inaccessible cardinals. This unfortunate clash of terms will not cause much trouble, since this paper deals only peripherally with the latter.)

Here we use an argument based on the special adjoint functor theorem, a basic tool of category theory, to show that there is a much more general construction that applies to any regular cardinal and specializes to the theorem of Aczel and Mendler when that cardinal is inaccessible.

1 The main theorems

For a set A we let $|A|$ denote the cardinality of A. We begin with a preliminary result. A functor $U\mathscr{A} \longrightarrow \mathscr{B}$ is said to *create* the colimit of a diagram $D\colon \mathscr{I} \longrightarrow \mathscr{A}$ is given a colimit cocone $UD \longrightarrow B$ in $\mathscr{B}$, there is a unique colimit cocone $D \longrightarrow A$ in $\mathscr{A}$ such that U applied to $D \longrightarrow A$ gives a cocone isomomorphic to $UD \longrightarrow B$. This usually happens when the category $\mathscr{B}$ is a category of objects of $\mathscr{A}$ with additional structure and U is the functor that forgets that structure. There is, by the way, a similar definition for limits.

1.1 Proposition. *For any category $\mathscr{C}$ and any endofunctor T on $\mathscr{C}$, the forgetful functor $U\colon \mathscr{C}_T \longrightarrow \mathscr{C}$ creates colimits.*

Proof. The argument is well known, but we sketch it. Given a diagram $D\colon \mathscr{I} \longrightarrow \mathscr{C}_T$ such that UD has a colimit $u\colon UD \longrightarrow C$, let $\psi\colon C \longrightarrow TC$ be the unique arrow such that for each object I of $\mathscr{I}$, the square

$$\begin{tikzcd} UDI \arrow[r, "\delta_I"] \arrow[d, "u_I"'] & TUDI \arrow[d, "Tu_I"] \\ C \arrow[r, "\psi"'] & TC \end{tikzcd}$$

commutes. The existence of ψ follows from the unique mapping property of a colimit. The maps δ_I are the structure maps on the DI. It is now routine to see that the structure so defined on C makes it a colimit of D. $\square$

In particular, U preserves epis so that if $\mathscr{C}$ has cointersections (the dual of intersections) of arbitrary families of quotients, so does $\mathscr{C}_T$.

The following theorem is implicit in the work of Makkai & Paré, but a direct proof is so easy that we include it.

1.2 Theorem. *If T is an accessible endofunctor on sets, the underlying functor $\mathbf{Set}_T \longrightarrow \mathbf{Set}$ has a right adjoint (and hence $\mathbf{Set}_T$ has a terminal object).*

Proof. According to the special adjoint functor theorem, we must show that the category of coalgebras is cocomplete, well-copowered and has a set of generators. The cocompleteness and the well-copoweredness follow immediately from the same facts for sets according to Proposition 1.1. As for the generators, this follows from the weighted bilimit Theorem 5.1.6 of [Makkai & Paré, 1990]. $\square$

In fact, a direct proof of all this is quite easy and we will give it in the next theorem because for that theorem we will require an estimate of the terminal object and that does not appear in the Makkai & Paré theorem.

We now prove the result from which the theorem of Aczel & Mendler follows. Although their result is (equivalent to one) stated for inaccessible cardinals, the argument is in fact valid for any cardinal κ that is is regular and if, in addition, $\lambda < \kappa$ implies $2^\lambda \leq \kappa$.

1.3 Proposition. *Let $\kappa > \aleph_0$ be such a cardinal as described above and $T\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be a κ-accessible functor. Suppose, in addition, that when $|A| < \kappa$, then $|TA| \leq \kappa$. Then $\mathbf{Set}_T$ has a terminal coalgebra of cardinality no larger than κ.*[1]

Proof. We claim that there is a set of generators each of cardinality less than κ. In fact, let $\alpha\colon A \longrightarrow TA$ be a coalgebra. Since inclusions of non-empty subsets split (have right inverses), T takes inclusions to injections. It will simplify notation to suppose that T takes subsets of A to subsets of TA. From the definition of accessible, there

[1] I would like to thank Peter Aczel for tightening up the statement of this theorem; the original form would not have implied his result

is, for each $a \in A$, a subset $A_a \subseteq A$ such that $|A_a| < \kappa$ and $\alpha(a) \in TA_a$. Let B_0 be a subset of A of cardinality less than κ. Let $B_1 = B_0 \cup \bigcup_{a \in B_0} A_a$. Then $B_0 \subseteq B_1$ and $\alpha(B_0) \subseteq TB_1$. This is a union of fewer than κ sets, each of cardinality less than κ and since κ is regular, it follows that $|B_1| < \kappa$. In this way we can build up a countable chain of subsets

$$B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots$$

of subsets of B of cardinality less than κ such that $\alpha(B_n) \subseteq TB_{n+1}$. If we let $B = \bigcup B_n$, it follows that $\alpha(B) \subseteq TB$, so that B is a subcoalgebra. Since each $|B_n| < \kappa$, we have $|B| < \kappa$ as well.

Thus there is a set of generators $G_i = (A_i, \alpha_i)$ with all $|A_i| \leq \kappa$. The cardinality of the set of all the A_i is at most κ. For each one, there are at most $|\mathrm{Hom}(A_i, TA_i)| \leq \kappa^\lambda$ coalgebra structures. But $\kappa^\lambda = \bigcup_{\mu<\kappa} \mu^\lambda$ since each function $\lambda \longrightarrow \kappa$ must, by regularity, factor through a smaller ordinal. For $\mu < \kappa$, $\mu^\lambda \leq (2^\mu)^\lambda = 2^{\mu\times\lambda} \leq \kappa$. Thus κ^λ is a union of κ sets of cardinality at most κ and hence the cardinality of this set of generators is at most κ. The coalgebra $G = \sum G_i$, whose underlying set is $\sum A_i$, has cardinality at most κ since it is the sum of at most κ many sets each of cardinality at most κ. Since the underlying functor creates colimits, it also creates epimorphisms, which are thereby surjective, and a quotient of this coalgebra also has size at most κ. We let G_0 be the colimit (cointersection) of all these quotients. Just as the intersection of subobjects of an object is still a subobject, this colimit is also an epimorphic image of G and hence $|G_0| \leq \kappa$.

Let C be any coalgebra. The defining property of generators implies that there is a surjection

$$\sum \lambda_i \cdot G_i \twoheadrightarrow C$$

where each λ_i is a cardinal (not necessarily less than κ) and $\lambda_i \cdot G_i$ denotes the sum of λ_i many copies of G_i. We now form the pushout diagram

$$\begin{tikzcd}
\sum \lambda_i \cdot G_i \arrow[r, two heads] \arrow[d, "f"'] & C \arrow[d] \\
G \arrow[r, two heads] & G'
\end{tikzcd}$$

where f is the morphism that is the identity on each copy of G_i in the sum $\sum \lambda_i \cdot G_i$. It is well-known (and easy) that the arrow opposite an epi in a pushout is an epi and so G' is a quotient of G. Since G_0 is the colimit of all the quotients, it follows that there is an arrow $G' \longrightarrow G_0$. This shows that every coalgebra has at least one arrow to G_0. Now suppose some object has two distinct arrows, say $f, g\colon C \longrightarrow G_0$. The coequalizer of those two arrows is an epimorphism $h\colon G_0 \twoheadrightarrow G_0'$. Let $k\colon G \twoheadrightarrow G_0$ be the arrow in the colimit cone. Then $h \circ k\colon G \twoheadrightarrow G_0'$ is an epi so that we have $l\colon G_0' \longrightarrow G_0$ such that

$l \circ h \circ k = k$. The diagram is

$$\begin{tikzcd} & G_0' \arrow[dl, "h \circ k"'] \arrow[d, "l"] & \\ G \arrow[r, two heads, "k"'] & G_0 \arrow[r, "h"] & G_0' \end{tikzcd}$$

From $l \circ h \circ k = k$ and k epi, we conclude that $l \circ h = \mathrm{id}$. But then $h \circ l \circ h = h$ and h is epi, so that $h \circ l = \mathrm{id}$ and we conclude that h is an isomorphism which contradicts $f \neq g$. □

2 Limits on countable chains

In Barr & Wells [1985], Proposition 7 of 9.4 states (in dual form) the following:

2.1 Theorem. *Suppose that the category $\mathscr{C}$ is cocomplete with finite limits and that $\mathscr{C}$ has and T preserves limits along countable chains. Then the underlying functor $\mathscr{C}_T \longrightarrow \mathscr{C}$ has an adjoint.*

The proof demonstrates that the terminal object of $\mathscr{C}_T$ is the limit of the chain

$$1 \xleftarrow{f} T1 \xleftarrow{Tf} T^2 1 \xleftarrow{T^2 f} \cdots \xleftarrow{T^n f} T^n 1 \xleftarrow{T^{n+1} f} T^{n+1} 1 \longleftarrow \cdots$$

where f is the unique arrow $T1 \longrightarrow 1$. In the present case of $\mathscr{C} = \mathbf{Set}$, the theorem applies as soon as T preserves limits along countable chains. Many functors do.

Let us call a functor ω*-continuous* if it preserves limits along countable chains, ω*-cocontinuous* if it preserves colimits along countable chains and ω*-bicontinuous* if it does both.

For the purposes of the next proposition, let us say that a finite equivalence relation E on a functor R is a subfunctor of $R \times R$ such that on any set X, EX is an equivalence relation on RX such that the equivalence class of any element is finite;

2.2 Proposition. *The class of ω-bicontinuous functors is stable under the following:*

1. *Finite limits;*

2. *Arbitrary sums;*

3. *Quotients modulo finite equivalence relations;*

Proof.

1. Any limit of ω-continuous functors is ω-continuous because limits commute with each other. It is easy to see that finite products and equalizers both commute with colimits along countable chains, so that ω-cocontinuous functors are stable under finite limits.

2. Since colimits commute with each other, ω-cocontinuous functors are stable under arbitrary colimits, in particular under arbitrary sums. Suppose we have an $\mathscr{I}$-indexed family of limit chains

$$X_{i1} \xleftarrow{f_{i1}} X_{i2} \longleftarrow \cdots \longleftarrow X_{in} \xleftarrow{f_{in}} X_{i\,n+1} \longleftarrow \cdots \longleftarrow X_i$$

with $g_{in}: X_i \longrightarrow X_{in}$ the maps from the limit. We want to show that

$$\sum X_{i1} \xleftarrow{\sum f_{i1}} \sum X_{i2} \longleftarrow \cdots \sum X_{in} \xleftarrow{\sum f_{in}} \sum X_{i\,n+1} \longleftarrow \cdots \longleftarrow \sum X_i$$

is also a limit chain. All sums in this argument are taken over the index set I. Suppose A is a set and we have a family of functions $h_n: A \longrightarrow \sum X_{in}$ such that

$$\left(\sum f_{in}\right) \circ h_{n+1} = h_n \tag{$*$}$$

Each h_n decomposes A as $\sum A_{in}$ where $A_{in} = h_n^{-1}(X_{in})$. From $(*)$, it follows that $A_{i\,n+1} \subseteq A_{in}$. If the inclusion were proper for any i, we would then have a proper inclusion $\sum A_{in} \subset \sum A_{i\,n+1}$, which contradicts the fact that both sums are A. Hence A_{ni} is independent of n and we denote it by A_i. Let $h_{ni}: A_i \longrightarrow X_{ni}$ be the restriction of h_i. From $(*)$ it obviously follows $f_{in} \circ h_{i\,n+1} = h_{in}$. This family gives a unique map $h_i: A_i \longrightarrow X_i$ such that $g_{in} \circ h_i = h_{in}$ and then $h = \sum h_i: A \longrightarrow \sum X_i$ is the unique arrow such that $\sum g_{in} \circ \sum h_i = \sum h_{in} = h_n$. Thus $\sum X_i$ is the limit of the chain of $\sum X_{in}$.

3. Again only the continuity is in question. Let us suppose we have a commutative diagram

$$\begin{tikzcd}
K_\infty \arrow[r] & \cdots \arrow[r] & K_n \arrow[r] & \cdots \arrow[r] & K_1 \arrow[r] & K_0 \\
R_\infty \arrow[r] \arrow[d] & \cdots \arrow[r] & R_n \arrow[r] \arrow[d] & \cdots \arrow[r] & R_1 \arrow[r] \arrow[d] & R_0 \arrow[d] \\
T_\infty \arrow[r] & \cdots \arrow[r] & T_n \arrow[r] & \cdots \arrow[r] & T_1 \arrow[r] & T_0
\arrow[from=1-1, to=2-1, shift left] \arrow[from=1-1, to=2-1, shift right]
\arrow[from=1-3, to=2-3, shift left] \arrow[from=1-3, to=2-3, shift right]
\arrow[from=1-5, to=2-5, shift left] \arrow[from=1-5, to=2-5, shift right]
\arrow[from=1-6, to=2-6, shift left] \arrow[from=1-6, to=2-6, shift right]
\end{tikzcd}$$

in which each of the three lines is a limit sequence and all the columns with finite indices are kernel pair/coequalizer sequences. Further suppose that the kernel pairs are such that each equivalence class is finite. We want to show that the left hand column is also a coequalizer.

First we show that $R_\infty \longrightarrow T_\infty$ is surjective. Let $t = \langle t_0, t_1, \ldots\rangle \in T_\infty$. For each n let $A_n \subseteq R_n$ be the inverse image of t_n under the vertical arrow. Since the vertical arrows are surjective, $A_n \neq 0$ for all n and the finiteness condition implies it is finite. The commutativity of the diagram implies that the horizontal arrow takes A_n into A_{n-1}. Thus we have a sequence of non-empty finite sets

$$\cdots \longrightarrow A_n \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0$$

whose inverse limit is therefore non-empty. Any element $r = \langle r_0, r_1, \ldots\rangle$ of the inverse limit maps to t. Suppose $r' = \langle r'_0, r'_1, \cdots\rangle$ is another element mapping to t. Then for each n, we have $\langle r_n, r'_n\rangle$ is the image of some element $k_n \in K_n$. Thus we have a sequence

$$k = \langle k_0, k_1, \cdots\rangle \in K_\infty$$

whose image in $R_\infty \times R_\infty$ is $\langle r, r'\rangle$. This shows that $K_\infty \rightrightarrows R_\infty \longrightarrow T_\infty$ is a coequalizer.

□

The identity functor is ω-bicontinuous and hence so are its powers. The symmetric nth power is the quotient modulo the action of the symmetric group S_n which determines a finite equivalence. Constant functors are also bicontinuous so the above proposition assures that any functor of the form

$$TX = \sum_{n>0}(A_n \times X^n + B_n \times X^n)/S_n$$

is ω-bicontinuous.

2.3 Quotients. One thing notably lacking in the stability properties of ω-bicontinuous functors are quotients in general. It turns out that if T is a quotient of the ω-bicontinuous functor R, then we can give a direct description of the a terminal T-coalgebra in terms of a terminal R-coalgebra. This construction depends heavily on the fact that we are in the category of sets, where all epimorphisms split.

Let $\pi\colon R \longrightarrow T$ be a surjective natural transformation between endofunctors on **Set**. We make no hypothesis about the nature of R except that it have a terminal coalgebra $\alpha\colon A \longrightarrow RA$. We will show that a quotient of A is a terminal T-coalgebra.

Let $A_0 = A$ and $\alpha_0 = \alpha$. Define $\gamma_0: A_0 \longrightarrow A_1$ to the be the image of $\pi \circ \alpha_0$ so that we have the commutative square

$$\begin{tikzcd} A_0 \arrow[r, two heads, "\gamma_0"] \arrow[d, "\alpha_0"'] & A_1 \arrow[d, tail, "\alpha_1"] \\ RA_0 \arrow[r, two heads, "\pi A_0"'] & TA_0 \end{tikzcd}$$

The function α_0 is an isomorphism and hence so is α_1, although we make no real use of this fact. We define the object A_n and functions $\alpha_n: A_n \longrightarrow TA_{n-1}$ and $\gamma_{n-1}: A_{n-1} \twoheadrightarrow A_n$ inductively so that γ_{n-1} is the image of $T\gamma_{n-2} \circ \alpha_{n-1}$. Thus we have the commutative square

$$\begin{tikzcd} A_{n-1} \arrow[r, two heads, "\gamma_{n-1}"] \arrow[d, "\alpha_{n-1}"'] & A_n \arrow[d, tail, "\alpha_n"] \\ TA_{n-2} \arrow[r, two heads, "T\gamma_{n-2}"'] & TA_{n-1} \end{tikzcd}$$

Again it follows from the fact that α_{n-1} is an isomorphism that α_n is.

Let $B = \operatorname{colim} A_n$ with transition functions $\delta_n: A_n \longrightarrow B$. We have $\delta_n \circ \gamma_{n-1} = \gamma_n$ for $n > 0$. Define $\beta: B \longrightarrow TB$ so that $\beta \circ \delta_0 = T\delta_0 \circ \pi A_0 \circ \alpha_0$ and $\beta \circ \delta_n = T\delta_{n-1} \circ \alpha_n$ for $n > 0$. This is a compatible family of functions for

$$\beta \circ \delta_1 \circ \gamma_0 = T\delta_0 \circ \alpha_1 \circ \gamma_0 = T\delta_0 \circ \pi A_0 \circ \alpha_0 = \beta \circ \delta_0$$

while for $n > 0$,

$$\begin{aligned} \beta \circ \delta_{n+1} \circ \gamma_n &= T\delta_n \circ \alpha_{n+1} \circ \gamma_n = T\delta_n \circ \gamma_{n-1} \circ T\alpha_n \\ &= \delta_n \circ T(\gamma_{n-1} \circ \alpha_n) = T\delta_{n-1} \circ \alpha_n = \beta \circ \delta_n \end{aligned}$$

Note that this construction does not suppose that T preserves the colimit that defines β. If it should preserve that colimit, then one easily infers that β is an isomorphism from the fact that all of the α_n are. However, as we will see in Example 4.6, (B, β) is not an initial T-algebra. It is weakly initial:

2.4 Proposition. *The object (B, β) is weakly initial in the category $\mathbf{Set}_T$.*

Proof. Let $\xi: X \longrightarrow TX$ be a T-algebra and let $\theta: TX \longrightarrow RX$ split πX so that $\xi = \pi X \circ \theta \circ \xi$. Then $\theta \circ \xi: X \longrightarrow RX$ is an R-algebra; hence there is a unique arrow

$f: X \longrightarrow A$ such that

$$\begin{tikzcd}
X \arrow[r, "\theta \circ \xi"] \arrow[d, "f"'] & RX \arrow[d, "\alpha"] \\
A \arrow[r, "Rf"'] & RA
\end{tikzcd}$$

commutes. I claim that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\theta \circ \xi"] \arrow[d, "f"'] & RX \arrow[r, "\pi X"] \arrow[d, "\alpha"] & TX \arrow[d, "Tf"] \\
A \arrow[r, "Rf"'] \arrow[d, "\delta_0"'] & RA \arrow[r, "\pi A"'] & TA \arrow[d, "T\delta_0"] \\
B \arrow[rr, "\beta"'] & & TB
\end{tikzcd}$$

In fact, the upper left square is the preceding square, the upper right commutes by the naturality of π and the bottom rectangle commutes from the definition of $\beta \circ \delta_0$. Together with $\pi X \circ \theta \circ \xi = \xi$ this implies that $\delta_0 \circ f: (X, \xi) \longrightarrow (B, \beta)$ is a morphism of $\mathbf{Set}_T$ and hence that (B, β) is weakly initial. □

It is now standard that the terminal object in $\mathbf{Set}_T$ is the coequalizer of all the endomorphisms of (B, β). Equivalently, it is the cointersection of all its quotients. As we will see in an example, this information may suffice to give a good handle on this object.

3 Initial algebras and terminal coalgebra

For any endofunctor T on any category $\mathscr{C}$, there is a connection between an initial T-algebra and a terminal T-coalgebra (assuming both exist). In fact, as is well known since [Lambek, 1970], if $\epsilon: TE \longrightarrow E$ is an initial T-algebra, the structure map ϵ is an isomorphism. Dually for terminal coalgebras. So if $\phi: F \longrightarrow TF$ is the terminal coalgebra, then $\phi^{-1}: TF \longrightarrow F$ is an algebra, so there is a unique algebra homomorphism $f: E \longrightarrow F$. Dually, ϵ^{-1} is a coalgebra structure and so there is a unique coalgebra homomorphism $E \longrightarrow F$. This turns out to be the same morphism and is characterized by the symmetric equation $\phi \circ f \circ \epsilon = Tf$. We will call this morphism the canonical morphism.

Freyd has studied categories in which this canonical morphism is always an isomorphism. This is not the case in $\mathbf{Set}$. The following theorem shows what is true and is almost as surprising. We note that any inverse limit in **Set** has a natural topology as a subobject of a product of discrete sets. In particular, the limit in Theorem 2.1 that defines a terminal coalgebra of an ω-bicontinuous functor has such a topology.

We begin with a preliminary result.

3.1 Proposition. *Let*

$$\begin{tikzcd}
X_0 \arrow[r, "j_0"] \arrow[d, "k_0"'] & X_1 \arrow[r] \arrow[d, "k_1"] & \cdots \arrow[r] & X_n \arrow[r, "j_n"] \arrow[d, "k_n"'] & X_{n+1} \arrow[r] \arrow[d, "k_{n+1}"] & \cdots \\
Y_0 & Y_1 \arrow[l, "t_1"] & \cdots \arrow[l] & Y_n \arrow[l] & Y_{n+1} \arrow[l, "t_{n+1}"] & \cdots \arrow[l]
\end{tikzcd}$$

be a diagram in $\mathscr{S}$ in which

P–1. $t_{n+1} \circ k_{n+1} \circ j_n = k_n$ *for all* n.

P–2. k_n *is injective for all* n.

P–3. $t_n \circ k_n$ *is surjective for all* n.

Then the limit of the lower sequence is the completion of the colimit of the upper in a natural metric.

Proof. Let $u_n : X_n \longrightarrow E$ be the transition map to the colimit and $p_n : F \longrightarrow Y_n$ be the transition map to the limit. Define a family of functions $f_{mn} : X_n \longrightarrow Y_m$ by the formulas

$$f_{mn} = \begin{cases} k_n & \text{if } n = m \\ k_m \circ j_{m-1} \circ \cdots \circ j_n & \text{if } n > m \\ t_{m+1} \circ \cdots \circ t_n \circ k_n & \text{if } n < m \end{cases}$$

Next we claim that $f_{m\,n+1} \circ j_n = f_{mn} = t_{m+1} \circ f_{m+1\,n}$. The first equation can be read off from one or the other of the following two diagrams:

$$\begin{tikzcd}
X_n \arrow[r, "j_n"] & X_{n+1} \arrow[r] & \cdots \arrow[r] & X_{m-1} \arrow[r, "j_{m-1}"] & X_m \arrow[r, "j_m"] \arrow[d, "k_m"'] & X_{m+1} \arrow[d, "k_{m+1}"] \\
 & & & & Y_m & Y_{m+1} \arrow[l, "t_{m+1}"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & X_n \arrow[d, "k_n"] \\
Y_m & Y_{m+1} \arrow[l, "t_{m+1}"'] & \cdots \arrow[l] & t_n \arrow[l] & Y_n \arrow[l, "t_n"']
\end{tikzcd}$$

The second one is dual to the first and is proved with similar diagrams.

These compatibilities imply, from the universal mapping properties of colimits and limits that there is a unique function $f\colon E \longrightarrow F$ such that $p_m \circ f \circ u_n = f_{mn}$.

Condition P–2 implies that each composite $X_n \xrightarrow{u_n} E \xrightarrow{f} F$ is injective and hence that $f\colon E \longrightarrow F$ is injective and P–3 implies that each composite $E \xrightarrow{f} F \xrightarrow{p_n} Y_n$ is surjective. We will use f to suppose that E is a subset of F.

Define a metric on F by saying that $d(y, y') = 2^{-n}$ for the largest n, if any, for which $p_n(y) = p_n(y')$. If there is no such n, then $d(y, y') = 2$. We easily see that the topology of this metric is the topology induced on the limit by the product topology. Now suppose that $x^{(0)}, x^{(1)}, \ldots, x^{(n)}$ is a Cauchy sequence in E in the induced metric. By thinning, we can suppose that $d(x^{(n)}, x^{(m)}) \le 2^{\min(n,m)}$. That is, for all $n < m$, we have $p_n \circ f(x^{(n)}) = p_n \circ f(x^{(m)})$. Thus if we let $y^{(n)} = p_n \circ f(x_n)$ we have a compatible family of elements of $\prod Y_n$. Thus there is a unique element $y \in F$ with $p_n(y) = y^{(n)} = p_n(x^{(m)})$ for all $m \ge n$ from which it is immediate that $\lim x^{(n)} = y$. The uniqueness of y is clear, so that F is the Cauchy completion of E. □

3.2 Theorem. *Let $T\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be an $\aleph_0$-accessible functor that is also ω-continuous (and hence ω-bicontinuous). Assume that $T\emptyset \ne \emptyset$. The the terminal T-coalgebra is the Cauchy completion of the initial T-algebra.*

Proof. We will first do this under the additional hypothesis that T preserves monics. In the diagram

$$
\begin{tikzcd}
\emptyset \arrow[r, "j"] \arrow[d, "k"'] & T\emptyset \arrow[r] \arrow[d, "Tk"] & \cdots \arrow[r] & T^n\emptyset \arrow[r, "T^n j"] \arrow[d, "T^n k"'] & T^{n+1}\emptyset \arrow[r] \arrow[d, "T^{n+1}k"] & \cdots \\
1 & T1 \arrow[l, "t"] & \cdots \arrow[l] & T^n 1 \arrow[l] & T^{n+1}1 \arrow[l, "T^n t"] & \cdots \arrow[l]
\end{tikzcd}
$$

the initial algebra (E, ϵ) is the colimit of the upper row and the terminal coalgebra (F, ϕ) is the limit of the bottom row. We note also that

1. $t \circ Tk \circ j = k$ and hence $T^n t \circ T^{n+1}k \circ T^n j = T^n k$ for all n.

2. k is injective and hence $T^n k$ is injective for all n.

3. $t \circ Tk$ is surjective and hence $T^n t \circ T^{n+1} k$ is surjective for all n.

Thus the preceding proposition applies and we conclude that F is the Cauchy completion of E in the metric of the limit.

Next we claim that the inclusion f is the canonical function from the initial algebra to the terminal coalgebra. We must show that $\phi \circ f \circ \epsilon = Tf$. Since TE is a colimit and TF a limit, this is equivalent to $Tp_m \circ \phi \circ f \circ \epsilon \circ Tu_m = Tp_m \circ Tf \circ Tu_n$. But the equations we have show that both sides are $f_{m+1\,n+1}$ and so f is the canonical map.

This finishes the proof under the additional assumption that T functors preserve injective functions. An injective with a non-empty domain splits and hence is automatically preserved. If $g: \emptyset \longrightarrow A$, then Tg is not necessarily injective. However, it turns out that we can modify T to produce a functor T^* that does preserve injectives, such that $T^*\emptyset \neq \emptyset$ and such that T and T^* have the same initial algebra and terminal coalgebra. We let $d_0, d: 1: 1 \longrightarrow 2$ be the two arrows and define

$$T^*\emptyset \xrightarrow{d} T1 \underset{Td_1}{\overset{Td_0}{\rightrightarrows}} T2$$

to be an equalizer. If the set A is non-empty, we let $T^*A = TA$. We have to define T^* on arrows. On arrows with non-empty domain, we let it be the same as T. On the identity of $\emptyset$, it is, of course, the identity. On the arrow $g: \emptyset \longrightarrow A$, with $A \neq 0$, choose an element $a: 1 \longrightarrow A$ and let $T^*g = Ta \circ d$. We first show this is independent of the element a. If $a': 1 \longrightarrow A$ is another, then there is a unique arrow $f: 2 \longrightarrow A$ such that $f \circ d_0 = a$ and $f \circ d_1 = a'$. Then $Ta \circ d = T(f \circ d_0) \circ d = Tf \circ Td_0 \circ d = Tf \circ Td_1 \circ d$ which reduces similarly to $Ta' \circ d$. This shows that T^*g does not depend on the choice of element, from which it is immediate that if $h: A \longrightarrow B$, then $Th \circ T^*g = T^*(h \circ g)$. Since there are no non-identity arrows to $\emptyset$, this shows that T^* is a functor.

Next we show that $T^*\emptyset \neq \emptyset$. Since $\emptyset \longrightarrow 1 \rightrightarrows 2$ is an equalizer, we have $T\emptyset \longrightarrow T1 \rightrightarrows T2$ commutes. But $T\emptyset \neq \emptyset$ so that the image of $T\emptyset \longrightarrow T1$ is non-empty and provides an element in the equalizer.

Also, since any $a: 1 \longrightarrow A$ is a split monic, the composite $T^*g = Ta \circ d$ is monic, and we see that T^* preserves all monics.

Since $T\emptyset \neq \emptyset$, the empty set does not allow either a T-algebra or T^*-algebra structure. Thus the categories of T-algebras and of T^*-algebras are isomorphic and have the same initial algebras (although the descriptions of it as a colimit will differ). As for coalgebras, the categories are also isomorphic, since the empty set bears a unique coalgebra structure (which is initial). In this case, the description of the terminal coalgebra is the same for T and T^*. $\square$

3.3 Remark. The argument used here is very similar to the construction of fixed points using embedding projection pairs (See [Barr & Wells, 1990] where they are called retract pairs. Also see [Smyth & Plotkin, 1983].) Indeed, if $T\emptyset = 1$ (which happens in many interesting cases), there is a *single* sequence made up of embedding projection pairs whose colimit is the initial algebra and whose limit is the terminal coalgebra. This observation suggests that embedding projection pairs are more than an *ad hoc* construction for finding fixed points.

3.4 Dependence on the axiom of choice. The above argument depends on the fact that functors on sets preserve injectives (with non-empty domain) and surjectives. The second depends on the axiom of choice and the first on the somewhat weaker property of being boolean. Neither assumption is a good one to make for computer science. So we consider briefly how necessary these assumptions are.

Interestingly, even without the axiom of choice, *every* $\aleph_0$-accessible functor preserves surjections. For T is $\aleph_0$-accessible if and only if the arrow $RX = \sum_{n\in\mathbf{N}} Tn \times X^n \longrightarrow TX$ is surjective for all X. In fact, this hypothesis is simply the explicit formulation of the property that every element of TX comes from $Tn \longrightarrow TX$ for some finite cardinal n. But in any topos, the functors X^n, for finite n, and constant multiples thereof, preserves epis and sums of such functors do as well. Thus if $X \twoheadrightarrow Y$, we have

$$\begin{tikzcd} RX \arrow[r, two heads] \arrow[d] & RY \arrow[d] \\ TX \arrow[r] & TY \end{tikzcd}$$

from which we see that $TX \twoheadrightarrow TY$ is epic as well.

The question of functors preserving monics seems more difficult. Certainly every polynomial functor does, since in any topos, sums preserve monics. Other functors, like the finite subsets functor can be shown directly to preserve monics. It seems likely that functors that arise in practice will have the property, but I know of no proof.

4 Examples

4.1 Suppose $T\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is the constant functor at some set S. Since the terminal coalgebra (A, α) has an isomorphism for structure map, the only possibility for the terminal coalgebra (or initial algebra) is, up to isomorphism, the identity arrow on A and it is a minute's work to see that it is.

4.2 Suppose we let $T\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be any functor for which $T1 = 1$. Since the terminal object is fixed, it is easy to conjecture—and just as easy to prove—that the terminal object is just 1 with the unique structure map. It is equally easy to show that any limit diagram that is preserved by (that is taken to a limit diagram by) T is created by the underlying functor $\mathbf{Set}_T \longrightarrow \mathbf{Set}$.

4.3 The easiest nontrivial example is the functor $T\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ defined by $TX = A + X$, with A a fixed set. This is a polynomial functor, so Proposition 2.2 applies and we conclude that the terminal object is the limit of the diagram

$$1 \xleftarrow{t_0} A + 1 \xleftarrow{t_1} A + A + 1 \longleftarrow \cdots \longleftarrow n \times A + 1 \xleftarrow{t_n} (n+1) \times A + 1 \longleftarrow \cdots$$

where t_0 is the unique map and $t_n = T(t_{n-1})$. It turns out that $t_n: (n+1) \times A + 1 \longrightarrow n \times A + 1$ takes the first n copies of A isomorphically to the n copies of A in the codomain, takes the remaining copy to 1 and is the identity on 1. The inverse limit of this sequence is $\mathsf{N} \times A + 1$.

A coalgebra $\alpha: X \longrightarrow A + X$ is made up of two partial functions $\alpha_1: X \longrightarrow A$ and $\alpha_2: X \longrightarrow X$. The terminal map $f: X \longrightarrow \mathsf{N} \times A + 1$ takes an element of X to 1 if it is in $\bigcap_n \mathrm{dom}(\alpha_2^n)$. If not, then $f(x) = \langle n, \alpha_1 \alpha_2^n(x) \rangle$ for the unique n such that $x \in \mathrm{dom}(\alpha_2^n) - \mathrm{dom}(\alpha_2^{n+1})$.

The initial algebra in this case is $\mathsf{N} \times A$ embedded in the obvious way. The topology is the finite complement topology.

4.4 Let $T: \mathbf{Set} \longrightarrow \mathbf{Set}$ be defined by $TX = 1 + A \times X$ for a fixed set A. This is again a polynomial functor so the terminal object is given by the inverse limit

$$1 \longleftarrow 1 + A \longleftarrow 1 + A + A^2 \longleftarrow \cdots \longleftarrow 1 + A + A^2 + \cdots + A^n \longleftarrow \cdots$$

with the arrow $t_n: 1 + A + \cdots + A^n + A^{n+1} \longrightarrow 1 + A + \cdots + A^n$ the identity on the first $n+1$ terms and the projection of $A^n \longrightarrow A^{n+1}$ on the last. An element of the inverse limit always begins with a sequence (possibly null) of finite sequences

$$\langle\rangle, \langle a_1 \rangle, \langle a_1, a_2 \rangle, \cdots, \langle a_1, a_2, \cdots, a_n \rangle$$

This can be extended either by the element $\langle a_1, a_2, \cdots, a_n \rangle \in A^n \subseteq T^{n+1}1$ or by any element of the form $\langle a_1, a_2, \cdots, a_n, a_{n+1} \rangle \in A^{n+1} \subseteq T^{n+1}1$. If the first choice is made, then there is no further choice and this element of the inverse limit is simply equivalent to the finite sequence $\langle a_1, a_2, \cdots, a_n \rangle$. If the second choice is made, then this process continues at least one more step. If it continues to add terms indefinitely, then we build an infinite sequence of elements of A. Thus the terminal coalgebra is equivalent to the set of finite and infinite strings of elements of A. This fact was first observed for the case $A = 1$ in [Mendler *et. al.*, 1986].

The topology on the terminal coalgebra is the one in which an infinite string is approximated by its initial segments. The set of finite strings is the initial algebra.

4.5 Let TX be the functor that assigns to each set S the set of subsets of X that have at most n elements. Then TX is a quotient of the functor $RX = 1 + X^n$ which identifies two n-tuples that include the same *set* of elements. The number of elements that are equivalent to any given n-tuple is evidently at most $n!$. We let $KX = 1 + \mathrm{Hom}(n, n) \times \mathrm{Hom}(n, n) \times X^n$. The two functions d^0, $d^1: KX \longrightarrow RX$ are defined by

$$d^0(\sigma, \tau, \langle x_1, \ldots, x_n \rangle) = \langle x_{\sigma 1}, \ldots, x_{\sigma n} \rangle$$

and

$$d^1(\sigma, \tau, \langle x_1, \ldots, x_n \rangle) = \langle x_{\tau 1}, \ldots, x_{\tau n} \rangle$$

The terminal coalgebra can be written down of course, but it is a little hard to see what the inverse limit is. Here is a direct computation of the terminal coalgebra.

Let A be the set (of isomorphism classes, but we ignore the distinction) of rooted trees which are at most n-way branching, including ones of infinite depth. Then there is an obvious coalgebra structure on α on A that takes a tree to 1 if the tree is a bare root and otherwise takes it to the set of its daughters, which is a finite subset of A. We show that A is a weak terminal object.

Let A_m denote the subset of A consisting of the trees of depth limited to m. Although A is not the union of the A_m, there is an obvious truncation function, which we denote $a \mapsto a|m$ of $A \longrightarrow A_m$ and an obvious topology in which each tree is a limit of its truncations. In addition, each element of A can be described by its truncations. Thus given a sequence of trees $a_0, a_1, \ldots, a_m, \ldots$ such that $a_m|m-1 = a_{m-1}$ for each m, there is a unique $a \in A$ such that $a|m = a_m$.

Now let (B, β) be a T-coalgebra. We will define a morphism $g\colon (B, \beta) \longrightarrow (A, \alpha)$ by defining for each $b \in B$ a sequence $g_0(b), g_1(b), \ldots$ of trees such that $g_m(b)$ is a tree of depth at most m and $g_m(b)|m-1 = g_{m-1}(b)$. We begin by defining $g_0(b)$ to be a bare root. Assume the functions $g_0, g_1, \ldots, g_{m-1}$ have been defined (for all coalgebras). Suppose that $\beta(b) = \{b_1, b_2, \ldots, b_k\}$, $k \le n$ are the elements of $\beta(b)$, listed without repetition. If $k = 0$, then $g_m(b)$ is defined to be the bare root. Otherwise, we define $g_m(b)$ to be the tree with a root and with the trees $g_{m-1}(b_1), g_{m-1}(b_2), \ldots, g_{m-1}(b_k)$ attached at the root.

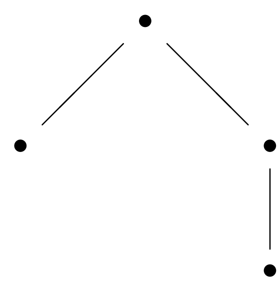
We can now show that $g_m(b)|m-1 = g_{m-1}(b)$. In fact, if $m = 1$, this is obvious. If we suppose that $g_{m-1}(b_i)|m-2 = g_{m-2}(b_i)$ for $i = 1, \ldots, k$, then $g_m(b)|m-1$ is simply a root with $g_{m-1}(b_1)|m-2$, $g_{m-1}(b_2)|m-2$, $\ldots$, $g_{m-1}(b_k)|m-2$ attached. This is a root with $g_{m-2}(b_1)$, $g_{m-2}(b_2)$, $\cdots$, $g_{m-2}(b_k)$ attached and that is $g_{m-1}(b)$. Thus we define $g(b)$. It is clear from the construction that $\alpha \circ g_m(b) = g_{m-1} \circ \tau(b)$ and hence $\alpha \circ g(b) = g \circ \tau(b)$ so that g is a coalgebra homomorphism and (A, α) is weakly terminal.

This is not, however, the terminal coalgebra. In fact, α isn't an isomorphism, since a tree with a root and two identical daughters has the same value under α as a tree with a root and one copy of that daughter.

Let us say that a binary rooted tree is **extensional** if no node has two identical daughters. It is clear that the set of extensional trees is a subcoalgebra. The image of the map g constructed above is not necessarily an extensional tree but since each tree has an extensional quotient, this causes no problem. This is done by first identifying any two identical daughters of any node of the tree. The resultant tree may again have nodes with identical daughters, so do it again. After a possibly transfinite number of steps, the tree will be extensional. This is assured because the quotient lattice is complete.

On the extensional trees, α is an isomorphism. This suggests that the set of extensional trees might be the terminal object. This is wrong, since the terminal coalgebra cannot have any congruences and we will see that there is a congruence on the set of extensional trees.

Begin by saying of two trees t, t' that $t \approx t'$ (read t is **extensionally equivalent** to t') if they reduce to the same extensional tree. Let us say of two trees t and t' that $t \sim t'$ (read t is **similar** to t') if $t|n \approx t'|n$ for all n. On finite trees this is the $\approx$ relation, but we will see by example that on infinite trees it is coarser. First we describe a pair of non-isomorphic extensional trees t and t' (so that $t \not\approx t'$). The tree t is simply an infinite chain. The tree t' is constructed as follows. Let t_1 be the tree



By a **leaf** we mean a node with no daughters. Evidently t_1 has two leaves. Let t_2 be the tree gotten from t_1 by replacing each leaf by a copy of t_1. Then t_2 has four leaves. Replace each one by a copy of t_1 to make t_3. Continue in this way to build a tree we denote by t' of infinite depth and with no leaves. (The tree we construct is actually the colimit of the finite approximations, treating trees as posets, hence categories.) This tree is extensional since at every branch the trees on the two nodes are different from each other. In each case one is an immediate branch and the other isn't.

On the other hand, any finite truncation of this tree is similar to a chain. In fact the tree has no leaves so that $t'|n$ has the property that every leaf is at depth exactly n. I claim that a tree with that property is extensionally equivalent to an $n+1$ element chain. In fact, the claim is evident when $n = 0$. Assume that the claim is true for $n-1$. If the given tree is not a bare root, then each of the non-empty set of nodes attached to the root has the property that all its leaves have depth exactly $n-1$ and are therefore extensionally equivalent to n element chains. Since all the daughters are extensionally equivalent to n element chains, the original tree is equivalent to an $n+1$ element chain. Thus $t \sim t'$ but $t \not\approx t'$.

We can now show that the $\sim$ classes of trees is the terminal coalgebra. In fact, suppose that $\equiv$ is a congruence relation on trees. We will show that $\equiv$ is included in $\sim$.

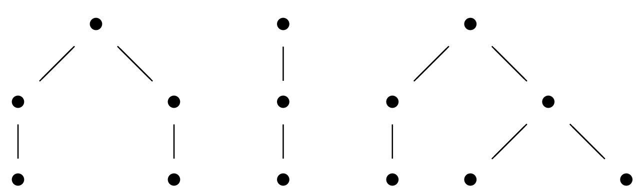
We must show that $a \equiv a'$ implies that $a|m \approx a'|m$ for all m. This is immediate for $m = 0$. Assume that this has been shown for $m-1$ and all pairs a and a'. Since $\equiv$ is a congruence, we must have that $\alpha(a) \equiv \alpha(a')$. Hence $\alpha(a) = \emptyset$ if and only if $\alpha(a') = \emptyset$ and in that case there is nothing to prove. So let us suppose that $\alpha(a) = \{a_1, a_2, \ldots, a_k\}$ and $\alpha(a') = \{a'_1, a'_2, \ldots, a'_{k'}\}$. The congruence classes of a_1, ..., a_k must coincide with those of a'_1, ..., $a'_{k'}$. Suppose that a_1, a_2, ..., a_l are a complete set of congruence classes in $\alpha(a)$ and similarly for a'_1, a'_2, ..., a'_l in $\alpha(a')$ (the numbers will be the same). We can further suppose that they are numbered so that $a_1 \equiv a'_1$, $a_2 \equiv a'_2$, ..., $a_l \equiv a'_l$. By the inductive hypothesis, we have $a_i|m-1 \approx a'_i|m-1$ for $i = 1, 2, \ldots, l$. Moreover if $l < j \le k$, we have that $a_j \equiv a_i$ for some $1 \le i \le l$ which implies that $a_i|m-1 \approx a_j|m-1$ and similarly for the daughters of a'. But this means that $a|m$ is extensionally equivalent to the tree with a_1, a_2, ..., a_l attached to the root and similarly $a'|m$ is extensionally equivalent to the tree with a'_1, a'_2, ..., a'_l attached to a root. Since these are equivalent in pairs, it follows that $a|m \approx a'|m$.

Thus $\sim$ is the top of the congruence lattice and the set of similarity classes is the terminal coalgebra.

4.6 Let $\mathbf{P}_{\text{fin}}: \mathbf{Set} \longrightarrow \mathbf{Set}$ denote the functor that associates to each set the set of its finite subsets with direct image on morphisms. A coalgebra consists of a pair (A, α) where A is a set and α is a function that assigns to each element of A a finite subset. The coalgebra exists by Theorem 1.2, but it is not given by the inverse limit since this functor is not ω-continuous. The reason is that a countable family of subsets might have more and more elements and in the limit correspond to an infinite subset of its domain.

On the other hand, $\mathbf{P}_{\text{fin}}$ is a quotient of the functor $RX = \sum_{n=0}^{\infty} X^n$ and so Proposition 2.4 applies and we get an terminal coalgebra for T as a quotient of an terminal coalgebra for R. It is not hard to see that an terminal coalgebra for R is the set of finitely branching ordered trees, possibly of infinite depth.

It is interesting to follow the construction of a weak terminal algebra by the process given in the proof of Proposition 2.4. The object A_0 is the set of finitely branching ordered trees. In A_1 two such trees are identified if the set of their first level branches are the same. So, for example, in the diagram below, the trees (1) and (2) are identified in A_1, but they are not identified with (3) because the two branches are distinct.



(1) (2) (3)

In A_2, the two lower branches of (3) are identified so that now all three become equal. In the limit we get the set of extensional finitely branching trees. However, just as in the preceding example, this is not the terminal object, although it is weakly terminal. The construction of the preceding example works without significant change. As before we let $t \sim t'$ if for all n we have $t|n$ and $t'|n$ have the same extensional reduction. Then $(T, \tau)/\sim$ is the terminal coalgebra. The argument is quite similar to the previous one. The details are left to the reader.

This is a good example because it illustrates what goes wrong with quotients of ω-bicontinuous functors. The functor R is ω-bicontinuous so its terminal coalgebra is given by the limit of the diagram

$$1 \longleftarrow R1 \longleftarrow \cdots \longleftarrow R^n 1 \longleftarrow \cdots$$

It is not hard to see that one can interpret $R^n 1$ as the set of rooted trees of depth at most n. But the trees are non-extensional. The morphism from $R^n 1$ to $R^{n-1} 1$ is truncation to one depth lower. This leaves width unchanged. That is, if a tree has depth n and it has a particular k-way branch at some depth $d < n$ then the image of that node under the truncation will still be k-way branching. When the extensional identity is imposed, a k-way branch can have some heretofore distinct branches identified when it is truncated.

The result is that in the inverse limit a particular branch can increase the nodes of nodes as you move out in the sequence and the limit tree can have a node with an infinite-way branch. This inverse limit is apparently not a coalgebra and certainly not a terminal coalgebra. The way to find the terminal coalgebra is as we have done here: first find the terminal R-coalgebra and then form the quotient.

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